A Birkhoff–Lewis type theorem for the nonlinear wave equation *

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Abstract

We give an extension of the celebrated Birkhoff–Lewis theorem to the nonlinear wave equation. Accordingly we find infinitely many periodic orbits with longer and longer minimal periods accumulating at the origin, which is an elliptic equilibrium of the associated infinite dimensional Hamiltonian system.

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1 Introduction and main results

The study of periodic solutions of the nonlinear wave equation has a long history. In particular, variational methods were successfully used, starting from the pioneering works of Rabinowitz at the end of the sixties, by many authors such as Brezis, Nirenberg, Coron, etc. The variational approach has the advantage of being global and of imposing mild restrictions on the strength of the nonlinearity. On the other hand it imposes a very strong restriction on the allowed periods: they have to be *rational* multiple of the string length, otherwise one faces with a *small divisors problem*. It must be remarked that in finite dimension small divisors appear only in searching quasi–periodic solutions and that the powerful machinery of KAM theory and, later, the Nash–Moser implicit function theorem were developed to deal with them. Starting from the late eighties Kuksin, Wayne, Craig, Bourgain, Pöschel, etc. used the fact that the wave equation is an *infinite dimensional Hamiltonian system* and modified the above finite dimensional "classical" ideas to work in this infinite dimensional context, finding periodic as well as quasi–periodic solutions. This alternative "Dynamical Systems" approach has the remarkable advantage of allowing irrational periods, but it is local, namely one finds only small amplitude solutions.

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Regarding this last approach, a first class of results was obtained in[1] [K87], [W90], [CW93], [Bou95], [Bou99], [P96], [Ba00], where the extension of the classical Lyapunov Center Theorem for finite dimensional Hamiltonian systems is proved. A latter class of results consists in the infinite dimensional extension of the Weinstein, Moser and Fadell–Rabinowitz resonant center theorems, see [LiS88], [BaPa01], [BeBo03], [GMPr04], [BeBo06], [Be]. All the previous results concern solutions that are continuations of the linear normal modes of oscillations and have periods close to the linear ones.

In the present paper we also use the "Dynamical Systems" approach, yet the kind of solutions we find is completely different. In [BirL34] (see also [L34], [Mo77]) Birkhoff and Lewis proved their celebrated theorem on the existence of infinitely many periodic orbits with larger and larger minimal period accumulating at elliptic equilibria of finite dimensional Hamiltonian systems. An extension of their result for some nonlinear PDEs, such as the beam equation and the regularizing NLS equation, was given in [BaBe05]; on the other hand, the possibility to include the case of wave equation was left as an open question, due to a more difficult small divisors problem, see Remark 1.4 below.

Here we extend the Birkhoff–Lewis theorem to the nonlinear wave equation. Namely we find infinitely many periodic orbits with longer and longer minimal periods accumulating at the origin, which is an elliptic equilibrium of the associated infinite dimensional Hamiltonian system.

Let us state more precisely our main result. We look for periodic in time solutions of the one–dimensional autonomous nonlinear wave equation with Dirichlet boundary conditions:

$$\begin{cases} u_{tt} - u_{xx} + \mu u + f(u) = 0 \\ u(t,0) = u(t,\pi) = 0\,, \end{cases} \tag{1}$$

where $\mu > 0$ is the "mass" and the nonlinearity f is a odd, real analytic function with $f'(0) = 0$, $f'''(0) \neq 0$. All the solutions of the linear equation $u_{tt} - u_{xx} + \mu u = 0$ are $\sum_{i\geq 1} A_i \cos(\omega_i t + \phi_i) \sin ix$, where $\omega_i := \sqrt{i^2 + \mu}$, $A_i \geq 0$, $\phi_i \in \mathbb{R}$. If $A_i = 0$ for every $i \neq i_0$, $i_0 \geq 1$, the solution is periodic; otherwise the solutions are not periodic[2] but are quasi–periodic or almost–periodic according if $N \geq 2$ or infinitely many A_i are different from zero.

In extending the Lyapunov Center Theorem to the wave equation, one fixes, for example, $i_0 := 1$ and finds $(2\pi/\tilde{\omega}_1)$–periodic solutions $u(t,x)$ of (1) that are ε–close to $\tilde{u}(t,x) := A_1 \cos(\tilde{\omega}_1 t + \phi_1) \sin x$, where $\tilde{\omega}_1 - \omega_1 \sim \varepsilon$ and $A_1 \sim \sqrt{\varepsilon}$. These solutions are continuations of the periodic solutions of the linear equation and their frequency $\tilde{\omega}_1$ tends to the linear frequency ω_1 when the perturbative parameter ε tends to zero.

On the contrary, we fix $N \geq 2$ and find $(2\pi/\varepsilon)$–periodic solutions $u(t,x)$ of (1) that are ε–close to $\tilde{u}(t,x) := \sum_{i\leq N} A_i \cos(\tilde{\omega}_i t) \sin x$, where $\tilde{\omega}_i - \omega_i \sim \varepsilon$ and $A_i \sim \sqrt{\varepsilon}$ for all $i \leq N$. Our solutions have frequency[3] ε tending to zero, that is, their period tends to infinity.

We stress that *these "Birkhoff–Lewis" periodic solutions are not continuations of periodic solutions of the linear equation*, where long period solutions do not exist at all, and *represent indeed a purely nonlinear phenomenon.*

We now state our formulation of the Birkhoff–Lewis theorem for the wave equation. It was announced in [BiDG07] and [BiV].

Theorem 1.1. *Fix $\mu > 0$ and let f be a real analytic, odd function of the form $f(u) = \sum_{m\geq 3} f_m u^m$, $f_3 \neq 0$. Let $N \geq 2$. Then there exists a Cantor like set $\mathcal{C}$ satisfying*

$$\lim_{\varepsilon_* \to 0^+} \frac{\mathrm{meas}\big(\mathcal{C} \cap (0, \varepsilon_*]\big)}{\varepsilon_*} = 1\,,$$

such that for all $\varepsilon = 2\pi/T \in \mathcal{C}$ there exists a T–periodic analytic solution $u(t,x)$ of (1), which is even

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[1] Actually [K87], [W90], [P96] are also devoted to quasi–periodic solutions.

[2] Except for a countable set of masses μ.

[3] By definition, the (maximal) frequency of a periodic orbit is the inverse of its (minimal) period.

in t, sine–series in x and satisfies

$$u(t,x) = \sqrt{\varepsilon} \left(\sum_{i \le N} a_i \, \cos(\tilde{\omega}_i t) \, \sin ix \;+\; O(\varepsilon) \right) , \tag{2}$$

in a suitable analytic norm (see Remark 1.2) and for suitable $a_i = a_i(\varepsilon) \ge \mathrm{ct} > 0$, $\tilde{\omega}_i = \tilde{\omega}_i(\varepsilon) = \varepsilon \kappa_i(\varepsilon)$, $\kappa_i(\varepsilon) \in \mathbb{N}^+$, $i \le N$,

$$\tilde{\omega}_i - \omega_i \;=\; O(\varepsilon) , \qquad\qquad \omega_i := \sqrt{i^2 + \mu} . \tag{3}$$

Moreover, for any fixed $0 < \rho < 1/2$, *except a zero measure set of* $\mu's$, *the minimal period* $T^{\min}$ *of the above* T*–periodic orbit satisfies*

$$T^{\min} \ge \mathrm{ct} / \varepsilon^{\rho} .$$

Remark 1.2. *Expanding in Fourier cosines and sines series with respect to t and x respectively, we write* $u(t,x) = \sum_{k \ge 0, i \ge 1} u_{ki} \, \cos(\varepsilon k t) \sin ix$. *Then u is close to the trigonometric polynomial* $\tilde{u}(t,x) = \sum_{k \ge 0, 1 \le i \le N} \tilde{u}_{ki} \cos(\varepsilon k t) \, \sin ix$, *where* $\tilde{u}_{ki} = \sqrt{\varepsilon} a_i$ *if* $k = \kappa_i$, *while* $\tilde{u}_{ki} = 0$ *otherwise. Finally u and $\tilde{u}$ are analytically ε–close*

$$\sum_{k \ge 0, i \ge 1} e^{2\bar{\alpha} k} |k|_*^{2\bar{\sigma}} e^{2\bar{a} i} i^{2\bar{s}} |u_{ki} - \tilde{u}_{ki}|^2 \le \mathrm{ct}\, \varepsilon^3 , \tag{4}$$

for some positive $\bar{a}, \bar{s}, \bar{\alpha}, \bar{\sigma}$ *(see (95)). Here and in the following* $|k|_* := \max\{|k|, 1\}$.

Remark 1.3. *Theorem 1.1 holds also substituting the summation over $i \le N$ in (2) with any other set of indexes* $\mathcal{I} = \{i_1, \ldots, i_N\}$ *with* $i_1 < \ldots < i_N$.

First we briefly recall the procedure used by Birkhoff and Lewis to prove their theorem. The first step consists in putting the Hamiltonian $H(p,q)$ in fourth order Birkhoff normal form, namely $H = Q + G + K$, where $Q = \sum_{i=1}^N \omega_i I_i$, $\omega_i \in \mathbb{R}$, $I_i := (p_i^2 + q_i^2)/2$ are the actions, G is a homogeneous quadratic polynomial in the "actions" and K is a remainder having a zero of fifth order at the origin. Close to the origin H is a perturbation of the integrable Hamiltonian $Q + G$. Assuming that the Hessian of G (w.r.t. I) is invertible (non–degeneracy "twist condition"), the action to frequency map of the integrable system is invertible; then one can find infinitely many completely resonant tori on which the flow of $Q + G$ is periodic. Can one expect some of these periodic orbits to persist for the complete Hamiltonian H? [BirL34] positively answered the question using *the implicit function theorem and topological methods.*

Sketch of the proof of Theorem 1.1

As in [BaBe05] we put the system in "Birkhoff seminormal[4] form". We fix a positive integer $N \ge 2$ and split the phase variables into two groups: the variables with index smaller or equal to N, the "low modes", and the variables with index larger than N, the "high modes". By a suitable canonical transformation we put the infinite dimensional Hamiltonian associated to the wave equation (see (7)) in the form $H = Q + \check{G} + \hat{G} + K$, where Q and $\check{G}$ are, respectively, homogeneous linear and quadratic terms in the actions, $\hat{G}$ is a homogeneous term of order four in the high modes, K has a zero of sixth order at the origin. Neglecting the term K, the $2N$–dimensional manifold obtained setting the high modes equal to zero is invariant and filled up by N–dimensional invariant tori. Under a non–degeneracy "twist" condition (which holds since $f_3 \neq 0$), the frequencies of the flow on such tori cover an open subset of $\mathbb{R}^N$. We focus on completely resonant tori that are filled up by periodic orbits. We want to prove that at least one periodic orbit persists for the complete Hamiltonian.

Then we put the low modes in action–angle variables and perform a Lyapunov–Schmidt reduction[5], inspired to [BeBiV04] (where the existence of periodic orbits accumulating on elliptic lower dimensional

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[4] Namely the sort of normal form used to construct lower dimensional tori (see [P96]). Complete Birkhoff normal forms were performed in [Ba03], [BaGr].

[5] Craig and Wayne first introduce the Lyapunov–Schmidt reduction method in studying PDEs in [CW93].

tori of a finite dimensional Hamiltonian system is proved) and also used in [BaBe05]. The bifurcation equation is defined for ϕ_0 belonging to a N–dimensional torus, whereas the range equation is infinite dimensional. Unlike the finite dimensional situation, here two new difficulties arise:

1. a small divisors problem in the range equation, that will be treated with Nash–Moser techniques,
2. solving the bifurcation equation on a Cantor set.

The small divisors problem. It is related to the resonance effects between the *linear* frequencies of the motion on the torus, $\{\omega_i\}_{i\leq N}$, and the infinite *normal* frequencies of the transversal oscillations $\{\omega_i\}_{i>N}$ (see (52)). Unlike the finite dimensional situation, one cannot expect to find solutions of any frequency[6] $\varepsilon > 0$, but just belonging to the Cantor like set of the “admissible frequencies” (for which the small divisors are not too small).

Remark 1.4. *To solve the small divisors problem, [BaBe05] uses an approach inspired to [Ba00]: one imposes a strong condition on the small divisors (see Remark 3.1) and uses the smoothing property of the nonlinearity to solve the range equation by the standard implicit function theorem (or the contraction mapping principle). Then the bifurcation equation is solved by topological arguments. We stress that for the PDEs considered in [BaBe05], one can find a positive measure (Cantor) set of admissible frequencies, such that the above strong condition on the small divisors is satisfied.*
In [BiDG], [BiDG06] we used the same approach for the nonlinear wave equation. In this case, however, it is immediate to notice that the strong condition on the small divisors could be satisfied at most by a zero measure set of admissible frequencies. Actually in [BiDG], [BiDG06] we were able to prove it only for a finite set of admissible (rational!) frequencies, finding only a finite number of Birkhoff–Lewis type solutions (see Remark 3.1).

In order to obtain a positive measure set of admissible frequencies we will use *a (analytic) Nash–Moser scheme*. As it is well known, the Nash–Moser technique requires *the invertibility of the linearized operator (allowing some loss of regularity) in a whole neighborhood of the origin* (this is always the most difficult point of the issue). Therefore the small divisors get more involved, as they depend also on the bifurcation parameter $\phi_0 \in \mathbb{T}^N$.

The bifurcation equation on a Cantor set. The standard procedure would be the following: one fixes ε (small) and $\phi_0 \in \mathbb{T}^N$, and solves the range equation by the Nash–Moser scheme. Due to the excision procedure used to control the small divisors at any steps, one can solve the range equation only for (ϕ_0, ε) belonging to a suitable positive measure Cantor set $\mathfrak{C} \subset \mathbb{T}^N \times \mathbb{R}^+$. Through a Whitney smoothing extension, the bifurcation equation takes the form: $\mathcal{F}(\phi_0; \varepsilon) = 0$ where $\mathcal{F} \in \mathcal{C}^\infty(\mathbb{T}^N \times \mathbb{R}^+, \mathbb{R}^N)$. We have to solve it on the Cantor set $\mathfrak{C}$. If $\phi_* \in \mathbb{T}^N$ is a *nondegenerate* solution of $\mathcal{F}(\phi_*; 0) = 0$, we can use the implicit function theorem to obtain a smooth curve $\big(\phi_0(\varepsilon); \varepsilon\big)$, $\phi_0(0) = \phi_*$, solving $\mathcal{F}\big(\phi_0(\varepsilon); \varepsilon\big) = 0$. Then it is simple to see that $\big(\phi_0(\varepsilon); \varepsilon\big) \in \mathfrak{C}$ for a positive measure set of ε.

Remark 1.5. *An analogous way to proceed would be to extend the finite dimensional Moser's proof of the Birkhoff-Lewis theorem (see [Mo77]) to this infinite dimensional context. This approach would consist in solving first the now infinite dimensional radial equation and then in solving a finite dimensional fixed point equation using the symplectic structure of the problem. As above, due to small divisors, the radial equation could be solved only for values of the parameters in a Cantor set. So the resulting fixed point equation should be solved on the above Cantor set.*

Remark 1.6. *In the discussed extensions of the Lyapunov Center Theorem to the wave equation, an analogous (with $N = 1$) problem appears and the above nondegeneracy is usually obtained assuming that the nonlinearity is odd and the cubic term is not vanishing. If one cannot exhibit a nondegenerate solution, the matter may become very difficult (see e.g. [BeBo06], [BeBo08]).*

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[6] The frequency of an orbit must be confused neither with the N linear frequencies of the motion on the resonant tori, nor with the infinitely many normal frequencies of the oscillations orthogonal to the above tori.

Solving the bifurcation equation by symmetry. To overcome the crucial point of solving the bifurcation equation on a Cantor set, one can try to show nondegeneracy. Yet, in the present case, this means, first of all, that one would have to evaluate the *sixth order term* in the Birkhoff normal form (these evaluation is very cumbersome, see for example the reckonings after Remark 4.4). We will not pursue this aim.

Our idea is to look for "symmetric solutions" thanks to the reversibility of the system. This means that we search solutions $u(t, x)$ that are sine–series in x (which is quite natural) and *even* in time. In terms of the associated Hamiltonian $H(p, q)$ this corresponds to look for solutions $(p(t), q(t))$ with *odd* $p(t)$ and *even* $q(t)$. This makes sense since the Hamiltonian satisfies the symmetry $H(p, q) = H(-p, q)$, namely it is *reversible*. Finding such symmetric solutions $(p(t), q(t))$ is equivalent to showing that $\phi_0 = 0$ is a solution of the bifurcation equation for any $\varepsilon > 0$. In the exposition we will actually invert the usual procedure: first we will solve the bifurcation equation by symmetries (using reversibility), then we will solve the range equation by a Nash–Moser scheme (see Remark 3.1).

Solving the range equation by Nash–Moser techniques. First we solve the range equation on the low modes taking the high modes as parameters; since the low modes are finite ($i \leq N$) no small divisors appear and the equations can be solved by the contraction mapping principle. Then we insert these solutions in the range equation for the infinite high modes ($i > N$).

To solve it, we use a Nash–Moser scheme inspired to the one developed in [BeBo06] (see also [BeBo07]). However here, in the crucial inversion of the linearized operator, we meet two further technical difficulties:

(i) *we have to diagonalize some not symmetric operators*,

(ii) *we have to deal with not Töplitz operators*[7] (namely not "product operators").

Let us briefly describe the scheme we have used. We first expand in time–Fourier series and, at any step $n \geq 1$ of the iterative algorithm, we truncate at an exponentially large rate (with respect to n). The truncated linearized operator we have to invert at every step is a (huge) matrix, whose entries are infinite dimensional spatial operators (namely operators acting on x–depending functions). We split this matrix into its diagonal part and its off–diagonal one. We want to show that: (a) the diagonal part is invertible, (b) the off–diagonal part is a small perturbation of the diagonal one.

To prove (a), we want to diagonalize any spatial operator corresponding to the entries of the diagonal part. These operators, even if they are close to diagonal, *are not symmetric*, due to some six order terms in the Birkhoff normal form (see Remark 4.4), and the usual Sturm–Liouville theory cannot be used. We diagonalize them and give estimates on their eigenvalues using the implicit function theorem. Assuming the "first order Melnikov non resonance condition" (see Definition 5.3) we show that the eigenvalues are polynomially bounded away from zero showing (a) with sufficiently good estimates (see subsection 6.2).

In showing (b), the small divisors come into play; it is enough to prove that the inverse of the product of two small divisors is polynomially bounded by the distance of the corresponding "singular sites" (see Lemma 7.1). Here some additional technical difficulties arise since the off–diagonal term *is not a Töplitz matrix*, namely it is not constant on the diagonals (this is due to the fact that in solving the range equation on the low modes are involved some integral operators, that are not product operators).

Remark 1.7. *The assumption that f is odd is necessary since we construct solutions which are real analytic sine–series, hence in a neighborhood of $x = 0$ they are defined, odd and satisfy the differential equation; adding the equation for $u(t, x)$ and $u(t, -x)$, one obtains $f(u) + f(-u) = 0$. This assumption is very natural from a physical point of view and it is satisfied, e.g., by the sine-Gordon, the sinh-Gordon and the ϕ^4 equation, taking*[8] *$\mu = 1$ and $f(u) = \sin u - u$, $\sinh u - u$, u^3 respectively. On the other hand let us note that we do not use the fact that f is odd in solving the bifurcation equation by symmetry.*

Scheme of the paper

In Section 2 we write equation (1) as an infinite dimensional Hamiltonian systems and perform the partial Birkhoff Normal Form, showing that it preserves the symmetry of the Hamiltonian. We put the

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[7] Not Töplitz operators were also considered in [EK05], [EK05bis], [GPr07]; see Remark 2.13.

[8] We note that for $\mu = 1$ the estimate on the minimal period of Theorem 1.1 holds (see Remark 10.3).

low modes in action–angle variables. We set the functional spaces in which we will find the solutions. In Section 3 we discuss the geometry of the problem and perform a Lyapunov–Schmidt decomposition, showing that, in the particular subspace of "symmetric" solutions we are considering (see (59)), the bifurcation equation has always the trivial solution $\phi_0 = 0$. Then we solve the range equation on the low modes. In section 4 we consider the linearized operator (on the high modes), expanding it in Fourier series with respect to time and split it into a diagonal term and an off–diagonal one. We also prove some symmetry proprieties. Postponing the proof of the invertibility of the linearized operator, in Section 5 we use a Nash-Moser scheme to solve the range equation. In Section 6 we study the above diagonal (in time) term. Its entries are not symmetric spatial operators. We diagonalize (in space) them by the implicit function theorem and we also prove useful estimates on their eigenvalues, as they appear in the small divisors. Assuming a suitable estimate on the small divisors that will be proved in Section 8, in Section 7 we show that the off-diagonal term is actually a small perturbation of the diagonal one, proving the invertibility of the linearized operator. In Section 9 we give a measure estimate on the set of admissible frequencies. We conclude the proof of Theorem 1.1 in Section 10, also showing the estimate on the minimal period. In the Appendix we prove some technical lemmata on the functional spaces introduced in Section 2.

Notations. $\mathrm{i} := \sqrt{-1}$. $\bar{z}$ denotes the complex conjugated of z. $\fint_0^{2\pi} := \frac{1}{2\pi}\int_0^{2\pi}$. For $i \in \mathbb{Z}$, $|i|_* := \max\{|i|, 1\}$. If E is a separable Hilbert space with basis e_i and $a = \sum_i a_i e_i$, $b = \sum_i b_i e_i$ then $a * b := \sum_i (a_i b_i) e_i \in E$; note that $\|a * b\|_E \leq \|a\|_E \, \|b\|_E$. Let $N, a, s, \alpha, \sigma > 0$. By ct we denote suitable constants depending only on N, a, s, α, σ; moreover if y belongs to a Banach space E and $x > 0$, then $y = O(x)$ means that $\|y\|_E \leq \text{ct}\ x$. Let E and F be complex Banach spaces, and let U be an open subset of E; we will denote by $\mathcal{A}(U, F)$ the space of analytic (namely continuously differentiable) functions from U into F. We will denote by $\mathcal{A}_0(E, F)$ the space of analytic functions from some neighborhood of the origin in E in some neighborhood of the origin in F. We will consider also real analytic functions: let E and F be real Banach spaces, and let U be an open subset of E; we will still denote by $\mathcal{A}(U, F)$ the space of functions from U into F such that for each point of U there exist a neighborhood $V \subset \mathbb{C}E$ and an analytic map $\tilde{f} : V \to \mathbb{C}F$ satisfying $\tilde{f} = f$ on $U \cap V$; $\mathbb{C}E$ and $\mathbb{C}F$ denote the complexifications of E and F.

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2 Set up

2.1 The Hamiltonian structure

We study the equation (1) as an infinite dimensional Hamiltonian system with coordinates u and $v = u_t$. Denoting $g(u) := \int_0^u f(s)ds$, the Hamiltonian is

$$H(v, u) = \int_0^\pi \left(\frac{v^2}{2} + \frac{u_x^2}{2} + \mu \frac{u^2}{2} + g(u) \right) dx \,.$$

The equations of motion are

$$u_t = \partial_v H = v, \qquad v_t = -\partial H_u = u_{xx} - \mu u - f(u) \,.$$

To rewrite it as a Hamiltonian in infinitely many coordinates, we make the ansatz

$$v = \mathcal{S}' p = \sum_{i \geq 1} \sqrt{\omega_i} p_i \chi_i \,, \quad u = \mathcal{S} q = \sum_{i \geq 1} \frac{q_i}{\sqrt{\omega_i}} \chi_i \,, \tag{5}$$

where $\omega_i = \sqrt{i^2+\mu}$ and $\chi_i = \sqrt{2/\pi}\sin ix$. The coordinates are taken from some Hilbert space

$$\ell^{a,s}(\mathbb{R}) := \left\{ q = (q_1, \ldots),\ q_i \in \mathbb{R}\,,\ i \geq 1 \ \text{ s.t. } \|q\|^2_{a,s} = \sum_{i\geq 1} |q_i|^2 i^{2s} e^{2ai} < \infty \right\}, \tag{6}$$

whit $a, s > 0$. We get

$$H(p,q) = Q + G = \frac{1}{2}\sum_{i\geq 1} \omega_i (q_i^2 + p_i^2) + \int_0^\pi g(\mathcal{S}q)\, dx\,. \tag{7}$$

The origin is an elliptic equilibrium. The equations of motion are:

$$\dot{p_i} = -\partial_{q_i} H = -\omega_i q_i - \partial_{q_i} G\,, \qquad \dot{q_i} = \partial_{p_i} H = \omega_i p_i\,, \qquad\qquad i \geq 1\,, \tag{8}$$

with respect to the standard symplectic structure $\sum dp_i \wedge dq_i$ on $\ell^{a,s}(\mathbb{R}) \times \ell^{a,s}(\mathbb{R})$. Instead of discussing the validity of this transformation, we take the Hamiltonian H as our new starting point. Indeed analytic orbits of H correspond to analytic solutions of (1).

Lemma 2.1 ([P96])**.** *Let $a > 0$ and s be arbitrary. If a curve $\mathbb{R} \ni t \longmapsto \big(p(t), q(t)\big) \in \ell^{a,s}(\mathbb{R}) \times \ell^{a,s}(\mathbb{R})$ is a real analytic orbit of (8), then*

$$u(t,x) := \sum_{i\geq 1} \frac{q_i(t)}{\sqrt{\omega_i}}\, \chi_i(x) \tag{9}$$

is a classical solution of (1) that is real analytic on $\mathbb{R} \times [0,\pi]$.

Let $\ell^{a,s}_b$ be the Hilbert space of all bi–infinite sequences with complex coefficients and finite norm $(\sum_{i\in\mathbb{Z}} |q_i|^2 |i|^{2s}_* e^{2a|i|})^{1/2}$. Through the inverse discrete Fourier transform $q \mapsto (2\pi)^{-1/2}\sum_{i\in\mathbb{Z}} q_i e^{\mathrm{i}ix}$, we identify $\ell^{a,s}_b$ with the space of all 2π–periodic functions which are analytic and bounded in the complex strip $|\mathrm{Im} z| < a$ with trace functions on $|\mathrm{Im} z| = a$ belonging to the usual Sobolev space H^s. If $s > 1/2$ the space $\ell^{a,s}_b$ is a Hilbert algebra with respect to the convolution of sequences (see [P96]).
Using the algebra property one gets that G *is smoothing of order one.*

Lemma 2.2 ([P96])**.** *For $a \geq 0$ and $s > 0$, the gradient $\nabla_q G := (\partial_{q_1} G, \partial_{q_2} G, \ldots)$ is a real analytic map from a neighborhood of the origin in $\ell^{a,s}(\mathbb{R})$ into $\ell^{a,s+1}(\mathbb{R})$. Moreover $\|\nabla_q G\|_{a,s+1} = O(\|q^3\|_{a,s})$.*

For the nonlinearity u^3 one finds

$$G = \frac{1}{4}\int_0^\pi |u(x)|^4\, dx = \frac{1}{4}\sum_{i,j,k,l} G_{ijkl} q_i q_j q_k q_l\,, \qquad \text{with} \qquad G_{ijkl} = \frac{1}{\sqrt{\omega_i\omega_j\omega_k\omega_l}} \int_0^\pi \chi_i\chi_j\chi_k\chi_l\, dx\,. \tag{10}$$

In [P96] it is proved that $G_{ijkl} = 0$ unless $i \pm j \pm k \pm l = 0$, for some combination of plus and minus signs. In particular $G_{iijj} = (2+\delta_{ij})/(2\pi\omega_i\omega_j)$. *From now on, we focus our attention on the nonlinearity $f(u) = u^3$, since terms of order five or more do not make any difference.*

For the rest of this paper we introduce the complex coordinates

$$\mathrm{z}_i = \frac{1}{\sqrt{2}}(q_i + \mathrm{i}p_i)\,, \quad \tilde{\mathrm{z}}_i = \frac{1}{\sqrt{2}}(q_i - \mathrm{i}p_i)\,, \tag{11}$$

that live in the now *complex* Hilbert space

$$\ell^{a,s} = \ell^{a,s}(\mathbb{C}) := \left\{ \mathrm{z} = (\mathrm{z}_1, \ldots),\ \mathrm{z}_i \in \mathbb{C}\,,\ i \geq 1 \ \text{ t.c. } \|\mathrm{z}\|^2_{a,s} = \sum_{i\geq 1} |\mathrm{z}_i|^2 i^{2s} e^{2ai} < \infty \right\}. \tag{12}$$

We are considering z and $\tilde{\mathrm{z}}$ as *independent variables*; note that, in general, $\tilde{\mathrm{z}} \neq \bar{\mathrm{z}}$, where $\bar{\mathrm{z}}$ denotes the complex conjugated of z. The phase space $\ell^{a,s} \times \ell^{a,s}$ is endowed with symplectic structure $-\mathrm{i} \sum_{i \geq 1} d\mathrm{z}_i \wedge d\tilde{\mathrm{z}}_i$. The Hamiltonian becomes

$$H = H(\mathrm{z}, \tilde{\mathrm{z}}) = Q + G = \sum_{i \geq 1} \omega_i \mathrm{z}_i \tilde{\mathrm{z}}_i + G(\mathrm{z}, \tilde{\mathrm{z}}) \in \mathcal{A}_0(\ell^{a,s} \times \ell^{a,s}, \mathbb{C}), \tag{13}$$

where, with abuse of notation, we have still denoted by $G = G(\mathrm{z}, \tilde{\mathrm{z}})$ the function $G\big((\mathrm{z} + \tilde{\mathrm{z}})/\sqrt{2}\big)$. The Hamilton's equations write $\dot{\mathrm{z}} = -\mathrm{i}\partial_{\tilde{\mathrm{z}}} H$, $\dot{\tilde{\mathrm{z}}} = \mathrm{i}\partial_{\mathrm{z}} H$.

We now show that the Hamiltonian H is *reversible*. Let us recall the definition of reversible system (see [MZ]). A system $\dot{w} = g(w)$ is called *reversible with respect to the reflection*[9] ρ *if*

$$g \circ \rho = -\rho g\,. \tag{14}$$

Condition (14) is equivalent to

$$\phi^t \circ \rho = \rho \phi^{-t} \tag{15}$$

where ϕ^t is the flow of $\dot{w} = g(w)$. From (15) it is clear that with $w = w(t) = \phi^t(w(0))$ also $\rho w(-t)$ is a solution of $\dot{w} = g(w)$. A solution $w(t)$ is called *symmetric* (w.r.t. ρ) if

$$w(t) = \rho w(-t)\,. \tag{16}$$

In view of (15) symmetric solutions are characterized by the condition

$$w(0) = \rho w(0)\,. \tag{17}$$

We now prove that the Hamiltonian H defined in (13) is reversible with respect to the reflection

$$\rho\,(\mathrm{z}, \tilde{\mathrm{z}}) := (\tilde{\mathrm{z}}, \mathrm{z})\,. \tag{18}$$

First we note that by (10) and (11) we get

$$H(\tilde{\mathrm{z}}, \mathrm{z}) = H(\mathrm{z}, \tilde{\mathrm{z}}) \qquad \text{namely} \qquad H \circ \rho = H\,. \tag{19}$$

Now, if $X_H := {}^{\mathsf{T}}(-\mathrm{i}\partial_{\tilde{\mathrm{z}}} H, \mathrm{i}\partial_{\mathrm{z}} H)$ is the hamiltonian vector field associated to H, by (19) we get

$$X_H = X_{H \circ \rho} = -\rho X_H \circ \rho\,. \tag{20}$$

Then, if $w := (\mathrm{z}, \tilde{\mathrm{z}})$, the hamiltonian system $\dot{w} = X_H(w)$ satisfies (14)

2.2 Partial Birkhoff normal form

From now on we denote by "ˆ" the high modes of a vector, for example $\hat{\mathrm{z}} := (\mathrm{z}_{N+1}, \mathrm{z}_{N+2}, \ldots)$.

Proposition 2.3 ([P96])**.** *There exists a close to the identity and symplectic change of coordinates* $(\mathrm{z}, \tilde{\mathrm{z}}) := \Gamma(\mathrm{z}_*, \tilde{\mathrm{z}}_*)$*,* $\Gamma \in \mathcal{A}_0(\ell^{a,s} \times \ell^{a,s}, \ell^{a,s} \times \ell^{a,s})$ *verifying*

$$\|\mathrm{z} - \mathrm{z}_*\|_{a,s+1} + \|\tilde{\mathrm{z}} - \tilde{\mathrm{z}}_*\|_{a,s+1} \leq \mathrm{ct}\, \left(\|\mathrm{z}_*\|^3_{a,s} + \|\tilde{\mathrm{z}}_*\|^3_{a,s}\right)\,, \tag{21}$$

transforming the Hamiltonian $H = Q + G$ *in (13) in seminormal form up to order six. That is*

$$H_* := H \circ \Gamma = Q + \check{G} + \hat{G} + K \in \mathcal{A}_0(\ell^{a,s} \times \ell^{a,s}, \mathbb{C})\,, \tag{22}$$

where

$$\nabla \check{G},\, \nabla \hat{G},\, \nabla K \in \mathcal{A}_0(\ell^{a,s} \times \ell^{a,s}, \ell^{a,s+1} \times \ell^{a,s+1})\,, \tag{23}$$

---

[9] A reflection ρ is a linear map from the phase space into itself such that $\rho^2 = Id$.

$$\check{G} = \frac{1}{2} \sum_{\min\{i,j\} \le N} \check{G}_{ij} \mathrm{z}_{*i} \tilde{\mathrm{z}}_{*i} \mathrm{z}_{*j} \tilde{\mathrm{z}}_{*j}$$

with uniquely determined coefficients $\check{G}_{ij} = (3/8\pi)\,(4 - \delta_{ij}/\omega_i\omega_j)$*,* $\hat{G}$ *is a fourth order homogeneous polynomial and*

$$\hat{G}(\mathrm{z}_*, \tilde{\mathrm{z}}_*) = O(\|\hat{\mathrm{z}}_*\|^4_{a,s} + \|\hat{\tilde{\mathrm{z}}}_*\|^4_{a,s}) \,, \qquad K(\mathrm{z}_*, \tilde{\mathrm{z}}_*) = O\left(\|\mathrm{z}_*\|^6_{a,s} + \|\hat{\tilde{\mathrm{z}}}_*\|^6_{a,s}\right) \,.$$

In the next subsection (see (26)) we will prove that H_* is still symmetric (recall (19)). To show it we briefly recall the “algebraic” part of the proof of the previous proposition given in [P96]. The transformation Γ is obtained as the time-1-map of the flow of the Hamiltonian vectorfield X_F given by a Hamiltonian

$$F = F(\mathrm{z}_*, \tilde{\mathrm{z}}_*) = {\sum_{i,j,k,\ell}}' F_{ijk\ell}\, \mathfrak{w}_i \mathfrak{w}_j \mathfrak{w}_k \mathfrak{w}_\ell \,, \tag{24}$$

where the prime means that the summation is over all nonzero integers, for $i \ge 1$, $\mathfrak{w}_i := \mathrm{z}_{*i}$ and $\mathfrak{w}_{-i} := \tilde{\mathrm{z}}_{*i}$,

$$F_{ijk\ell} = \begin{cases} \dfrac{\mathrm{i} G_{ijk\ell}}{\omega'_i + \omega'_j + \omega'_k + \omega'_\ell} & \text{for } (i,j,k,\ell) \in S_N \\ 0 & \text{otherwise} \,. \end{cases} \tag{25}$$

Here $G_{ijk\ell}$ are defined for arbitrary integers by setting $G_{ijk\ell} := G_{|i||j||k||\ell|}$; moreover $\omega'_i := \operatorname{sign} i \cdot \omega_{|i|}$ and S_N is the set of $(i,j,k,\ell) \in \mathbb{Z}^4$ such that $\min\{|i|, |j|, |k|, |\ell|\} \le N$, $i \pm j \pm k \pm \ell = 0$ but $(i,j,k,\ell) \ne (p,-p,q,-q)$ or some permutation of it.

2.3 Symmetry of the Hamiltonian

We now prove that after the Birkhoff Normal Form the new Hamiltonian H_* in (22) still verifies the symmetry (19), namely

$$H_* \circ \rho = H_* \,. \tag{26}$$

First we note that F defined in (24)-(25) satisfies $-F \circ \rho = F$, since $-F_{-i,-j,-k,-\ell} = F_{i,j,k,\ell}$ and therefore

$$-F(\tilde{\mathrm{z}}_*, \mathrm{z}_*) = -{\sum_{i,j,k,\ell}}' F_{i,j,k,\ell}\, \mathfrak{w}_{-i} \mathfrak{w}_{-j} \mathfrak{w}_{-k} \mathfrak{w}_{-\ell} = -{\sum_{i,j,k,\ell}}' F_{-i,-j,-k,-\ell}\, \mathfrak{w}_i \mathfrak{w}_j \mathfrak{w}_k \mathfrak{w}_\ell = F(\mathrm{z}_*, \tilde{\mathrm{z}}_*) \,.$$

Then we have

$$(\nabla F) \circ \rho = -\rho \nabla F \,. \tag{27}$$

Developing in Lie series we get $H_* := H \circ \Gamma = \sum_{j \ge 0} L^j_F H / j!$, where $L^0_F H := H$ and $L^j_F H := \{L^{j-1}_F H, F\}$. Then (26) follows if we prove that $(L^j_F H) \circ \rho = L^j_F H$, $\forall\, j \ge 0$; we prove it by induction. For $j = 0$ it follows by (19). If $G := L^{j-1}_F H$ satisfies $G \circ \rho = G$ we have that

$$L^j_F \circ \rho = \{G, F\} \circ \rho = (X_G \cdot \nabla F) \circ \rho = -\rho X_G \cdot (\nabla F \circ \rho) = -\rho X_G \cdot (-\rho \nabla F) = X_G \cdot \nabla F = \{G, F\} = L^j_F \,,$$

using that $X_G \circ \rho = -\rho X_G$ (recall (20)) and (27). The proof of (26) is now completed.

2.4 Action–angle variables on the low modes

We now consider new variables $(I, \phi, \hat{z}, \hat{\tilde{z}})$, where (I, ϕ) are action–angle variables on the low modes; moreover we also connect the (small) amplitude of the searched solution with its frequency ε, rescaling the variables:

$$\begin{aligned} \mathrm{z}_{*i} &= \sqrt{\varepsilon}\sqrt{I_i/2}(\cos\phi_i + \mathrm{i}\sin\phi_i) \,, \qquad \tilde{\mathrm{z}}_{*i} = \sqrt{\varepsilon}\sqrt{I_i/2}(\cos\phi_i - \mathrm{i}\sin\phi_i) \,, \qquad i \le N \,, \\ \mathrm{z}_{*i} &= \sqrt{\varepsilon}\hat{z}_i \,, \qquad \tilde{\mathrm{z}}_{*i} = \sqrt{\varepsilon}\hat{\tilde{z}}_i \,, \qquad i > N \,. \end{aligned} \tag{28}$$

Here the phase space is[10] $P^N_r \times \mathbb{T}^N_r \times B_r(\ell^{a,s}) \times B_r(\ell^{a,s}) \ni (I, \phi, \hat{z}, \hat{\bar{z}})$, where $P^N_r := \{I \in \mathbb{C}^N\,,\ |I| < r\,,\ \mathrm{Re} I_i > 0\,, \forall\, i \le N\}$, $\mathbb{T}^N_r := \{\phi \in \mathbb{C}^N/2\pi\mathbb{Z}^N\,,\ |\mathrm{Im}\phi_i| < r\,, \forall\, i \le N\}$, $B_r(\ell^{a,s}) := \{\hat{z} \in \ell^{a,s}\,, \|\hat{z}\|_{a,s} < r\}$; here $r > 0$ can be chosen large at will, taking ε small enough.
Instead of looking for T–periodic solutions, we rescale time

$$t \to \varepsilon t\,, \qquad\qquad T = 2\pi/\varepsilon\,, \tag{29}$$

and search 2π–periodic solutions (at the same moment dividing the Hamiltonian by ε).
After (28) and (29), the new Hamiltonian is

$$\mathcal{H}(I, \phi, \hat{z}, \hat{\bar{z}}; \varepsilon) := \varepsilon^{-2} H_*(\mathrm{z}_*, \tilde{\mathrm{z}}_*)\,. \tag{30}$$

The symplectic structure is $\sum_{i\le N} dI_i \wedge d\phi_i - \mathrm{i} \sum_{i>N} d\hat{z}_i \wedge d\hat{\bar{z}}_i$. Recalling the definitions of $\check{G}, \hat{G}, K$ given in Proposition 2.3, we can rewrite $\mathcal{H}$ in the form

$$\mathcal{H} := \varepsilon^{-1}\omega \cdot I + \varepsilon^{-1}\Omega \cdot \hat{z} * \hat{\bar{z}} + \frac{1}{2} A I \cdot I + B I \cdot \hat{z} * \hat{\bar{z}} + \hat{G}(\hat{z}, \hat{\bar{z}}) + \varepsilon \widetilde{K}(I, \phi, \hat{z}, \hat{\bar{z}}; \varepsilon)\,, \tag{31}$$

where

$$\omega := (\omega_1, \ldots, \omega_N) \qquad \text{and} \qquad \Omega := (\omega_{N+1}, \ldots) \tag{32}$$

are respectively the vectors of the *linear and elliptic frequencies*, A is the $N \times N$ matrix $A := \big(\check{G}_{ij}\big)_{i,j\le N}$, B is the $\infty \times N$ matrix $B := \big(\check{G}_{ij}\big)_{i>N,\, j\le N}$, with $\check{G}_{ij}$ defined in Proposition 2.3 and[11]

$$\tilde{K}(I, \phi, \hat{z}, \hat{\bar{z}}; \varepsilon) := \varepsilon^{-3} K(\mathrm{z}_*, \tilde{\mathrm{z}}_*)\,.$$

We note that

$$\det A \neq 0 \qquad\qquad \text{("twist condition")} \tag{33}$$

and

$$|B_{ij}| \le \mathrm{ct}\,/|i|\,, \qquad \forall\, i > N\,,\ j \le N\,. \tag{34}$$

By (23) we get

$$\nabla\mathcal{H}(\cdot;\varepsilon) \in \mathcal{A}\Big(P^N_r \times \mathbb{T}^N_r \times B_r(\ell^{a,s}) \times B_r(\ell^{a,s}), \mathbb{C}^N \times \mathbb{C}^N \times \ell^{a,s+1} \times \ell^{a,s+1}\Big)\,. \tag{35}$$

Remark 2.4. *By continuity we can choose $r_\sharp = r_\sharp(a,s) > 0$ and $\varepsilon_\sharp = \varepsilon_\sharp(a,s) > 0$ such that $\nabla\hat{G}$, $\nabla\tilde{K}$ and their derivatives up to order three are uniformly bounded on $P^N_{r_\sharp} \times \mathbb{T}^N_{r_\sharp} \times B_{r_\sharp}(\ell^{a,s}) \times B_{r_\sharp}(\ell^{a,s}) \times [0, \varepsilon_\sharp]$.*

Let ρ_* be the reflection in (18) in the new variable, namely

$$\rho_* := (\hat{\rho}, \rho)\,, \quad \text{where } \hat{\rho}(I, \phi) := (I - \phi)\,, \quad \text{namely } \rho_*(I, \phi, \hat{z}, \hat{\bar{z}}) := (I, -\phi, \hat{\bar{z}}, \hat{z})\,. \tag{36}$$

By (26), it is immediate to see that

$$\mathcal{H} \circ \rho_* = \mathcal{H}\,, \tag{37}$$

$$-\rho_* X_{\mathcal{H}} = X_{\mathcal{H}} \circ \rho_*\,, \tag{38}$$

where $X_{\mathcal{H}} := {}^{\intercal}(-\partial_\phi\mathcal{H}, \partial_I\mathcal{H}, -\mathrm{i}\partial_{\hat{\bar{z}}}\mathcal{H}, \mathrm{i}\partial_{\hat{z}}\mathcal{H})$.

---

[10] Here and in the following, with a little abuse of notation, we still denote by $\ell^{a,s}$ the subspace of $\ell^{a,s}$ formed by the sequences $\{\hat{z}_i\}_{i>N}$.

[11] Note that $\tilde{K}$ is analytic in ε since K in Proposition 2.3 is of order six and it is even in the variables $(\mathrm{z}_*, \tilde{\mathrm{z}}_*)$.

2.5 Functional setting

Let E be a separable complex Hilbert space. We will denote by $\mathcal{C}_{\rm per}(\mathbb{R}, E)$ the space of all the continuous and 2π–periodic functions from $\mathbb{R}$ into E. For $f \in \mathcal{C}_{\rm per}(\mathbb{R}, E)$ we can define the "Fourier coefficients"

$$f_k := \fint_0^{2\pi} e^{-\mathrm{i}kt} f(t)dt \in E \tag{39}$$

(recall that $\fint_0^{2\pi} := \frac{1}{2\pi}\int_0^{2\pi}$). Let $\alpha > 0$ and $\sigma > 1/2$. Let

$$H_E^{\alpha,\sigma} := \Big\{ f \in \mathcal{C}_{\rm per}(\mathbb{R}, E) \quad \text{s.t.} \quad \|f\|^2_{H_E^{\alpha,\sigma}} := \sum_k e^{2\alpha|k|}|k|_*^{2\sigma}\|f_k\|_E^2 < \infty \Big\} . \tag{40}$$

The functions of $H_E^{\alpha,\sigma}$ are actually analytic (or, better, have an analytic extension) on the complex strip $\mathbb{T}_\alpha := \Big\{ t \in \mathbb{C}, \ \text{s.t.} \ |\mathrm{Im}\, t| < \alpha \Big\}$.
The proofs of all the results of this subsection are contained in the Appendix.

Proposition 2.5. *Every $f \in H_E^{\alpha,\sigma}$ is continuous on $\overline{\mathbb{T}_\alpha}$ and analytic on $\mathbb{T}_\alpha$ and*

$$f(t) = \sum_k f_k e^{\mathrm{i}kt} , \tag{41}$$

with $f_k \in E$ defined in (39).

For $f, g \in H_E^{\alpha,\sigma}$ we set $\langle f, g\rangle_{H_E^{\alpha,\sigma}} := \sum_k e^{2\alpha|k|}|k|_*^{2\sigma}\langle f_k, g_k\rangle_E$.

Proposition 2.6. *$\langle\cdot,\cdot\rangle_{H_E^{\alpha,\sigma}}$ is a scalar product on $H_E^{\alpha,\sigma}$. With the norm $\|\cdot\|_{H_E^{\alpha,\sigma}}$ inducted by it, $H_E^{\alpha,\sigma}$ is an Hilbert space. Moreover there exists a constant $c = c(\sigma) > 0$ such that $\sup_{t\in\mathbb{T}_\alpha} \|f(t)\|_E \le c\|f\|_{H_E^{\alpha,\sigma}}$.*

We note that a Hilbert basis of $H_E^{\alpha,\sigma}$ is $\{e^{\mathrm{i}kt}e_j\}_{k,j\in\mathbb{Z}}$ where $\{e_j\}_{j\in\mathbb{Z}}$ is a Hilbert basis of E. We also remark that the series in (41) converges in $H_E^{\alpha,\sigma}$ and, therefore, uniformly converges on $\mathbb{T}_\alpha$.
One can consider also the case in which E is a real Hilbert space.

Proposition 2.7. *Let be $f : E \longrightarrow F$, $f(x) = \sum_{n\ge n_0} \frac{1}{n!} d^n f(0)[x, \ldots, x]$ analytic for $\|x\|_E < r_0$. Let $\alpha > 0$, $\sigma > 1/2$. Then there exists $r = r(\sigma) > 0$ such that $f_* : H_E^{\alpha,\sigma} \longrightarrow H_F^{\alpha,\sigma}$, $h \longmapsto f(h)$ (where $[f(h)](t) := f(h(t))$) is analytic on $\|h\|_{H_E^{\alpha,\sigma}} < r$. Moreover*

$$\|f(h)\|_{H_F^{\alpha,\sigma}} \le M \sum_{n\ge n_0} \Big(\|h\|_{H_E^{\alpha,\sigma}}/2r \Big)^n ,$$

where $M > 0$ is a constant independent of α and σ.

Special norms of linear operators

Let E, F be two separable complex Hilbert spaces and $\mathcal{L}(E, F)$ be the space of linear and continuous operator from E to F. Let $\mathcal{L}(H_E^{\alpha,\sigma}, H_F^{\alpha,\sigma})$ be the space of linear and continuous operator from the Hilbert spaces $H_E^{\alpha,\sigma}$ and $H_F^{\alpha,\sigma}$ defined in (40); $\alpha > 0$, $\sigma > 1/2$. Then for every $L \in \mathcal{L}(H_E^{\alpha,\sigma}, H_F^{\alpha,\sigma})$, $k, \ell \in \mathbb{Z}$, are well defined $L_{k\ell} \in \mathcal{L}(E, F)$ by

$$\forall\, x \in E , \qquad\qquad L_{k\ell}[x] := \fint_0^{2\pi} e^{-\mathrm{i}kt} L\big[e^{\mathrm{i}\ell t}x\big]\, dt \, \in \, F . \tag{42}$$

Definition 2.8. *Let us define the space*

$$\mathcal{L}^{\alpha,\sigma}(E,F) := \Big\{ \ L \in \mathcal{L}(H_E^{\alpha,\sigma}, H_F^{\alpha,\sigma}) \ \ s.t. \ \ \|L\|^2_{\mathcal{L}^{\alpha,\sigma}(E,F)} := \sup_\ell \sum_k e^{2\alpha|k-\ell|}|k-\ell|_*^{2\sigma} \|L_{k\ell}\|^2_{\mathcal{L}(E,F)} < \infty \Big\} . \tag{43}$$

Remark 2.9. *Let us consider the L's belonging to* $\mathcal{L}^{\alpha,\sigma}(E,F)$ *for which there exist* $L_n^\sharp \in \mathcal{L}(E,F)$, $n \in \mathbb{Z}$, *such that* $L_{k\ell} = L^\sharp_{k-\ell}$, *for every* $k,\ell \in \mathbb{Z}$. *By (43) it is defined* $L^\sharp \in H^{\alpha,\sigma}_{\mathcal{L}(E,F)}$ *by* $L^\sharp(t) := \sum_n e^{\mathrm{i}nt} L_n^\sharp$, $\forall\, t \in \mathbb{R}$. *Then L is a Töplitz operator, that is, a "product operator", since it acts as the product by* $L^\sharp(t)$, *namely* $\big(L[f]\big)(t) = \big(L^\sharp(t)\big)[f(t)]$, $\forall\, f \in H_E^{\alpha,\sigma}$, $t \in \mathbb{R}$. *Moreover* $\|L^\sharp\|_{H^{\alpha,\sigma}_{\mathcal{L}(E,F)}} = \|L\|_{\mathcal{L}^{\alpha,\sigma}(E,F)}$.

Proposition 2.10. *If* $L \in \mathcal{L}^{\alpha,\sigma}(E,F)$ *then for every* $f \in H_E^{\alpha,\sigma}$

$$\big(L[f]\big)(t) = \sum_k e^{\mathrm{i}kt} g_k \,, \qquad \textit{where} \quad g_k := \sum_\ell L_{k\ell}[f_\ell] \ \in \ F \tag{44}$$

with $L_{k\ell} \in \mathcal{L}(E,F)$ *and* $f_\ell \in E$ *defined in (42) and (39) respectively. In particular the former series converges in* $H_F^{\alpha,\sigma}$, *the latter in* F.

Proposition 2.11. $\mathcal{L}^{\alpha,\sigma}(E,F)$ *is a Banach space with respect to the norm* $\|\cdot\|_{\mathcal{L}^{\alpha,\sigma}(E,F)}$. *Moreover there exists a constant* $c = c(\sigma) > 0$ *such that* $\|\cdot\|_{\mathcal{L}(H_E^{\alpha,\sigma},H_F^{\alpha,\sigma})} \le c\,\|\cdot\|_{\mathcal{L}^{\alpha,\sigma}(E,F)}$. *Finally* $\|L_{kk}\|_{\mathcal{L}(E,F)} \le \|L\|_{\mathcal{L}(H_E^{\alpha,\sigma},H_F^{\alpha,\sigma})}$.

Proposition 2.12. *Let* E,F,G *be separable complex Hilbert spaces. Let* $L^{(1)} \in \mathcal{L}^{\alpha,\sigma}(E,F)$ *and* $L^{(2)} \in \mathcal{L}^{\alpha,\sigma}(F,G)$. *Then* $L^{(2)} \circ L^{(1)} \in \mathcal{L}^{\alpha,\sigma}(E,G)$ *and* $\|L^{(2)} \circ L^{(1)}\|_{\mathcal{L}^{\alpha,\sigma}(E,G)} \le \mathrm{ct}\,\|L^{(1)}\|_{\mathcal{L}^{\alpha,\sigma}(E,F)}\,\|L^{(2)}\|_{\mathcal{L}^{\alpha,\sigma}(F,G)}$.

Remark 2.13. *Not Töplitz operators with scalar entries were also considered in [EK05], [EK05bis], [GPr07]. They essentially use the norm* $\sup_{k,\ell} e^{\alpha|k-\ell|}|L_{k\ell}|$.

3 Lyapunov-Schmidt decomposition and symmetries

Let us write $\mathcal{H} =: H_{\mathrm{int}} + \hat{G} + \varepsilon\widetilde{K}$ where

$$H_{\mathrm{int}}(I,\phi,\hat{z},\hat{\bar{z}};\varepsilon) = \varepsilon^{-1}\omega \cdot I + \varepsilon^{-1}\Omega \cdot \hat{z} * \hat{\bar{z}} + \frac{1}{2}AI \cdot I + BI \cdot \hat{z} * \hat{\bar{z}}$$

is integrable. The orbits of the *integrable Hamiltonian* H_{int} are

$$I(t) = I_0 \,, \quad \phi_0(t) = \phi_0 + \varepsilon^{-1}\tilde{\omega}t + B^t \hat{z}_0 * \hat{\bar{z}}_0 t \,, \quad \hat{z}(t) = e^{-\mathrm{i}\tilde{\Omega}t/\varepsilon}\hat{z}_0 \,, \quad \hat{\bar{z}}(t) = e^{\mathrm{i}\tilde{\Omega}t/\varepsilon}\hat{\bar{z}}_0 \,. \tag{45}$$

where $\tilde{\omega} := \omega + \varepsilon A I_0$ is the vector of the *shifted linear frequencies* and $\tilde{\Omega} := \mathrm{diag}_{i>N}\Big(\omega_i + \varepsilon\big(BI_0\big)_i\Big)$ is the matrix of the *shifted elliptic frequencies*.

The manifold $\{\hat{z} = \hat{\bar{z}} = 0\}$ is invariant for the H_{int}–flow (and for the $(H_{\mathrm{int}}+\hat{G})$–flow too) and it is filled up by the N–dimensional tori $\big\{I = I_0,\ \phi_0 \in \mathbb{T}^N,\ \hat{z} = \hat{\bar{z}} = 0\big\}$, on which the flow $t \longmapsto (I_0, \phi_0 + \varepsilon^{-1}\tilde{\omega}t, 0, 0)$ is 2π–periodic, if and only if

$$\varepsilon^{-1}\tilde{\omega} = \kappa \in \mathbb{Z}^N \,. \tag{46}$$

Thanks to the "twist condition" (33), we can satisfy (46) choosing

$$\begin{aligned} I_0 &:= I_0(\varepsilon) = A^{-1}\left(\tilde{\kappa} - \{\omega/\varepsilon\}\right) && (47)\\ \kappa &:= \kappa(\varepsilon) = [\omega/\varepsilon] + \tilde{\kappa} \,, && (48)\end{aligned}$$

where $\{\cdot\}$ and $[\cdot]$ denotes, respectively, the fractional and integer part; moreover $\tilde{\kappa} \in \mathbb{Z}^N$ is a suitable ε–independent vector[12] such that $1 < \mathrm{Re}(I_{0i})$, for $i \le N$. Then the shifted linear and elliptic frequencies depend on the one dimensional parameter ε:

$$\tilde{\omega} \ = \ \tilde{\omega}(\varepsilon) = \omega + \varepsilon A I_0(\varepsilon) = \varepsilon\kappa(\varepsilon) \,, \tag{49}$$

---

[12] Note that $\mathbb{R}^N \ni \tilde{\kappa} \mapsto A^{-1}\tilde{\kappa} - A^{-1}\{\omega/\varepsilon\}$ is invertible and $|A^{-1}\{\omega/\varepsilon\}|$ is bounded by some constant independent of ε.

$$\tilde\Omega \; = \; \tilde\Omega(\varepsilon) = \mathrm{diag}_{i>N}\tilde\Omega_i(\varepsilon)\,, \qquad \text{where} \qquad \tilde\Omega_i(\varepsilon) := \omega_i + \varepsilon\big(BI_0(\varepsilon)\big)_i\,,\ \forall\, i > N\,. \tag{50}$$

We also introduce

$$\mathcal{Z} := \{\varepsilon \in (0,1) \text{ s.t. } \exists\, i \le N \text{ with } \omega_i/\varepsilon \in \mathbb{Z}\}\,.$$

$\mathcal{Z}$ is a sequence tending to zero. Then $I_0(\varepsilon), \kappa(\varepsilon), \tilde\omega(\varepsilon), \tilde\Omega(\varepsilon)$, are smooth for $\varepsilon \notin \mathcal{Z}$ and

$$I_0'(\varepsilon) = \varepsilon^{-2}A^{-1}\omega\,, \qquad \kappa'(\varepsilon) = 0\,, \qquad\qquad \forall\,\varepsilon \notin \mathcal{Z}\,, \tag{51}$$

where, here and in the following, we denote by "$\,'\,$" the (total) derivative with respect to ε.

For any fixed ε, the infinite 2π–periodic H_{int}–orbits of the family

$$\mathcal{F} := \big\{I(t) = I_0(\varepsilon), \ \ \phi(t) = \phi_0 + \kappa(\varepsilon)\,t, \ \ \hat z(t) = \hat{\tilde z}(t) = 0\big\} \approx \mathbb{T}^N\,,$$

will not all persist for the flow of the complete Hamiltonian $\mathcal{H}$. But if $\mathcal{F}$ is *isolated* (namely there are no other 2π–periodic H_{int}–solutions close to it), i.e.

$$\varepsilon k - \tilde\Omega_i(\varepsilon) = \varepsilon k - \omega_i + \varepsilon\big(BA^{-1}\{\omega/\varepsilon\}\big)_i - \varepsilon\big(BA^{-1}\tilde\kappa\big)_i \neq 0\,, \qquad \forall\, k \in \mathbb{Z},\ i > N\,, \tag{52}$$

(recall (45)) we can hope to prove existence of solutions of $\mathcal{H}$ bifurcating from $\mathcal{F}$. The "geometric" condition (52) clearly appears as a *non–resonance condition* between the linear frequencies $\{\omega_i\}_{i\le N}$ and the elliptic ones $\{\omega_i\}_{i>N}$.

We will look for 2π–periodic orbits of $\mathcal{H}$ of the form

$$\big(I(t), \phi(t), \hat z(t), \hat{\tilde z}(t)\big) = \big(I_0(\varepsilon), \phi_0 + \kappa(\varepsilon)t, 0, 0\big) + \varepsilon\mathfrak{u}(t)\,, \tag{53}$$

where $\mathfrak{u}(t) := \big(J(t), \psi(t), z(t), \tilde z(t)\big)$ belongs to

$$H_{a,s,\alpha,\sigma} := H^{\alpha,\sigma}_{\mathbb{C}^N\times\mathbb{C}^N\times\ell^{a,s}\times\ell^{a,s}}$$

with $\int_0^{2\pi}\psi = 0$ and $\phi_0 \in \mathbb{T}^N$ is a parameter to determinate. From the Hamilton's equations of $\mathcal{H}$, it results that $\mathfrak{u}$ and ϕ_0 must satisfy

$$\left\{\begin{array}{lcl} \varepsilon\dot J & = & -\varepsilon\partial_\phi\widetilde K(\star) =: \mathfrak{N}_J(J,\psi,z,\tilde z;\phi_0;\varepsilon) \\ \varepsilon\dot\psi - \varepsilon AJ & = & \varepsilon^2B^tz * \tilde z + \varepsilon\partial_I\tilde K(\star) =: \mathfrak{N}_\psi(J,\psi,z,\tilde z;\phi_0;\varepsilon) \\ \varepsilon\mathrm{i}\dot z - \tilde\Omega z & = & \varepsilon^2BJ * z + \varepsilon^3\partial_{\tilde z}\hat G(z,\tilde z) + \varepsilon\partial_{\hat{\tilde z}}\widetilde K(\star) =: \mathfrak{N}_z(J,\psi,z,\tilde z;\phi_0;\varepsilon) \\ \varepsilon\mathrm{i}\dot{\tilde z} + \tilde\Omega\tilde z & = & -\varepsilon^2BJ * \tilde z - \varepsilon^3\partial_z\hat G(z,\tilde z) - \varepsilon\partial_{\hat z}\widetilde K(\star) =: \mathfrak{N}_{\tilde z}(J,\psi,z,\tilde z;\phi_0;\varepsilon)\,, \end{array}\right. \tag{54}$$

where $\star := (I_0, \phi_0 + k\,t, 0, 0) + \varepsilon\mathfrak{u}$.

The right hand side of (54) defines the Nemitskij operator $\mathfrak{N} := (\mathfrak{N}_J, \mathfrak{N}_\psi, \mathfrak{N}_z, \mathfrak{N}_{\tilde z})$ acting on $\mathfrak{u}$; by (35) and Proposition 2.7, $\mathfrak{N}$ "gains one derivative", namely

$$\mathfrak{N}(\cdot;\phi_0;\varepsilon) \in \mathcal{A}_0(H_{a,s,\alpha,\sigma}, H_{a,s+1,\alpha,\sigma})\,, \tag{55}$$

with $a, s, \alpha > 0$, $\sigma > 1/2$. Moreover $\mathfrak{N}$ is also smooth for $\varepsilon \notin \mathcal{Z}$ and $\phi_0 \in \mathbb{T}^N$.

The left hand side of (54) defines a linear operator $\mathfrak{L}$ acting on $\mathfrak{u}$. Then (54) reduces to the functional equation $\mathfrak{L}\mathfrak{u} = \mathfrak{N}(\mathfrak{u};\phi_0)$. The kernel $\mathcal{K}$ and the range $\mathcal{R}$ of $\mathfrak{L}$ are, respectively, $\mathcal{K} = \big\{\psi \equiv \mathrm{const}\big\}$ and $\mathcal{R} = \Big\{\int_0^T\tilde\psi = 0\Big\}$. Performing a Lyapunov-Schmidt reduction, equation $\mathfrak{L}\mathfrak{u} = \mathfrak{N}(\mathfrak{u};\phi_0)$ splits into the *kernel equation*

$$0 = \Pi_{\mathcal{K}}\mathfrak{N}(\mathfrak{u};\phi_0) \tag{56}$$

and the *range equation*

$$\mathfrak{L}\mathfrak{u} = \Pi_{\mathcal{R}}\mathfrak{N}(\mathfrak{u};\phi_0)\,. \tag{57}$$

The "geometric" condition (52) means that $\mathfrak{L}$ is *formally* invertible on $\mathcal{R}$. To make it quantitative, one chooses the parameter ε such that the diophantine estimate

$$|\varepsilon k - \tilde\Omega_i(\varepsilon)| \ge \mathrm{ct}\,/|i|^\tau\,, \quad \tau \ge 1\,, \qquad \forall\, k \in \mathbb{Z}\,,\ i > N \tag{58}$$

holds, then $\mathfrak{L}$ is actually invertible $\mathfrak{L}^{-1} : H_{a,s+\tau,\alpha,\sigma} \to H_{a,s,\alpha,\sigma}$, loosing τ derivatives. The range equation becomes $\mathfrak{u} = \mathfrak{L}^{-1}\Pi_{\mathcal{R}}\mathfrak{N}(\mathfrak{u};\phi_0)$.

Remark 3.1. *In [BaBe05] the nonlinearity gains* $\tilde{s} > 1$ *derivatives, so, taking* $1 < \tau \leq \tilde{s}$, *one obtains a positive measure set of parameters satisfying the diophantine condition and the range equation can be solved by contractions.*
In [BiDG], [BiDG06], to have that $\mathfrak{L}^{-1}\Pi_{\mathcal{R}}\mathfrak{N}$ *is a contraction, we had to impose* $\tau = 1$. *This means that the set of* ε *for which condition (58) is satisfied must have zero measure. Actually there we were able to find only a finite set of admissible* ε.
In the present paper, to obtain a positive measure set of ε, *we take* $\tau > 1$. *Now the operator* $\mathfrak{L}^{-1}\Pi_{\mathcal{R}}\mathfrak{N}$ *cannot be a contraction, "loosing* $\tau - 1 > 0$ *derivatives". A Nash–Moser scheme is necessary. One should linearize equation (57) in a whole neighborhood of the origin; then the small divisors get more involved depending also on* ϕ_0. *Then one could solve the range equation for a Cantor set of parameters* (ϕ_0, ε) *and solve the bifurcation equation on it; as we discussed in the introduction, this problem is difficult and here we use a complete different strategy. We look for solutions* $\mathfrak{u}$ *of (54) in the "symmetric space"*

$$S := \Big\{ \ \mathfrak{u} = (J, \psi, z, \tilde{z}) \quad s.t. \quad \rho_* \mathfrak{u} = \mathfrak{I}\mathfrak{u} \ \Big\}, \tag{59}$$

where

$$(\mathfrak{I}\mathfrak{u})(t) := \mathfrak{u}(-t) \tag{60}$$

denotes the time inversion. It is simple to see that if $\mathfrak{u} \in S$ *the kernel equation (56) is always satisfied with* $\phi_0 = 0$ *(see Proposition 3.4). So, from now on, we take* $\phi_0 = 0$.

For brevity we introduce the Hilbert space

$$\ell_*^{a,s} := \Big\{ w = (\ldots, w_{-2}, w_{-1}, w_1, w_2, \ldots),\ w_i \in \mathbb{C}\,, \quad \text{s.t.} \ \ \|w\|_{a,s}^2 := \sum_{i \neq 0} |w_i|^2 |i|^{2s} e^{2a|i|} < \infty \Big\}.$$

We will systematically identify $\ell^{a,s} \times \ell^{a,s}$ with $\ell_*^{a,s}$, namely $(z, \tilde{z})$ with w, setting $w_j = z_j$ if $j \geq 1$ and $w_j = \tilde{z}_{-j}$ if $j \leq -1$. With a little abuse of notation we will write, e.g., $w = (z, \tilde{z})$, $\hat{G}(w) = \hat{G}(z, \tilde{z})$, etc. Finally let $W_{a,s,\alpha,\sigma} := H^{\alpha,\sigma}_{\ell_*^{a,s}}$, with norm $\| \cdot \|_{a,s,\alpha,\sigma} := \| \cdot \|_{H^{\alpha,\sigma}_{\ell_*^{a,s}}}$ (defined in (40)).

The strategy to solve the system (54) is the following. By the Fixed Point Theorem (see Proposition 3.4 below), we will solve the first two equations in (54) finding $J \in H^{\alpha,\sigma}_{\mathbb{C}^N}$ even in t and $\psi \in H^{\alpha,\sigma}_{\mathbb{C}^N}$ odd in t for every fixed $w = w(t)$ belonging to $W_{a,s,\alpha,\sigma} \cap \mathfrak{W}^+$ where

$$\mathfrak{W}^{\pm} := \Big\{ \ w = \sum_{k \in \mathbb{Z}} w_k e^{\mathrm{i}kt} \quad \text{s.t.} \quad \rho w = \pm \mathfrak{I} w \text{ or, equivalently, } \rho w_k = \pm w_{-k} \ \Big\}. \tag{61}$$

Then we will find w solving the last two equations in (54) with $J = J(w)$, $\psi = \psi(w)$. This is the most difficult task and it will be overcome by a Nash–Moser procedure.

Remark 3.2. *We have to control that the solutions we find are indeed real. This means that if*

$$\mathfrak{R}^{\pm} := \Big\{ \ w = (z, \tilde{z}) \ s.t. \ \tilde{z}(t) = \pm \overline{z(t)} \ \Big\}, \tag{62}$$

we must show that $w \in \mathfrak{R}^+$. *This fact is not surprising since our Hamiltonian is real analytic. So in the following we will always write the "reality conditions" in the statements, but we will often omit the (straightforward) proofs.*

3.1 Solving the low modes

Let us consider the linear operator $L^\sharp[J, \psi] := (\dot{J}, \dot{\psi} - AJ)$; its kernel is $\{(J, \psi) = (0, \mathrm{c}t)\}$ and its range is $\{(f, g) \text{ s.t. } \int_0^{2\pi} g = 0\}$. Therefore $L^\sharp$ is invertible in the space of functions with $\int_0^{2\pi} g = 0$. Indeed we define $\mathfrak{B} \in \mathcal{L}^{\alpha,\sigma}(\mathbb{C}^{2N}, \mathbb{C}^{2N})$ acting on $f = \sum_\ell f_\ell e^{\mathrm{i}\ell t}$, $g = \sum_\ell g_\ell e^{\mathrm{i}\ell t}$ by $\mathfrak{B}[f, g] := (J, \psi)$ where

$$J := -\sum_{\ell \neq 0} \left(\frac{\mathrm{i} f_\ell}{\ell} + A \frac{g_\ell}{\ell^2} \right) e^{\mathrm{i}\ell t}\,, \qquad \psi := -A^{-1} f_0 - \mathrm{i} \sum_{\ell \neq 0} \frac{g_\ell}{\ell} e^{\mathrm{i}\ell t}\,. \tag{63}$$

Note that the integral operator $\mathfrak{B}$ is not a Töplitz operator.

Lemma 3.3. *If $g_0 = 0$, then $(J, \psi) := \mathfrak{B}[f, g]$ solves $L^\sharp[J, \psi] = (f, g)$. Moreover $\|\mathfrak{B}\|_{\mathcal{L}^{\alpha,\sigma}(\mathbb{C}^{2N},\mathbb{C}^{2N})} \leq \mathrm{ct}$ (with constant independent of α, σ) and $\hat\rho\mathfrak{B} = -\mathfrak{I}\mathfrak{B}\mathfrak{I}\hat\rho$. Finally $\mathfrak{B}(H^{\alpha,\sigma}_{\mathbb{R}^{2N}}) \subset H^{\alpha,\sigma}_{\mathbb{R}^{2N}}$.*

Proposition 3.4. *Let $a, s, \alpha > 0$ and $\sigma > 1/2$. For every $c_1 > 0$ there exists $\varepsilon_0 > 0$ independent of α, such that $\forall\, \varepsilon \leq \varepsilon_0$ there exist $\chi(\cdot;\varepsilon) = \big(J(\cdot;\varepsilon), \psi(\cdot;\varepsilon)\big) \in \mathcal{A}(W_{a,s,\alpha,\sigma}, H^{\alpha,\sigma}_{\mathcal{C}^{2N}})$, defined for every $\|w\|_{a,s,\alpha,\sigma} \leq c_1$ and satisfying*

$$\chi \circ \rho\mathfrak{I} = \hat\rho\mathfrak{I}\chi\,, \tag{64}$$

such that, if $w \in \mathfrak{W}^+$, then (by (64) $J(w;\varepsilon)$ is even in time and $\psi(w;\varepsilon)$ is odd and) $\chi(w;\varepsilon)$ solves the first two equations in (54). Moreover $\|\chi(w;\varepsilon)\|_{H^{\alpha,\sigma}_{\mathbb{C}^N}} \leq \mathrm{ct}$ and, if $\varepsilon \notin \mathcal{Z}$, $\|\partial_\varepsilon\chi(w;\varepsilon)\|_{H^{\alpha,\sigma}_{\mathbb{C}^{2N}}}, \leq \mathrm{ct}\,\varepsilon^{-2}$. Finally if $w \in \mathfrak{R}^+$, then $J(w;\varepsilon), \psi(w;\varepsilon) \in H^{\alpha,\sigma}_{\mathbb{R}^N}$.

PROOF. Let $\Phi \in \mathcal{A}_0(H^{\alpha,\sigma}_{\mathbb{C}^{2N}} \times W_{a,s,\alpha,\sigma}, H^{\alpha,\sigma}_{\mathbb{C}^{2N}})$ be defined by

$$\Phi(\chi, w;\varepsilon) := \varepsilon^{-1}\mathfrak{B}\hat N(\chi, w;\varepsilon)\,, \quad \text{where } \hat N := (\mathfrak{N}_J, \mathfrak{N}_\psi)\,.$$

We claim that there exists $c_2 = c_2(a, s) > 0$ (large enough) such that, $\forall\, c_1 > 0$ and $\|w\|_{a,s,\alpha,\sigma} \leq c_1$, there exists $\varepsilon_0 = \varepsilon_0(a, s, \sigma, c_1) > 0$ (but independent of α) such that if $\varepsilon \leq \varepsilon_0$ then $\Phi(\cdot, w;\varepsilon)$ is a contraction on the closed ball $B := \overline{B_{c_2}(H^{\alpha,\sigma}_{\mathbb{C}^{2N}})}$. By Remark 2.4, Proposition 2.7 and (54), there exists $c_3 > 0$ independent of α, σ, such that for every $c_1, c_2 > 0$, if $\varepsilon \leq \varepsilon_0$ dependent[13] on σ but not on α,

$$\begin{aligned} \|\hat N(\chi, w;\varepsilon)\|_{H^{\alpha,\sigma}_{\mathbb{C}^{2N}}} &\leq c_3\varepsilon\,, & (65)\\ \Big\|\partial_\chi \hat N(\chi, w;\varepsilon)\Big\|_{\mathcal{L}(H^{\alpha,\sigma}_{\mathbb{C}^{2N}}, H^{\alpha,\sigma}_{\mathbb{C}^{2N}})} &\leq c_3\varepsilon^2\,, & (66)\\ \Big\|\partial_w \hat N(\chi, w;\varepsilon)\Big\|_{\mathcal{L}(W_{a,s,\alpha,\sigma}, H^{\alpha,\sigma}_{\mathbb{C}^{2N}})} &\leq c_3\varepsilon^2\,, & (67) \end{aligned}$$

for any $\chi \in B$ and $\|w\| \leq c_1$. Then for any $\chi, \chi^* \in B$, $\|w\| \leq c_1$, $\varepsilon \leq \varepsilon_0$, taking $c_2 := c_3\|\mathfrak{B}\|$ we get $\|\Phi(\chi, w;\varepsilon)\| \leq \varepsilon^{-1}\|\mathfrak{B}\|c_3\varepsilon = c_2$, (namely Φ maps B on itself) and

$$\|\Phi(\chi, w;\varepsilon) - \Phi(\chi^*, w;\varepsilon)\| \leq \varepsilon^{-1}\|\mathfrak{B}\|\big\|\hat N(\chi, w;\varepsilon) - \hat N(\chi^*, w;\varepsilon)\big\| \leq \mathrm{ct}\,\varepsilon^{-1}\varepsilon^2\|\chi - \chi^*\|\,,$$

(namely Φ is a contraction on B). Therefore, by the Contraction Mapping Theorem and the analytic dependence of Φ on w, there exists $\chi(\cdot;\varepsilon) \in \mathcal{A}_0(W_{a,s,\alpha,\sigma}, H^{\alpha,\sigma}_{\mathbb{C}^{2N}})$, defined for $\|w\|_{a,s,\alpha,\sigma} < c_1$, solution of

$$\chi(w;\varepsilon) = \Phi\big(\chi(w;\varepsilon); w;\varepsilon\big)\,. \tag{68}$$

Let us prove (64). Since $\chi(w)$ is the limit of the sequence $\chi^{j+1}(w) := \Phi(\chi^j(w), w)$ with $\chi^0(w) := 0$, it is enough to prove by induction that $\chi^j \circ \rho\mathfrak{I} = \hat\rho\mathfrak{I}\chi^j$. This directly follows noticing that $\hat N(\hat\rho\chi, \rho w) = -\hat\rho\hat N(\chi, w)$ (by (38)) and $\hat\rho\mathfrak{B} = -\mathfrak{I}\mathfrak{B}\mathfrak{I}\hat\rho$ (recall Lemma 3.3).
By (64), the solution of (68) has the property that $t \mapsto \big(J(w)\big)(t)$ is even and $t \mapsto \big(\psi(w)\big)(t)$ is odd when $w \in \mathfrak{W}^+$. Then $t \mapsto \big(\mathfrak{N}_J\big(J(w), \psi(w), w\big)\big)(t)$ is odd, its average is zero and, by Lemma 3.3, χ solves the first two equation of (54).
If $J, \psi \in H^{\alpha,\sigma}_{\mathbb{R}^N}$ and $w \in W_{a,s,\alpha,\sigma} \cap \mathfrak{R}^+$, the, by (37) and (54), $\mathfrak{N}_J(J, \psi, w;\varepsilon)$, $\mathfrak{N}_\psi(J, \psi, w;\varepsilon) \in H^{\alpha,\sigma}_{\mathbb{R}^N}$; moreover, by Lemma 3.3, $\Phi(J, \psi; w;\varepsilon) \in H^{\alpha,\sigma}_{\mathbb{R}^{2N}}$. Hence the above solution of (68) also belongs to $H^{\alpha,\sigma}_{\mathbb{R}^{2N}}$.
Deriving (68) with respect to ε, we get

$$\partial_\varepsilon\chi = -\varepsilon^{-1}\chi + \varepsilon^{-1}\mathfrak{B}(\mathfrak{N}'_J, \mathfrak{N}'_\psi)\,, \tag{69}$$

with (recall (51))

$$\begin{aligned} &\mathfrak{N}'_J := -\varepsilon^2\partial^2_{\phi I}\tilde K[\partial_\varepsilon J] - \varepsilon^2\partial^2_{\phi\phi}\tilde K[\partial_\varepsilon\psi] + U_J\,, \qquad \mathfrak{N}'_\psi := \varepsilon^2\partial^2_{II}\tilde K[\partial_\varepsilon J] + \varepsilon^2\partial^2_{I\phi}\tilde K[\partial_\varepsilon\psi] + U_\psi\,,\\ &U_J := -\partial_\phi\tilde K - \varepsilon\partial^2_{\phi I}\tilde K[I'_0 + J] - \varepsilon\partial^2_{\phi\phi}\tilde K[\psi] - \varepsilon^2(\partial^2_{\phi z}\tilde K, \partial^2_{\phi\tilde z}\tilde K)[w] - \varepsilon\partial^2_{\phi\varepsilon}\tilde K\,, \end{aligned}$$

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[13] Here the choice of ε_0 is related to $r_\sharp$ of Remark 2.4 and to $c(s)$ of Proposition 2.7.

$$U_\psi := \partial_I \tilde K + \varepsilon \partial^2_{II} \tilde K [I_0' + J] + \varepsilon \partial^2_{I\phi} \tilde K [\psi] + \varepsilon^2 (\partial^2_{Iz} \tilde K, \partial^2_{I\tilde z} \tilde K)[w] + \varepsilon \partial^2_{I\varepsilon} \tilde K + 2\varepsilon B^t z * \tilde z \,.$$

Since the linear operator

$$M : \partial_\varepsilon \chi \mapsto \partial_\varepsilon \chi - \varepsilon \mathfrak{B}(\partial^2_{II} \tilde K \partial_\varepsilon J + \partial^2_{I\phi} \tilde K \partial_\varepsilon \psi, -\varepsilon \partial^2_{\phi I} \tilde K \partial_\varepsilon J - \varepsilon \partial^2_{\phi\phi} \tilde K \partial_\varepsilon \psi)$$

is invertible (for ε small enough), by (69) we find $\partial_\varepsilon \chi = \varepsilon^{-1} M^{-1} \big[-\chi + \mathfrak{B}(U_J, U_\psi) \big]$ by which the final estimates of the proposition follow, noting that the dominant terms in U_J, U_ψ are the ones containing I_0' and $|I_0'(\varepsilon)| \le \mathrm{ct}\,/\varepsilon^2$ (recall (51)). $\square$

Summarizing, we have solved the first two equations in (54) finding $J(w;\varepsilon)$ and $\psi(w;\varepsilon)$; we have to find w that solves the last two equations in (54) with $J = J(w;\varepsilon)$ and $\psi = \psi(w;\varepsilon)$ namely

$$L_\varepsilon w + \varepsilon \mathcal{N}(w;\varepsilon) = 0 \,. \tag{70}$$

Here $L_\varepsilon : W_{a,s,\alpha,\sigma} \to W_{a,s-1,\alpha,\sigma-1}$ is defined by

$$L_\varepsilon w := (-\varepsilon \mathrm{i} \dot z + \tilde\Omega(\varepsilon) z, -\varepsilon \mathrm{i} \dot{\tilde z} - \tilde\Omega(\varepsilon) \tilde z) = -\varepsilon \mathrm{i} \dot w + \tilde\Omega(\varepsilon) w \,, \tag{71}$$

where, with a little abuse of notation[14], we still denote by $\tilde\Omega$ the bi–infinite matrix

$$\mathrm{diag}_{|i|>N} \tilde\Omega_i \,, \qquad \text{with} \quad \tilde\Omega_i := \mathrm{sign}\,(i)\ \tilde\Omega_{|i|} \,, \quad \forall\, |i| > N \,; \tag{72}$$

finally

$$\mathcal{N}(w;\varepsilon) := \Big(\varepsilon B J(w;\varepsilon) * z + \varepsilon^2 \partial_{\tilde z} \hat G(w) + \partial_{\hat{\tilde z}} \tilde K(y_\natural), -\varepsilon B J(w;\varepsilon) * \tilde z - \varepsilon^2 \partial_z \hat G(w) - \partial_{\hat z} \tilde K(y_\natural) \Big) \,, \tag{73}$$

where $y_\natural := \big(I_0(\varepsilon) + \varepsilon J(w;\varepsilon), k(\varepsilon) t + \varepsilon \psi(w;\varepsilon), \varepsilon w; \varepsilon \big)$. Note that by (55) and Proposition 3.4

$$\mathcal{N}(\cdot;\varepsilon) \in \mathcal{A}_0(W_{a,s,\alpha,\sigma}, W_{a,s+1,\alpha,\sigma}) \,. \tag{74}$$

For $\varepsilon \notin \mathcal{Z}$

$$\begin{aligned} \partial_\varepsilon \mathcal{N}(w;\varepsilon) := \Big(& B J * z + \varepsilon B J * z + 2\varepsilon \partial_{\tilde z} \hat G(w) + \partial^2_{\hat{\tilde z} I} \tilde K(y_\natural)[I_0' + J + \varepsilon \partial_\varepsilon J] \\ & + \partial^2_{\hat{\tilde z}\phi} \tilde K(y_\natural)[\psi + \varepsilon \partial_\varepsilon \psi] + \partial^2_{\hat{\tilde z}\hat w} \tilde K(y_\natural)[w] + \partial^2_{\hat{\tilde z}\varepsilon} \tilde K(y_\natural), \\ & - B J * \tilde z - \varepsilon B J * \tilde z - 2\varepsilon \partial_z \hat G(w) - \partial^2_{\hat z I} \tilde K(y_\natural)[I_0' + J + \varepsilon \partial_\varepsilon J] \\ & - \partial^2_{\hat z \phi} \tilde K(y_\natural)[\psi + \varepsilon \partial_\varepsilon \psi] - \partial^2_{\hat z \hat w} \tilde K(y_\natural)[w] - \partial^2_{\hat z \varepsilon} \tilde K(y_\natural) \Big) \,. \end{aligned} \tag{75}$$

We note that by (38) and (64) we get

$$\mathcal{N} \circ \rho \mathfrak{I} = -\rho \mathfrak{I} \mathcal{N} \,. \tag{76}$$

Then

$$L_\varepsilon(\mathfrak{W}^+ \cap \mathfrak{R}^+) \,,\ \mathcal{N}(\mathfrak{W}^+ \cap \mathfrak{R}^+) \subseteq \mathfrak{W}^- \cap \mathfrak{R}^- \,. \tag{77}$$

4 The linearized operator

Let us linearize the operator (70) in $w_* \in W_{a,s,\alpha,\sigma}$ (close to the origin). We get $\mathcal{L}(w_*;\varepsilon) : W_{a,s,\alpha,\sigma} \to W_{a,s-1,\alpha,\sigma-1}$,

$$\mathcal{L}(w_*;\varepsilon)[w] := L_\varepsilon w + \varepsilon D\mathcal{N}(w_*;\varepsilon)[w] =: L_\varepsilon w + \varepsilon^2 \Lambda(w_*;\varepsilon)[w] \,, \tag{78}$$

where[15]

$$\Lambda(w_*;\varepsilon)[w] = \varepsilon^{-1} D\mathcal{N}(w_*;\varepsilon)[w] = \big(\Lambda^1(w_*;\varepsilon)[w], \Lambda^2(w_*;\varepsilon)[w] \big) = \tag{79}$$

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[14] Recall definition (50).

[15] With $\hat w := (\hat z, \hat{\tilde z})$.

$$\Big(B\check J * z_* + BJ(w_*;\varepsilon) * z + \varepsilon\partial^2_{\tilde z w}\hat G(w_*)w + \partial^2_{\hat{\tilde z} I}\tilde K(y_\flat)\check J + \partial^2_{\hat{\tilde z}\phi}\tilde K(y_\flat)\check\psi + \partial^2_{\hat{\tilde z}\hat w}\tilde K(y_\flat)w\,,$$
$$-B\check J * \tilde z_* - BJ(w_*;\varepsilon) * \tilde z - \varepsilon\partial^2_{zw}\hat G(w_*)w - \partial^2_{\hat z I}\tilde K(y_\flat)\check J - \partial^2_{\hat z\phi}\tilde K(y_\flat)\check\psi - \partial^2_{\hat z\hat w}\tilde K(y_\flat)w\Big)\,,$$

$$\check J := DJ(w_*;\varepsilon)[w]\,, \qquad \check\psi := D\psi(w_*;\varepsilon)[w]\,, \qquad y_\flat := (I_0 + \varepsilon J(w_*;\varepsilon), \kappa(\varepsilon)t + \varepsilon\psi(w_*;\varepsilon), \varepsilon w_*;\varepsilon)\,. \quad (80)$$

We now prove that $\Lambda(w_*;\varepsilon) \in \mathcal{L}^{\alpha,\sigma}(\ell^{a,s}_*, \ell^{a,s+1}_*)$ (recall Definition 2.8). First of all we note that

$$\check J \mapsto B\check J * z_*\,,\ \check J \mapsto \partial^2_{\hat{\tilde z} I}\tilde K(y_\flat)\check J\,,\ \check\psi \mapsto \partial^2_{\hat{\tilde z}\phi}\tilde K(y_\flat)\check\psi\,,\ \check J \mapsto B\check J * \tilde z_*\,,\ \check J \mapsto \partial^2_{\hat z I}\tilde K(y_\flat)\check J\,,\ \check\psi \mapsto \partial^2_{\hat z\phi}\tilde K(y_\flat)\check\psi\,,$$

belonging to $\mathcal{L}(H^{\alpha,\sigma}_{\mathbb{C}^N}, H^{\alpha,\sigma}_{\ell^{a,s+1}})$, and

$$w \mapsto BJ(w_*;\varepsilon) * z\,,\ w \mapsto \partial^2_{\tilde z w}\hat G(w_*)w\,,\ w \mapsto \partial^2_{\hat{\tilde z}\hat w}\tilde K(y_\flat)w\,,\ w \mapsto BJ(w_*;\varepsilon) * \tilde z\,,\ w \mapsto \partial^2_{zw}\hat G(w_*)w\,,$$
$$w \mapsto \partial^2_{\hat z\hat w}\tilde K(y_\flat)w\,,$$

belonging to $\mathcal{L}(W_{a,s,\alpha,\sigma}, H^{\alpha,\sigma}_{\ell^{a,s+1}})$, are all product operators; namely they belong to $\mathcal{L}^{\alpha,\sigma}(\mathbb{C}^N, \ell^{a,s+1})$ and $\mathcal{L}^{\alpha,\sigma}(\ell^{a,s}_*, \ell^{a,s+1})$ respectively; moreover, for r_0 and ε_0 sufficiently small but independent of α, the norm of all the above operators are all uniformly bounded on $\{\|w_*\|_{a,s} \le r_0\} \times [0,\varepsilon_0]$ by some constant independent of α. On the other hand the operators $\check J, \check\psi$ are not Töplitz operators.

Lemma 4.1. *$D\chi(w_*;\varepsilon)[\cdot] \in \mathcal{L}(W_{a,s,\alpha,\sigma}, H^{\alpha,\sigma}_{\mathbb{C}^{2N}})$ actually belong to $\mathcal{L}^{\alpha,\sigma}(\ell^{a,s}_*, \mathbb{C}^{2N})$. Moreover the following estimates hold*

$$\|D\chi(w_*;\varepsilon)\|_{\mathcal{L}^{\alpha,\sigma}(\ell^{a,s}_*,\mathbb{C}^{2N})} \le \mathrm{ct}\,\varepsilon\,,$$
$$\|D^2\chi(w_*;\varepsilon)\| + \|D^3\chi(w_*;\varepsilon)\| \le \mathrm{ct}\,\varepsilon\,, \qquad \varepsilon\|\partial_\varepsilon D\chi(w_*;\varepsilon)\| + \|\partial_\varepsilon D^2\chi(w_*;\varepsilon)\| \le \mathrm{ct}$$

for every $\|w_*\|_{a,s} \le r_0$ and $\varepsilon \in [0,\varepsilon_0] \setminus \mathcal{Z}$. All the constant above are independent of α.

PROOF. Deriving with respect to w the identity in (68) we get

$$(\check J, \check\psi) = M_1[\check J, \check\psi] + M_2 w\,, \tag{81}$$

where

$$\begin{aligned} M_1[\check J, \check\psi] &:= \varepsilon^{-1}\mathfrak{B}\Big[\partial_J\mathfrak{N}_J(y_\sharp)\check J + \partial_\psi\mathfrak{N}_J(y_\sharp)\check\psi\,,\ \partial_J\mathfrak{N}_\psi(y_\sharp)\check J + \partial_\psi\mathfrak{N}_\psi(y_\sharp)\check\psi\Big] \\ M_2 w &:= \varepsilon^{-1}\mathfrak{B}\Big[\partial_w\mathfrak{N}_J(y_\sharp)w\,,\ \partial_w\mathfrak{N}_\psi(y_\sharp)w\Big] \qquad\qquad y_\sharp := \big(\chi(w_*;\varepsilon), w_*\big)\,. \end{aligned}$$

We note that $M_1 \in \mathcal{L}(H^{\alpha,\sigma}_{\mathbb{C}^{2N}}, H^{\alpha,\sigma}_{\mathbb{C}^{2N}})$ satisfies

$$\|M_1\|_{\mathcal{L}^{\alpha,\sigma}(\mathbb{C}^{2N},\mathbb{C}^{2N})} \le \mathrm{ct}\,\varepsilon^{-1}\|\mathfrak{B}\|_{\mathcal{L}^{\alpha,\sigma}(\mathbb{C}^{2N},\mathbb{C}^{2N})}\varepsilon^2 \ll 1,$$

by (66) and Lemma 3.3. Therefore M_1^m, $\sum_{m\ge0} M_1^m = (Id - M_1)^{-1} \in \mathcal{L}^{\alpha,\sigma}(\mathbb{C}^{2N},\mathbb{C}^{2N})$, by Propositions 2.11 and 2.12. By (81)

$$(\check J, \check\psi) = (Id - M_1)^{-1}[M_2 w]\,. \tag{82}$$

Finally $M_2 \in \mathcal{L}(H^{\alpha,\sigma}_{\ell^{a,s}_*}, H^{\alpha,\sigma}_{\mathbb{C}^{2N}})$ actually satisfies $\|M_2\|_{\mathcal{L}^{\alpha,\sigma}(\ell^{a,s}_*,\mathbb{C}^{2N})} \le \mathrm{ct}\,\varepsilon^{-1}\varepsilon^2 \le \mathrm{ct}\,\varepsilon$, by (67). By Proposition 2.12 we get the estimate on $D\chi$. The estimate on the operatorial norms of $D^2\chi$, $D^3\chi$ follow deriving (82) with respect to w.

Let us derive (81) with respect to $\varepsilon \notin \mathcal{Z}$, obtaining

$$(\partial_\varepsilon\check J, \partial_\varepsilon\check\psi) = (Id - M_1)^{-1}\big[M_1'[\check J, \check\psi] + M_2'[w]\big]\,.$$

We have

$$M_1' = -\varepsilon^{-1}M_1 + \varepsilon^{-1}\mathfrak{B}\Big[\big(\partial_J\mathfrak{N}_J(y_\sharp)\big)' + \big(\partial_\psi\mathfrak{N}_J(y_\sharp)\big)', \big(\partial_J\mathfrak{N}_\psi(y_\sharp)\big)' + \big(\partial_\psi\mathfrak{N}_\psi(y_\sharp)\big)'\Big]\,,$$

where (recall (54)), e.g.,

$$\big(\partial_J \mathfrak{N}_J(y_\sharp)\big)' = 2\varepsilon \partial^2_{II}\tilde{K}(y_\sharp) + \varepsilon^2 \partial^3_{III}\tilde{K}(y_\sharp)[I_0' + J + \varepsilon\partial_\varepsilon J] + \varepsilon^2 \partial^3_{II\phi}\tilde{K}(y_\sharp)[\psi + \varepsilon\partial_\varepsilon \psi] + \varepsilon^2 \partial^3_{II\varepsilon}\tilde{K}(y_\sharp)\,,$$

in which the dominant term is that containing I_0'. We get $\big\|\big(\partial_J \mathfrak{N}_J(y_\sharp)\big)'\big\|_{\mathcal{L}(H^{\alpha,\sigma}_{\mathbb{C}^{2N}}, H^{\alpha,\sigma}_{\mathbb{C}^{2N}})} \le \mathrm{ct}\,$. The other terms being analogous, we get $\|M_1'\|_{\mathcal{L}(H^{\alpha,\sigma}_{\mathcal{C}^{2N}}, H^{\alpha,\sigma}_{\mathcal{C}^{2N}})}$, $\|M_2'\|_{\mathcal{L}(H^{\alpha,\sigma}_{\ell^{a,s}_*}, H^{\alpha,\sigma}_{\mathcal{C}^{2N}})} \le \mathrm{ct}\,\varepsilon^{-1}$ and the estimate on $\partial_\varepsilon DJ$ follows.
To obtain the estimates on $\partial_\varepsilon D^2\chi$ we proceed analogously noting that $\|DM_1'\|_{\mathcal{L}(H^{\alpha,\sigma}_{\mathcal{C}^{2N}} \times H^{\alpha,\sigma}_{\ell^{a,s}_*}, H^{\alpha,\sigma}_{\mathcal{C}^{2N}})}$, $\|DM_2'\|_{\mathcal{L}(H^{\alpha,\sigma}_{\ell^{a,s}_*} \times H^{\alpha,\sigma}_{\ell^{a,s}_*}, H^{\alpha,\sigma}_{\mathcal{C}^{2N}})} \le \mathrm{ct}\,$. ☐

By the above considerations and Lemma 4.1, the following result follows.

Proposition 4.2. *$\Lambda(w_*;\varepsilon) \in \mathcal{L}^{\alpha,\sigma}(\ell^{a,s}_*, \ell^{a,s+1}_*)$ with estimate $\|\Lambda(w_*;\varepsilon)\|_{\mathcal{L}^{\alpha,\sigma}(\ell^{a,s}_*, \ell^{a,s+1}_*)} \le \mathrm{ct}$ for any $\|w_*\|_{a,s,\alpha,\sigma} \le r_0$, $\varepsilon \in [0,\varepsilon_0] \setminus \mathcal{Z}$. The constant, r_0 and ε_0 are independent of α.*

Then, by Proposition 2.10, there exist

$$\Lambda_{k\ell} = \Lambda_{k\ell}(w_*;\varepsilon) \in \mathcal{L}(\ell^{a,s}_*, \ell^{a,s+1}_*)\,, \qquad k,\ell \in \mathbb{Z}\,, \tag{83}$$

such that for any $w \in W_{a,s,\alpha,\sigma}$ (recall that $w(t) = \sum_\ell w_\ell e^{\mathrm{i}\ell t}$, with $w_\ell \in \ell^{a,s}_*$, by Proposition 2.5) we have

$$\big(\Lambda(w_*;\varepsilon)[w]\big)(t) = \sum_k e^{\mathrm{i}kt} \sum_\ell \Lambda_{k\ell}(w_*;\varepsilon)[w_\ell]\,, \tag{84}$$

with

$$\|\Lambda(w_*;\varepsilon)\|^2_{\mathcal{L}^{\alpha,\sigma}(\ell^{a,s}_*, \ell^{a,s+1}_*)} = \sup_\ell \sum_k e^{2\alpha|k-\ell|}|k-\ell|_*^{2\sigma} \|\Lambda_{k,\ell}(w_*;\varepsilon)\|^2_{\mathcal{L}(\ell^{a,s}_*, \ell^{a,s+1}_*)} \le \mathrm{ct}\ , \tag{85}$$

$\forall\, \|w_*\|_{\ell^{a,s}_*} \le r_0$.

We are going to prove some "symmetry" properties of the operators $\Lambda_{k\ell}$, that will be used to prove that the solution of (70) we are searching belongs to $\mathfrak{W}^+ \cap \mathfrak{R}^+$ (recall Proposition 3.4). In particular they will used to prove the last assertion in the statement of Lemma 5.4. For brevity we will often omit the explicit dependence on ε in the following.

Lemma 4.3. *If $w_* \in \mathfrak{W}^+$ then $\forall\, k,\ell \in \mathbb{Z}$, $\Lambda_{k\ell}(w_*) = -\rho\Lambda_{-k-\ell}(w_*)\rho$; moreover if $w_* \in \mathfrak{R}^+$ then $\Lambda_{k\ell}(w_*) \in \mathcal{L}\big(\ell^{a,s}_*(\mathbb{R}), \ell^{a,s+1}_*(\mathbb{R})\big)$.*

PROOF. First we note that by (76) we have

$$D\mathcal{N}(\rho\mathfrak{I}w_*)[\rho\mathfrak{I}w] = \frac{d}{d\xi}_{|\xi=0} \mathcal{N}\big(\rho\mathfrak{I}(w_* + \xi w)\big) = -\frac{d}{d\xi}_{|\xi=0} \big(\rho\mathfrak{I}\mathcal{N}(w_* + \xi w)\big) = -\rho\mathfrak{I}\big(D\mathcal{N}(w_*)[w]\big)\,,$$

namely

$$\Lambda(\rho\mathfrak{I}w_*)[\rho\mathfrak{I}w] = -\rho\mathfrak{I}\big(\Lambda(w_*)[w]\big)\,,$$

and, therefore, for $w_* \in \mathfrak{W}^+$,

$$\Lambda(w_*)[\rho\mathfrak{I}w] = -\rho\mathfrak{I}\big(\Lambda(w_*)[w]\big)\,.$$

Then for any $x \in \ell^{a,s}_*$ we get

$$\begin{aligned}
\Lambda_{k\ell}(w_*)[x] \quad &:= \quad \fint_0^{2\pi} e^{-\mathrm{i}kt}\Lambda(w_*)[e^{\mathrm{i}\ell t}x]\,dt = \fint_0^{2\pi} e^{-\mathrm{i}kt}\mathfrak{I}\rho\rho\mathfrak{I}\Big(\Lambda(w_*)[e^{\mathrm{i}\ell t}x]\Big)\,dt \\
&= \quad -\fint_0^{2\pi} e^{-\mathrm{i}kt}\mathfrak{I}\rho\Big(\Lambda(w_*)[\rho\mathfrak{I}e^{\mathrm{i}\ell t}x]\Big)\,dt = -\rho\fint_0^{2\pi} e^{-\mathrm{i}kt}\mathfrak{I}\Big(\Lambda(w_*)[e^{-\mathrm{i}\ell t}\rho x]\Big)\,dt \\
&= \quad -\rho\fint_0^{2\pi} e^{\mathrm{i}k\tau}\Lambda(w_*)[e^{-\mathrm{i}\ell\tau}\rho x]\,d\tau = -\rho\Lambda_{-k-\ell}(w_*)[\rho x]
\end{aligned}$$

□

Recollecting

$$\mathcal{L}w = \mathcal{L}(w_*;\varepsilon)[w] = \sum_k e^{\mathrm{i}kt}\left(\varepsilon k w_k + \tilde\Omega w_k + \varepsilon^2 \sum_\ell \Lambda_{k\ell}(w_*;\varepsilon)[w_\ell]\right). \tag{86}$$

We split $\mathcal{L}$ into a diagonal (in time) term and an off–diagonal one:

$$\mathcal{L}(w_*;\varepsilon) = D(w_*;\varepsilon) + \varepsilon^2 T(w_*;\varepsilon)\,, \tag{87}$$

where the diagonal operator is $\big(D(w_*;\varepsilon)[w]\big)(t) := \sum_k e^{\mathrm{i}kt} D_k(w_*;\varepsilon)[w_k]$ with

$$D_k(w_*;\varepsilon) := \varepsilon k Id_{\ell^{a,s}_*} + \tilde\Omega + \varepsilon^2 \mathcal{M}_k(w_*;\varepsilon) : \ell^{a,s}_* \to \ell^{a,s-1}_* \tag{88}$$

and

$$\mathcal{M}_k = \mathcal{M}_k(w_*;\varepsilon) := \Lambda_{kk}(w_*;\varepsilon) \in \mathcal{L}(\ell^{a,s}_*, \ell^{a,s+1}_*)\,, \tag{89}$$

and the off–diagonal operator $T = T(w_*;\varepsilon) \in \mathcal{L}^{\alpha,\sigma}(\ell^{a,s}_*, \ell^{a,s+1}_*)$ is

$$\big(T(w_*;\varepsilon)[w]\big)(t) := \sum_k e^{\mathrm{i}kt} \sum_{\ell\neq k} \Lambda_{k\ell}(w_*;\varepsilon)[w_\ell]\,. \tag{90}$$

By (85) there exists ct > 0 independent of k such that

$$\|\mathcal{M}_k(w_*;\varepsilon)\|_{\mathcal{L}(\ell^{a,s}_*,\ell^{a,s+1}_*)} \le \mathrm{ct}\,, \qquad \forall\, \|w_*\|_{a,s,\alpha,\sigma} \le r_0\,,\ \varepsilon \le \varepsilon_0\,. \tag{91}$$

Remark 4.4. *The operators $\mathcal{M}_k$ defined in (89), are not symmetric. This lack of symmetry could surprise. Considering, instead of $\mathcal{L}$, the operator $\mathcal{J}\mathcal{L}$, with $\mathcal{J} := \begin{pmatrix} 0_{\ell^{a,s}} & -Id_{\ell^{a,s}} \\ Id_{\ell^{a,s}} & 0_{\ell^{a,s}} \end{pmatrix}$, one would have that $\mathcal{J}\mathcal{L}$ is symmetric on $W_{a,s,\alpha,\sigma}$, since it is of the form $-\varepsilon\mathrm{i}\mathcal{J}\partial_t$+Hessian. However note that the time Fourier coefficients of $\mathcal{J}\mathcal{L}$ are not symmetric on $\ell^{a,s}_*$, in particular $(\mathcal{J}\mathcal{L})_{kk}$ is not symmetric since $(-\varepsilon\mathrm{i}\mathcal{J}\partial_t)_{kk} = \varepsilon k\mathcal{J}$. On the other hand, the decomposition in (86)-(90) is crucial for the method[16] we use to find the eigenvalues, as we are going to explain. The eigenvalues μ_{ki} of D_k accumulate to zero. Conversely, for every k, the eigenvalues λ_{ki} of $\tilde\Omega + \varepsilon^2\mathcal{M}_k$ are large, well-separated and weakly depending on k (see (147)), so that they are relatively easy to find (see Section 6). Finally, by (88), $\mu_{ki} = \varepsilon k + \lambda_{ki}$. The smallness of μ_{ki} is due to the interaction between the term εk, which depends only on the time-Fourier-index k, and the term λ_{ki}, which substantially depends only on the space-Fourier-index i. Roughly speaking, we could say that the above decomposition allows us to "decouple the time and space effects".*

For completeness we now show that $\mathcal{M}_k$ is not symmetric. By (79) and Lemma 4.1, for ε small, the dominant term in Λ is[17] $\Big(BJ(w_*)*z + \partial^2_{\hat{\tilde z}\hat w}\tilde K(y_0)w,\ -BJ(w_*)*\tilde z - \partial^2_{\hat z\hat w}\tilde K(y_0)w\Big)$, where $y_0 := (I_0, \tilde\omega t, 0)$. Let $\hat{\mathcal{M}}$ be the dominant term in $\mathcal{M}_k$. $\hat{\mathcal{M}}$ is independent of k. We have

$$\hat{\mathcal{M}} = \begin{pmatrix} \hat{\mathcal{M}}^{11} & \hat{\mathcal{M}}^{12} \\ \hat{\mathcal{M}}^{21} & \hat{\mathcal{M}}^{22} \end{pmatrix}, \qquad \text{with} \qquad \hat{\mathcal{M}}^{21}_{ij} := -\fint_0^{2\pi} \partial^2_{\hat z_i \hat z_j}\tilde K(y_0)dt\,, \qquad \hat{\mathcal{M}}^{12}_{ij} := \fint_0^{2\pi} \partial^2_{\hat{\tilde z}_i \hat{\tilde z}_j}\tilde K(y_0)dt\,,$$

$$\hat{\mathcal{M}}^{11}_{ij} := \fint_0^{2\pi} \Big(\delta_{ij}(BJ(w_*))_i + \partial^2_{\hat z_i \hat{\tilde z}_j}\tilde K(y_0)\Big)dt\,, \qquad \hat{\mathcal{M}}^{22}_{ij} := -\fint_0^{2\pi} \Big(\delta_{ij}(BJ(w_*))_i + \partial^2_{\hat z_i \hat{\tilde z}_j}\tilde K(y_0)\Big)dt\,.$$

The operators $\hat{\mathcal{M}}^{nm} \in \mathcal{L}(\ell^{a,s}, \ell^{a,s+1})$ have got the following properties. $\hat{\mathcal{M}}^{12}$ and $\hat{\mathcal{M}}^{21}$ are obviously symmetric. By (37) and (37), $\hat{\mathcal{M}}^{11}$ and $\hat{\mathcal{M}}^{22}$ are also symmetric, $\hat{\mathcal{M}}^{22} = -\hat{\mathcal{M}}^{11}$ and $\hat{\mathcal{M}}^{12} = -\hat{\mathcal{M}}^{21}$; therefore $\hat{\mathcal{M}}$ is symmetric if and only if $\hat{\mathcal{M}}^{12} = 0 = \hat{\mathcal{M}}^{21}$. Actually $\hat{\mathcal{M}}^{21}$ does not vanish in general.

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[16] Analogously in [BeBo06].

[17] For the rest of this section we omit the explicit dependence on ε.

We now give an example of this fact in a simple case: let us take $N := 1$ and prove that $\hat{\mathcal{M}}^{21}_{33} = -\fint_0^{2\pi} \partial^2_{\hat{z}_3\hat{z}_3} \tilde{K}(y_0)dt \neq 0$. The only term of $\tilde{K}$ that does not identically vanish when we evaluate $\partial^2_{\hat{z}_3\hat{z}_3}\tilde{K}(y_0)$ comes from the following term in K (defined in (22)): $\mathrm{z}^2_{*3}(a_1\mathrm{z}^4_{*1} + a_2\mathrm{z}^3_{*1}\tilde{\mathrm{z}}_{*1} + a_3\mathrm{z}^2_{*1}\tilde{\mathrm{z}}^2_{*1} + a_4\mathrm{z}_{*1}\tilde{\mathrm{z}}^3_{*1} + a_5\tilde{\mathrm{z}}^4_{*1})$. By the Hamiltonian symmetry (recall (26)) we get $a_1 = a_5$ and $a_2 = a_4$; then (recall (28)) the only relevant term in $\tilde{K}$ is $\hat{z}^2_3\frac{I_1^2}{4}\ (2a_1\cos 4\phi_1 + 2a_2\cos 2\phi_1 + a_3)$. Finally $\hat{\mathcal{M}}^{21}_{33} = -\frac{I^2_{01}}{2}a_3$. It remains only to show that $a_3 \neq 0$. We must evaluate the sixth order term of K; recalling the analysis carried out in subsection 2.2, we obtain that it is equal to $\{G,F\} + \frac{1}{2}\{\{Q,F\},F\}$, namely to

$$-8 \sum_{m,b,c,d,j,k,\ell}{}^{'} \operatorname{sign} m \ \frac{G_{mbcd}G_{mjk\ell}}{\omega'_m + \omega'_j + \omega'_k + \omega'_\ell}\ \delta_{mbcdjk\ell}\ \mathfrak{w}_b\mathfrak{w}_c\mathfrak{w}_d\mathfrak{w}_j\mathfrak{w}_k\mathfrak{w}_\ell\,,$$

where the prime means that the summation is over all nonzero integers and

$$\delta_{mbcdjk\ell} := \begin{cases} 0 & \text{if } (m,j,k,\ell) \notin S_N \text{ or } m \neq \pm b \pm c \pm d, \\ 1 & \text{if } (m,j,k,\ell) \in S_N \text{ and } (-m,b,c,d) \in S_N, \\ 2 & \text{if } (m,j,k,\ell) \in S_N,\ m = \pm b \pm c \pm d \text{ but } (-m,b,c,d) \notin S_N\ . \end{cases}$$

Then $a_3 = -72(b_1G_{1111}G_{3311} + b_2G^2_{3111} + b_3G^2_{3311} + b_4G^2_{5311})$, where

$$\begin{aligned} b_1 &:= \frac{1}{\omega_1} - \frac{1}{2(\omega_3-\omega_1)} + \frac{1}{2(\omega_3+\omega_1)}\,, \qquad b_3 := \frac{2}{\omega_3} + \frac{1}{2(\omega_3-\omega_1)} + \frac{1}{2(\omega_3+\omega_1)}\,, \\ b_2 &:= \frac{1}{\omega_3+3\omega_1} - \frac{1}{\omega_3-3\omega_1} - \frac{3}{\omega_3-\omega_1} + \frac{3}{\omega_3+\omega_1}\,, \\ b_4 &:= \frac{1}{\omega_5+\omega_3-2\omega_1} + \frac{1}{\omega_5-\omega_3+2\omega_1} + \frac{1}{\omega_5+\omega_3+2\omega_1} + \frac{1}{\omega_5-\omega_3-2\omega_1} + \frac{4}{\omega_5+\omega_3} + \frac{4}{\omega_5-\omega_3}\,. \end{aligned}$$

The coefficient a_3 analytically depends on the "mass" $\mu > 0$, moreover $\lim_{\mu\to 0^+} a_3 = -\infty$ since $\lim_{\mu\to 0^+} \mu a_3 = -54\pi^{-2}$. We conclude that, a part from (at most) a finite set of μ, $a_3 \neq 0$ and so $\hat{\mathcal{M}}$ and, hence, $\mathcal{M}_k$ are not symmetric.

5 Nash–Moser scheme

Let us define the projection P_n defined for $w \in W_{a,s,\alpha,\sigma}$ by

$$P_n\Big(\sum_k w_k e^{\mathrm{i}kt}\Big) := \sum_{|k|\le 4^n} w_k e^{\mathrm{i}kt}\,. \tag{92}$$

The following estimates are straightforward. Let $\alpha > \alpha'$, $\sigma' > \sigma$ and $w \in W_{a,s,\alpha,\sigma}$, then

$$\|w\|_{a,s,\alpha',\sigma'} \le c_\flat \Big(\frac{1}{\alpha-\alpha'}\Big)^{(\sigma'-\sigma)} \|h\|_{a,s,\alpha,\sigma}\,, \tag{93}$$

with $c_\flat := e^{(\sigma-\sigma')}(\sigma'-\sigma)^{(\sigma'-\sigma)}$, and

$$P_n w = 0 \qquad \Longrightarrow \qquad \|w\|_{a,s,\alpha',\sigma} \le e^{-(\alpha-\alpha')\,(4^n+1)}\|w\|_{a,s,\alpha,\sigma}\,. \tag{94}$$

Remark 5.1. *Up to now we have considered arbitrary a, s, α, σ and the various constants* ct *continuously depended on them. Now we fix*

$$\bar{a} > 0\,, \qquad \bar{s} > 2\,, \qquad \bar{\alpha} > 0\,, \qquad \bar{\sigma} > 1/2 \tag{95}$$

and we will use all the previous results with a, s, a, σ equal to some functions of $\bar{a}, \bar{s}, \bar{\alpha}, \bar{\sigma}$; for example we will take $a := \bar{a}$, $s := \bar{s}$ or $s := \bar{s} - 1$ etc.

Let us define

$$\alpha_0 := 2\bar{\alpha} \qquad \text{and} \qquad \alpha_n := \alpha_{n-1} - \alpha_0 2^{-n-1} = \alpha_0 - \alpha_0 \sum_{j=2}^{n+1} 2^{-j} \ \longrightarrow\ \alpha_0/2 = \bar{\alpha}\,. \tag{96}$$

Let, for $s \in \mathbb{R}$,

$$W_s^{(n)} := P_n W_{\bar{a},\bar{s}+s,\alpha_n,\bar{\sigma}} \qquad \text{and} \qquad W^{(n)} := W_0^{(n)}\,. \tag{97}$$

Let us consider the truncation of the linearized operator $\mathcal{L}$ defined in (78), namely

$$\mathcal{L}^{(n)} := P_n \mathcal{L}\,|_{W^{(n)}} \colon W^{(n)} \to W_{-1}^{(n)}\,. \tag{98}$$

We will find the (right) inverse $\mathcal{G}^{(n)}$ of $\mathcal{L}^{(n)}$, see Lemma 5.4 below. We will use the decomposition in (87), namely

$$\mathcal{L}^{(n)} = D^{(n)} + \varepsilon^2 T^{(n)}\,, \quad \text{where} \quad D^{(n)} := P_n D\,|_{W^{(n)}} \quad \text{and} \quad T^{(n)} := P_n T\,|_{W^{(n)}}\,. \tag{99}$$

By Lemma 4.3

$$w_* \in \mathfrak{W}^+ \qquad \Longrightarrow \qquad D^{(n)}(\mathfrak{W}^\pm)\,,\ T^{(n)}(\mathfrak{W}^\pm) \subseteq \mathfrak{W}^\mp\,. \tag{100}$$

5.1 Melnikov condition and invertibility

Let us define

$$1 < \tau < 2 \qquad \gamma := \varepsilon^\delta\,, \qquad \gamma_n := (1 + 2^{-n-1})\gamma > \gamma \qquad 0 < \delta < \tau - 1 < 1 \tag{101}$$

(δ will be fixed in (197)).

Remark 5.2. *The restriction $\tau < 2$ is technical. We will use it in estimating the eigenvalues of the linearized operator (see (163)). It also simplifies the estimates on the small divisors.*

Definition 5.3 (First order Melnikov conditions)**.** *Let $n \in \mathbb{N}$, $\varepsilon_0 > 0$, $w_* \in W^{(n)}$. Let us define*

$$\Delta_n^\tau(w_*) := \left\{ \varepsilon \in (0,\varepsilon_0) \text{ s.t. } \left|\varepsilon k - \tilde{\Omega}_i - \varepsilon^2 \big(\mathcal{M}_k(w_*;\varepsilon)\big)_{ii}\right| > \frac{\varepsilon\gamma_n}{i^\tau}\,, \quad |\varepsilon k - i| > \frac{\varepsilon\gamma}{i^\tau}\,,\ \forall\, i > N\,,\ 0 \le k \le 4^n \right\},$$

where $\big(\mathcal{M}_k(w_*;\varepsilon)\big)_{ii} := \langle \mathcal{M}_k(w_*;\varepsilon) e_i, e_i\rangle_{\ell_*^{\bar{a},\bar{s}}}$ with e_i the standard orthonormal basis of $\ell_*^{\bar{a},\bar{s}}$. Note that $\Delta_n^\tau(w_*)$ is an open set, since nothing changes if we substitute "$\forall\, i > N$" with "$\forall\, N < i \le 4^{n+1}$" in the above definition.

The following Lemma is crucial for the iterative scheme. Its proof is the real core of the issue and it will be proved in the following sections (see, in particular, pg. 31).

Lemma 5.4 (Invertibility)**.** *Let $n \ge 1$. If $\|w_*\|_{\bar{a},\bar{s}-1,\alpha_n,\bar{\sigma}+2} \le r_0$, with r_0 defined in Proposition 4.2, then for all $\varepsilon \in \Delta_n^\tau(w_*)$ there exists $\mathcal{G}^{(n)}(\cdot;\varepsilon) : W_{-1}^{(n)} \longrightarrow W^{(n)}$ satisfying*

$$\big\|\mathcal{G}^{(n)}(w_*;\varepsilon)\big\|_{\mathcal{L}(W_{-1}^{(n)},W^{(n)})} \le \mathrm{ct}\, \gamma^{-1} \varepsilon^{\tau-2} 4^{n(\tau-1))}\,, \tag{102}$$

such that

$$\mathcal{L}^{(n)} \mathcal{G}^{(n)} h = h\,, \qquad \forall\, h \in W_{-1}^{(n)}\,, \tag{103}$$

namely $\mathcal{G}^{(n)}$ is the right inverse of $\mathcal{L}^{(n)}$. Finally, if $w_* \in \mathfrak{W}^+ \cap \mathfrak{R}^+$ then $\mathcal{G}^{(n)}(w_*;\varepsilon) : \mathfrak{W}^- \cap \mathfrak{R}^- \to \mathfrak{W}^+ \cap \mathfrak{R}^+$.

5.2 Iteration

In this subsection $|\cdot|_n := \|\cdot\|_{\bar a,\bar s,\alpha_n,\bar\sigma}$, for brevity.

Lemma 5.5 (Iterative Lemma)**.** *Let be $w^{(0)}=0$, $A_0 := (0,\varepsilon_0)\setminus\mathcal{Z}$. Then there exists a sequence of open sets $A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_0$ and $w^{(n)} = w^{(n)}(\varepsilon)$, $w^{(n)} \in W^{(n)} \cap \mathfrak{W}^+ \cap \mathfrak{R}^+$ defined for $\varepsilon \in A_n$ with*

$$A_n := \big\{\varepsilon \in A_{n-1} \ \ s.t. \ \ \varepsilon \in \Delta_n^\tau\big(w^{(n-1)}(\varepsilon)\big)\big\}\,. \tag{104}$$

$w^{(n)}(\varepsilon) = \sum_{i=1}^n h_i(\varepsilon)$, where $h^{(n)}(\varepsilon) \in W^{(n)} \cap \mathfrak{W}^+ \cap \mathfrak{R}^+$ is defined as

$$h^{(n)}(\varepsilon) := -\mathcal{G}^{(n)}(w^{(n-1)};\varepsilon)\big[L_\varepsilon w^{(n-1)} + \varepsilon P_n \mathcal{N}(w^{(n-1)};\varepsilon)\big]\,, \tag{105}$$

$h^{(0)}(\varepsilon) = w^{(0)}(\varepsilon) = 0$ and

$$|h^{(n)}(\varepsilon)|_n \le c_0\, \varepsilon^{\tau-1-\delta} e^{-\chi^n \alpha_0/8}\,, \qquad 1<\chi<2\,, \tag{106}$$

for a large enough constant c_0. Moreover $h^{(n)}(\varepsilon)$ is differentiable in the open set A_n with

$$|(h^{(n)})'(\varepsilon)|_n \le c_0\, \varepsilon^{-\zeta}\, e^{-\tilde\chi^n \alpha_0/8}\,, \qquad 0<\zeta := 2(2+\delta-\tau)<2\,, \qquad 1<\tilde\chi<\chi\,. \tag{107}$$

PROOF. We proceed by induction. Let us suppose the construction holds for n, and prove it for $n+1$. First we note that $h^{(n+1)}$ is well defined since $\|w^{(n)}\|_{\bar a,\bar s,\alpha_{n+1},\bar\sigma+p} \le r_0$, so that, by Lemma 5.4, is well defined $\mathcal{G}^{n+1}(\varepsilon, w^{(n)})$ for $\varepsilon \in A_{n+1}$. Indeed by Lemma 93, (106) and $\alpha_{n+1} - \alpha_i \le -\alpha_0/2^{i+1}$, we get

$$\|w^{(n)}\|_{\bar a,\bar s,\alpha_{n+1},\bar\sigma+2} \le \mathrm{ct} \sum_{i=1}^n |h^{(i)}|_i (\alpha_i - \alpha_{n+1})^{-2} \le \mathrm{ct} \sum_{i=1}^n 2^{2i} |h^{(i)}|_i \le \mathrm{ct}\, \varepsilon^{\tau-1-\delta} \le r_0$$

for ε small enough.
Notice that from the definition of $\mathcal{L}^{(n)}$ in (78) and (98) and using (103) and (105), we get

$$-L_\varepsilon h^{(n)} - \varepsilon P_n D\mathcal{N}(w^{(n-1)})[h^{(n)}] = -\mathcal{L}^{(n)}(w^{(n-1)})h^{(n)} = L_\varepsilon w^{(n-1)} + \varepsilon P_n \mathcal{N}(w^{(n-1)})\,. \tag{108}$$

We are going to prove that (106) and (107) holds for $h^{(n+1)}$ by induction. Let

$$R_n := P_{n+1}\mathcal{N}(w_n) - P_{n+1}\mathcal{N}(w^{(n-1)}) - P_{n+1} D\mathcal{N}(w^{(n-1)})[h^{(n)}]\,, \tag{109}$$

then

$$\begin{aligned}
h^{(n+1)} &= -\mathcal{G}^{(n+1)}(w^{(n)})\big[L_\varepsilon w^{(n)} + \varepsilon P_{n+1}\mathcal{N}(w^{(n)})\big] \\
&= -\mathcal{G}^{(n+1)}(w^{(n)})\Big[L_\varepsilon w^{(n)} + \varepsilon P_{n+1}\mathcal{N}(w^{(n-1)}) + \varepsilon P_{n+1} D\mathcal{N}(w^{(n-1)})[h^{(n)}] + \varepsilon R_n\Big] \\
&= -\mathcal{G}^{(n+1)}(w^{(n)})\Big[L_\varepsilon w^{(n-1)} + \varepsilon P_n \mathcal{N}(w^{(n-1)}) + L h_n + \varepsilon P_n D\mathcal{N}(w^{(n-1)})[h^{(n)}] + \varepsilon R_n \\
&\qquad -\varepsilon P_n \mathcal{N}(w^{(n-1)}) + \varepsilon P_{n+1}\mathcal{N}(w^{(n-1)}) - \varepsilon P_n D\mathcal{N}(w^{(n-1)})[h^{(n)}] + \varepsilon P_{n+1} D\mathcal{N}(w^{(n-1)})[h^{(n)}]\Big] \\
&= -\mathcal{G}^{(n+1)}(w^{(n)})\Big[-\varepsilon P_n \mathcal{N}(w^{(n-1)}) + \varepsilon P_{n+1}\mathcal{N}(w^{(n-1)}) \\
&\qquad -\varepsilon P_n D\mathcal{N}(w^{(n-1)})[h^{(n)}] + \varepsilon P_{n+1} D\mathcal{N}(w^{(n-1)})[h^{(n)}] + \varepsilon R_n\Big] \\
&= -\varepsilon\mathcal{G}^{(n+1)}(w^{(n)})\Big[P_n^{n+1}\mathcal{N}(w^{(n-1)}) + P_n^{n+1} D\mathcal{N}(w^{(n-1)})[h^{(n)}] + R_n\Big]\,,
\end{aligned} \tag{110}$$

where $P_n^{n+1} := P_{n+1} - P_n$ and we used (108). Notice that

$$(4^n+1)(\alpha_n - \alpha_{n+1}) = (4^n+1)\frac{\alpha_0}{2^{n+2}} \ge \frac{\alpha_0}{4} 2^n\,. \tag{111}$$

Taking $a := \bar{a}$, $s := \bar{s}$, $\alpha := \alpha_n$, $\sigma := \bar{\sigma}$ in (74) we have $\mathcal{N}(\cdot;\varepsilon) \in \mathcal{A}_0(W_{\bar{a},\bar{s},\alpha_n,\bar{\sigma}}, W_{\bar{a},\bar{s}+1,\alpha_n,\bar{\sigma}})$, $\forall\, n \geq 1$. By (73) and recalling that all the estimates carried out in (34), Remark 2.4, Propositions 3.4 and 2.7, Lemma 4.1, do not depend on α, we conclude that for $r_0, \varepsilon_0 > 0$ small enough

$$\begin{aligned}\sup_{\varepsilon\in[0,\varepsilon_0]} \sup_{n\geq 1} \sup_{|w|_n \leq r_0} &\big(\|\mathcal{N}(w;\varepsilon)\|_{\bar{a},\bar{s}+1,\alpha_n,\bar{\sigma}} + \varepsilon^{-1}\|D\mathcal{N}(w;\varepsilon)\|_{\mathrm{op}} \\ &+\varepsilon^{-2}\|D^2\mathcal{N}(w;\varepsilon)\|_{\mathrm{op}} + \varepsilon^{-2}\|D^3\mathcal{N}(w;\varepsilon)\|_{\mathrm{op}}\big) \leq \mathrm{ct}\,, \end{aligned} \tag{112}$$

where $\|\cdot\|_{\mathrm{op}}$ denotes the appropriate operatorial norms. Reasoning as above, by (75) we get analogous estimate for $\partial_\varepsilon \mathcal{N}$, namely

$$\sup_{\varepsilon\in[0,\varepsilon_0]\setminus\mathcal{Z}} \sup_{n\geq 1} \sup_{|w|_n \leq r_0} \big(\varepsilon^2\|\partial_\varepsilon\mathcal{N}(w;\varepsilon)\|_{\bar{a},\bar{s}+1,\alpha_n,\bar{\sigma}} + \varepsilon\|D\partial_\varepsilon\mathcal{N}(w;\varepsilon)\|_{\mathrm{op}} + \|D^2\partial_\varepsilon\mathcal{N}(w;\varepsilon)\|_{\mathrm{op}}\big) \leq \mathrm{ct}\,. \tag{113}$$

By (109) and (112) we get

$$|R_n|_{n+1} \leq |R_n|_n \leq \mathrm{ct}\,|h^{(n)}|_n^2 \leq \mathrm{ct}\, c_0^2 \varepsilon^{2(\tau-1-\delta)} e^{-\chi^n\alpha_0/4}\,. \tag{114}$$

By Lemma 5.4, Lemma 94, (112), (114) and $|w^{(n-1)}|_n \leq \sum_{1\leq i\leq n-1}|h_i|_i \leq \mathrm{ct}\,\varepsilon^{\tau-1-\delta} \leq r_0$, we have

$$\begin{aligned}|h^{(n+1)}|_{n+1} &\leq \varepsilon\|\mathcal{G}^{(n+1)}(w^{(n)})\|_{\mathcal{L}(W_{-1}^{(n+1)},W^{(n+1)})}|P_n^{n+1}\mathcal{N}(w^{(n-1)}) + P_n^{n+1}D\mathcal{N}(w^{(n-1)})[h^{(n)}] + R_n|_{n+1} \\ &\leq \mathrm{ct}\,\varepsilon^{\tau-1-\delta}\, 4^{n(\tau-1)}\Big(e^{-(4^n+1)(\alpha_n-\alpha_{n+1})}|\mathcal{N}(w^{(n-1)}) + D\mathcal{N}(w^{(n-1)})[h^{(n)}]|_n + |h^{(n)}|_n^2\Big) \\ &\leq \mathrm{ct}\,\varepsilon^{\tau-1-\delta}\, 4^{n(\tau-1)}\Big(e^{-2^n\alpha_0/4} + c_0^2\varepsilon^{2(\tau-1-\delta)}e^{-\chi^n\alpha_0/4}\Big) \\ &\leq \mathrm{ct}\,\varepsilon^{\tau-1-\delta}\, 4^{n(\tau-1)}e^{-\chi^n\alpha_0/4} \leq c_0\,\varepsilon^{\tau-1-\delta}\, e^{-\chi^{n+1}\alpha_0/8}\,, \end{aligned} \tag{115}$$

for c_0 large enough (note that $4^{n(\tau-1)}e^{-\chi^n\alpha_0/4+\chi^{n+1}\alpha_0/8} \to 0$ for $n \to \infty$ since $1 < \chi < 2$), proving (106).

Let us suppose that (107) holds up to n and prove it for $n+1$. Using (103) in (110) we get

$$-\mathcal{L}^{(n+1)}(w^{(n)})[h^{(n+1)}] = \varepsilon\big(P_n^{n+1}\mathcal{N}(w^{(n-1)}) + P_n^{n+1}D\mathcal{N}(w^{(n-1)})[h^{(n)}] + R_n\big)\,. \tag{116}$$

Deriving (116) with respect to ε, we have

$$-\mathcal{L}^{(n+1)}(w^{(n)})[(h^{(n+1)})'] = \big(\mathcal{L}^{(n+1)}(w^{(n)})\big)'[h^{(n+1)}] + P_n^{n+1}\tilde{R}_n + \varepsilon R_n' + R_n\,, \tag{117}$$

where

$$\begin{aligned}\tilde{R}_n \;:=\; &\mathcal{N}(w^{(n-1)}) + D\mathcal{N}(w^{(n-1)})[h^{(n)}] + \varepsilon\partial_\varepsilon\mathcal{N}(w^{(n-1)}) + \varepsilon D\mathcal{N}(w^{(n-1)})[(w^{(n-1)})'] \\ &+\varepsilon\partial_\varepsilon D\mathcal{N}(w^{(n-1)})[h^{(n)}] + \varepsilon D^2\mathcal{N}(w^{(n-1)})[h^{(n)},(w^{(n-1)})']\,. \end{aligned} \tag{118}$$

We are going to estimate all the four terms in the right hand side of (117). By (51), (71), (78) and (98)

$$\big(\mathcal{L}^{(n+1)}(w^{(n)})\big)' = P_{n+1}\big(-\mathrm{i}\partial_t + BI_0 + \varepsilon^{-1}BA^{-1}\omega + 2\varepsilon\Lambda + \varepsilon^2\partial_\varepsilon\Lambda(w^{(n)};\varepsilon) + \varepsilon^2 D\Lambda(w^{(n)};\varepsilon)[(w^{(n)})']\big)\,|_{W^{(n+1)}}\,.$$

We have $|\partial_t h^{(n+1)}|_{n+1} \leq 4^{n+1}|h^{(n+1)}|_{n+1}$. By (78), $\Lambda = \varepsilon^{-1}D\mathcal{N}$ and, therefore, $\partial_\varepsilon\Lambda = -\varepsilon^{-2}D\mathcal{N} + \varepsilon^{-1}D\partial_\varepsilon\mathcal{N}$; then $|\varepsilon^2\partial_\varepsilon\Lambda(w^{(n)};\varepsilon)| \leq \mathrm{ct}$ by (112) and (113). By inductive hypothesis

$$|(w^{(n)})'|_n \leq \sum_{1\leq i\leq n} |(h^{(i)})'|_i \leq \mathrm{ct}\, c_1\varepsilon^{-\zeta}\,. \tag{119}$$

Again by (112), $|\varepsilon^2 D\Lambda(w^{(n)};\varepsilon)[(w^{(n)})']|_{n+1} \leq \mathrm{ct}\, c_1\varepsilon^{2-\zeta}$. By (101) $2-\zeta > -1$; then by (115)

$$\big|\big(\mathcal{L}^{(n+1)}(w^{(n)})\big)'[h^{(n+1)}]\big|_{n+1} \leq \mathrm{ct}\, c_0(4^{n+1} + \varepsilon^{-1})\varepsilon^{\tau-1-\delta}e^{-\chi^{n+1}\alpha_0/8}\,. \tag{120}$$

We now estimate $\tilde{R}_n$. By (106), (112), (113) and (119) we get[18] $|\tilde{R}_n|_n \le \mathrm{ct}\,\varepsilon^{-1}$, since again $2-\zeta > -1$. By Lemma 94 and (111)

$$|P_n^{n+1}\tilde{R}_n|_{n+1} \le e^{(4^n+1)(\alpha_n-\alpha_{n+1})}|P_{n+1}\tilde{R}_n|_{n+1} \le e^{-2^n\alpha_0/4}|\tilde{R}_n|_n \le \mathrm{ct}\, e^{-2^n\alpha_0/4}\varepsilon^{-1}\,. \tag{121}$$

Deriving (109) with respect to ε, we have $R_n' = R_n^1 + R_n^2 + R_n^3$ with

$$\begin{aligned}
R_n^1 &:= P_{n+1}\big(D\mathcal{N}(w^{(n)})[(h^{(n)})'] - D\mathcal{N}(w^{(n-1)})[(h^{(n)})']\big)\\
R_n^2 &:= P_{n+1}\big(D\mathcal{N}(w^{(n)})[(w^{(n-1)})'] - D\mathcal{N}(w^{(n-1)})[(w^{(n-1)})'] - D^2\mathcal{N}(w^{(n-1)})[(w^{(n-1)})', h^{(n)}]\big)\\
R_n^3 &:= P_{n+1}\big(\partial_\varepsilon\mathcal{N}(w_n) - \partial_\varepsilon\mathcal{N}(w^{(n-1)}) - \partial_\varepsilon D\mathcal{N}(w^{(n-1)})[h^{(n)}]\big)\,.
\end{aligned}$$

From (112), (119), (113) and $|w^{(n)}|_n \le \sum_{1\le i\le n}|h_i|_i \le \mathrm{ct}\,\varepsilon^{\tau-1-\delta} \le r_0$, we get

$$\begin{aligned}
|R_n^1|_{n+1} &\le \|D^2\mathcal{N}\|_{\mathrm{op}}|h^{(n)}|_n|(h^{(n)})'|_n \le \mathrm{ct}\,c_0c_1\varepsilon^{\tau+1-\zeta-\delta}e^{-(\chi^n+\tilde{\chi}^n)\alpha_0/8}\\
|R_n^2|_{n+1} &\le \|D^3\mathcal{N}\|_{\mathrm{op}}|(w^{(n-1)})'|_{n-1}|h^{(n)}|_n^2 \le \mathrm{ct}\,c_0^2c_1\varepsilon^{2\tau-\zeta-2\delta}e^{-\chi^n\alpha_0/4}\\
|R_n^3|_{n+1} &\le \|D^2\partial_\varepsilon\mathcal{N}\|_{\mathrm{op}}|h^{(n)}|_n^2 \le \mathrm{ct}\,c_0^2\varepsilon^{2(\tau-1-\delta)}e^{-\chi^n\alpha_0/4}\,,
\end{aligned}$$

and then

$$|\varepsilon R_n'|_{n+1} \le \mathrm{ct}\,c_0c_1\varepsilon^{\tau+2-\zeta-\delta}e^{-(\chi^n+\tilde{\chi}^n)\alpha_0/8}\,. \tag{122}$$

Recalling (120), (121), (122), (114), we get

$$\big|\big(\mathcal{L}^{(n+1)}(w^{(n)})\big)'[h^{(n+1)}] + P_n^{n+1}\tilde{R}_n + \varepsilon R_n' + R_n\big|_{n+1} \le \mathrm{ct}\,c_0\varepsilon^{\tau-2-\delta}4^{n+1}e^{-(\chi^n+\tilde{\chi}^n)\alpha_0/8}\,. \tag{123}$$

Applying $\mathcal{G}^{n+1}(w^{(n)})$ to both sides of (117) and using Lemma 5.4 and (123), we have

$$|(h^{(n+1)})'|_{n+1} \le \mathrm{ct}\,c_0\varepsilon^{2(\tau-2-\delta)}4^{n(\tau-1)}4^{n+1}e^{-(\chi^n+\tilde{\chi}^n)\alpha_0/8} \le c_1\varepsilon^{-\zeta}e^{-\tilde{\chi}^{n+1}\alpha_0/8}\,, \tag{124}$$

for c_1 large enough (note that $4^{n(\tau-1)}4^{n+1}e^{-(\chi^n+\tilde{\chi}^n-\tilde{\chi}^{n+1})\alpha_0/8} \to 0$ for $n\to\infty$ since $1<\tilde{\chi}<\chi$). We finally prove by induction that $h^{(n+1)} \in \mathfrak{W}^+\cap\mathfrak{R}^+$. Indeed by inductive hypothesis $w^{(n)} \in \mathfrak{W}^+\cap\mathfrak{R}^+$ and, by (77), $L_\varepsilon w^{(n)} + \varepsilon P_n\mathcal{N}(w^{(n)};\varepsilon) \in \mathfrak{W}^-\cap\mathfrak{R}^-$, so we conclude by (105) and Lemma 5.4. □

Corollary 5.6. *Define*

$$\mathcal{C} := \bigcap_{n\ge 0} A_n\,. \tag{125}$$

If $\varepsilon\in\mathcal{C}$, *then* $w^{(n)}(\varepsilon)$ *converge in* $W_{\bar{a},\bar{s},\bar{\alpha},\bar{\sigma}} = H^{\bar{\alpha},\bar{\sigma}}_{\ell_*^{\bar{a},\bar{s}}}$ *to a solution* $w(\varepsilon)\in\mathfrak{W}^+\cap\mathfrak{R}^+$ *of (70) with*

$$\|w(\varepsilon)\|_{\bar{a},\bar{s},\bar{\alpha},\bar{\sigma}} \le \mathrm{ct}\,\varepsilon^{\tau-1-\delta}\,. \tag{126}$$

PROOF. By Lemma 5.5 if $\varepsilon\in\mathcal{C}$ there exists a sequence $w^{(n)}(\varepsilon)\in W^{(n)}\supset W_{\bar{a},\bar{s},\bar{\alpha},\bar{\sigma}}$, $n\ge 0$. By (106) $w^{(n)}(\varepsilon)$ is a Cauchy sequence in $W_{\bar{a},\bar{s},\bar{\alpha},\bar{\sigma}}$, indeed

$$\|w^{(n+m)} - w^{(n)}\|_{\bar{a},\bar{s},\bar{\alpha},\bar{\sigma}} \le \sum_{i=n+1}^{n+m}\|h^{(i)}\|_i \le c_0\varepsilon^{\tau-1-\delta}\sum_{i=n+1}^{\infty}e^{-\chi^i\alpha_0/8} \longrightarrow 0\,, \qquad \text{when } n\to\infty\,.$$

Therefore $w^{(n)}(\varepsilon)$ converge in $W_{\bar{a},\bar{s},\bar{\alpha},\bar{\sigma}}$ to a suitable $w(\varepsilon)$ satisfying (126). To prove that w satisfies (70) we take the limit for $n\to\infty$ in the space $W_{\bar{a},\bar{s}-1,\bar{\alpha},\bar{\sigma}-1}$ into the equation (108). For the right hand side of (108) we simply have

$$L_\varepsilon w^{(n-1)} + \varepsilon P_n\mathcal{N}(w^{(n-1)}) \longrightarrow L_\varepsilon w + \varepsilon\mathcal{N}(w)\,.$$

Since $\|h^{(n)}\|_{\bar{a},\bar{s},\bar{\alpha},\bar{\sigma}}\to 0$, by (112) we get for the left hand side of (108)

$$-L_\varepsilon h^{(n)} - \varepsilon P_n D\mathcal{N}(w^{(n-1)})[h^{(n)}] \longrightarrow 0\,,$$

proving (70). □

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[18] The dominant term is $\varepsilon\partial_\varepsilon\mathcal{N}(w^{(n-1)})$.

6 Diagonal term

Notations. From now on[19] $\ell_*^{\bar a,\bar s} = \ell_*^{\bar a,\bar s}(\mathbb{R})$, $\ell_*^{\bar a,\bar s+1} = \ell_*^{\bar a,\bar s+1}(\mathbb{R})$, etc. For brevity, the basis, the norm and the scalar product of $\ell_*^{\bar a,\bar s+s}$ will be denoted by $\{e_i^{(s)}\}_{i\in\mathbb{Z}_*}$ (where $\mathbb{Z}_* := \mathbb{Z}\setminus\{0\}$), $|\cdot|_s$ and $\langle\cdot,\cdot\rangle_s$ respectively; if $s=0$ we will omit it, namely $e_i := e_i^{(0)}$, $|\cdot| := |\cdot|_0$ and $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_0$. We will also denote by $\|\cdot\|_{s_1,s_2}$ the operatorial norm on $\mathcal{L}^{s_1,s_2} := \mathcal{L}(\ell_*^{\bar a,\bar s+s_1}, \ell_*^{\bar a,\bar s+s_2})$.
If $U\in\mathcal{L}^{s_1,s_2}$ we denote

$$U_{ij} := U_{ij}^{(s_1,s_2)} := \langle e_i^{(s_2)}, U e_j^{(s_1)}\rangle_{s_2}\,,$$

then if $a = \sum_j a_j e_j^{(s_1)} \in \ell_*^{\bar a,\bar s+s_1}$ we have $Ua = \sum_i(\sum_j U_{ij}a_j)e_i^{(s_2)} \in \ell_*^{\bar a,\bar s+s_2}$. Finally if $p_1\ge 0$ and $\tilde U := U|_{\ell_*^{\bar a,\bar s+s_1+p_1}} \in \mathcal{L}^{s_1+p_1,s_2+p_2}$, then

$$\tilde U_{ij} = i^{p_2} j^{-p_1} U_{ij}\,. \tag{127}$$

In order to invert the operator $D^{(n)}$, we now analyze the spectral properties of the operators

$$S_k = S_k(w_*;\varepsilon) := \tilde\Omega + \varepsilon^2\mathcal{M}_k(w_*;\varepsilon) : \ell_*^{\bar a,\bar s} \to \ell_*^{\bar a,\bar s-1}\,. \tag{128}$$

All the difficulties arise since $\mathcal{M}_k$ is not symmetric, recall Remark 4.4.
In the following subsection we will study a generic operator

$$S := \tilde\Omega + \mathfrak{e}\mathcal{M} \in \mathcal{L}^{0,-1}\,, \tag{129}$$

where $\mathfrak{e}$ is a small parameter and $\mathcal{M}\in\mathcal{L}^{0,1}$ is a not symmetric operator. Let $M_* := \|\mathcal{M}\|_{0,1} < \infty$. If $\mathcal{M}_{ij} := \mathcal{M}_{ij}^{0,0}$ and $\tilde{\mathcal{M}}_{ij} := \tilde{\mathcal{M}}_{ij}^{0,1}$, we have that $i\mathcal{M}_{ij} = \tilde{\mathcal{M}}_{ij}$. Then by definition of M_* we have

$$\sum_i i^2 |\sum_k \mathcal{M}_{ik} a_k|^2 \le M_*^2 \sum_k a_k^2\,. \tag{130}$$

In particular, for every fixed $i\in\mathbb{Z}_*$ and every sequence $\{a_k\}_{k\in\mathbb{Z}_*}$

$$|\sum_k \mathcal{M}_{ik} a_k|^2 \le M_*^2 i^{-2} \sum_k a_k^2 \tag{131}$$

and for every fixed $j\in\mathbb{Z}_*$, choosing $a_k := \delta_{kj}$ in (130), we get

$$\sup_j \sum_i i^2 |\mathcal{M}_{ij}|^2 \le M_*^2\,. \tag{132}$$

6.1 "Diagonalization" of S

Proposition 6.1. *There exist $C_0, \mathfrak{e}_0 > 0$ (depending only on M_* and on the mass μ) such that for every $\mathfrak{e}\le\mathfrak{e}_0$ there exist $V\in\mathcal{L}^{-1,-1}$ and, $\forall\,|i|>N$, $\tilde\lambda_i\in\mathbb{R}$ with $|\tilde\lambda_i| \le \mathrm{ct}/i^2$ such that*

$$\|V - Id_{\ell_*^{\bar a,\bar s-1}}\|_{-1,-1} \le C_0\mathfrak{e}\,, \qquad \textit{moreover} \quad V|_{\ell_*^{\bar a,\bar s}} \in \mathcal{L}^{0,0} \quad \textit{and} \quad \|V|_{\ell_*^{\bar a,\bar s}} - Id_{\ell_*^{\bar a,\bar s}}\|_{0,0} \le C_0\mathfrak{e} \tag{133}$$

and

$$SV|_{\ell_*^{\bar a,\bar s}} = V(\tilde\Omega + \tilde\Lambda) \in \mathcal{L}^{0,-1}\,, \tag{134}$$

where $\tilde\Lambda\in\mathcal{L}^{0,1}$ is the diagonal operator with entries $\tilde\Lambda_{ij} := \tilde\Lambda_{ij}^{0,0} := \mathfrak{e}\delta_{ij}(\mathcal{M}_{ii} + \mathfrak{e}\tilde\lambda_i)$.

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[19] Recall that by Lemma 4.3 the involved operators are "real on the reals".

PROOF. For $U \in \mathcal{L}^{0,0}$ let

$$|||U|||^2 := \sup_j j^2 \sum_i |U_{ij}|^2 \,.$$

Note that $\|U\|_{0,0} \leq \mathrm{ct}\, |||U|||$. Let

$$E := \big\{ U \in \mathcal{L}^{0,0} \text{ s.t. } U_{ii} = 0 \ \forall\, i \text{ and } |||U||| < \infty \big\} \,.$$

E is a Banach space with the norm $|||\cdot|||$. For $r > 0$ let $B_r := \{U \in E \text{ s.t. } |||U||| \leq r\}$. Let us introduce the operator Φ with entries

$$\big(\Phi(U)\big)_{ij} := \Phi_{ij}(U) := \mathfrak{e}\, \frac{1-\delta_{ij}}{\tilde{\Omega}_i - \tilde{\Omega}_j} \Big(-\mathcal{M}_{ij} - \sum_k \mathcal{M}_{ik} U_{kj} + U_{ij}\mathcal{M}_{jj} + U_{ij} \sum_k \mathcal{M}_{jk} U_{kj} \Big) \,. \tag{135}$$

We claim that

$$\exists\, C\,,\ \mathfrak{e}_0 > 0 \text{ s.t. } \forall\, \mathfrak{e} \leq \mathfrak{e}_0\,,\ \Phi \text{ is a contraction on } B_{C\mathfrak{e}} \,. \tag{136}$$

We postpone the proof of (136) and assume that $U \in B_{C\mathfrak{e}}$ is the fixed point of the equation $U = \Phi(U)$. It is simple to prove that $U \in \mathcal{L}^{0,0}$ can be naturally extended to an operator $\tilde{U} \in \mathcal{L}^{-1,-1}$ defining $\tilde{U}_{ij} := \frac{j}{i} U_{ij}$. Indeed, by (135), $\tilde{U}$ is formed by four addenda. For example, the first addendum $\tilde{U}^{(1)}$ has entries

$$\tilde{U}^{(1)}_{ij} := -\mathfrak{e}\, \frac{1-\delta_{ij}}{\tilde{\Omega}_i - \tilde{\Omega}_j} \frac{j}{i} \mathcal{M}_{ij} \,.$$

We now show that $\|\tilde{U}^{(1)}\|_{-1,-1} < \mathrm{ct}\, \mathfrak{e}$. Indeed for every sequence b_k with $\sum_k b_k^2 = 1$ we have that, for any fixed i, defining $a_k := -\mathfrak{e}\, \frac{(1-\delta_{ik})k}{(\tilde{\Omega}_i - \tilde{\Omega}_k) i} b_k$, by (131) we get

$$|\sum_k \tilde{U}^{(1)}_{ik} b_k|^2 = |\sum_k \mathcal{M}_{ik} a_k|^2 \leq \mathfrak{e}^2 M_*^2 i^{-2} \sum_{k \neq i} \frac{k^2}{(\tilde{\Omega}_i - \tilde{\Omega}_k)^2 i^2} b_k^2 \leq 16 \mathfrak{e}^2 M_*^2 i^{-2} \,, \tag{137}$$

where in the last inequality we have used that

$$\frac{|k|}{|\tilde{\Omega}_i - \tilde{\Omega}_k||i|} \leq 4 \,, \qquad \forall\, i \neq k \,, \tag{138}$$

as it follows by[20]

$$|\tilde{\Omega}_i - \tilde{\Omega}_k| \geq |i-k|/2 \,. \tag{139}$$

Then by (137)

$$\|\tilde{U}^{(1)}\|^2_{-1,-1} := \sup_{\sum_k b_k^2 = 1} \sum_i |\tilde{U}^{(1)}_{ik} b_k|^2 \leq \sum_i 16 \mathfrak{e}^2 M_*^2 i^{-2} \leq \mathrm{ct}\, \mathfrak{e}^2 \,.$$

Since the other addenda can be treated analogously we conclude that

$$\|\tilde{U}\|_{-1,-1} < \mathrm{ct}\, \mathfrak{e} \,. \tag{140}$$

Let

$$V := Id_{\ell_*^{\bar{a},\bar{s}-1}} + \tilde{U} \in \mathcal{L}^{-1,-1} \,, \qquad \text{then} \quad U = V\,|_{\ell_*^{\bar{a},\bar{s}}} - Id_{\ell_*^{\bar{a},\bar{s}}}$$

and (133) follows by (140) and $\|U\|_{0,0} \leq \mathrm{ct}\ |||U||| \leq \mathrm{ct}\, \mathfrak{e}$. Let

$$\tilde{\lambda}_i := \frac{1}{\mathfrak{e}} \sum_{k \neq i} \mathcal{M}_{ik} U_{ki} \,. \tag{141}$$

---

[20] Note we can always assume (139) taking ε small (recall (50) and (72)).

We prove that $|\tilde\lambda_i| \le \mathrm{ct}\,/i^2$. Indeed by (135) $\tilde\lambda_i$ is formed by four terms; we will only consider the first one, $\tilde\lambda_i^{(1)} := \sum_{k\neq i}(\tilde\Omega_i - \tilde\Omega_k)^{-1}\mathcal{M}_{ik}\mathcal{M}_{ki}$, since the other ones can be treated analogously. For any fixed i let us define $a_k := (\tilde\Omega_i - \tilde\Omega_k)^{-1}\mathcal{M}_{ki}\delta_{ik}$; by (131), (138) and (132) we get

$$\begin{aligned}
|\tilde\lambda_i^{(1)}|^2 &= \Big|\sum_{k\neq i}\mathcal{M}_{ki}a_k\Big|^2 \le \frac{M_*^2}{i^2}\sum_{k\neq i}\frac{\mathcal{M}_{ki}^2}{(\tilde\Omega_i - \tilde\Omega_k)^2} = \frac{M_*^2}{i^4}\sum_{k\neq i}\frac{i^2}{k^2(\tilde\Omega_i - \tilde\Omega_k)^2}k^2\mathcal{M}_{ki}^2 \\
&\le 16\frac{M_*^2}{i^4}\sum_{k\neq i} k^2\mathcal{M}_{ki}^2 \le 16\frac{M_*^2}{i^4}\,.
\end{aligned}$$

We now show that

$$\tilde\Omega U - \tilde U\tilde\Omega - \tilde\Lambda + \mathfrak{e}\mathcal{M} + \mathfrak{e}\mathcal{M}U - \tilde U\tilde\Lambda = 0 \tag{142}$$

(viewed as an operator belonging to $\mathcal{L}^{0,-1}$) from which (134) follows. Indeed we have

$$\begin{aligned}
i\big(\tilde\Omega U - \tilde U\tilde\Omega - \tilde\Lambda + \mathfrak{e}\mathcal{M} + \mathfrak{e}\mathcal{M}U - \tilde U\tilde\Lambda\big)_{ij}& \\
= \Big(\tilde\Omega_i U_{ij} - \tilde U_{ij}\tilde\Omega_j - \tilde\Lambda_{ij} + \mathfrak{e}\mathcal{M}_{ij} + \mathfrak{e}\sum_k \mathcal{M}_{ik}U_{kj} - U_{ij}\tilde\Lambda_{jj}\Big)\,.&
\end{aligned} \tag{143}$$

Taking $i = j$ in (143) we get $-\tilde\Lambda_{ii} + \mathfrak{e}\mathcal{M}_{ii} + \mathfrak{e}\sum_k \mathcal{M}_{ik}U_{ki}$ which is null by (141). On the other hand, for $i \neq j$ we have

$$(\tilde\Omega_i - \tilde\Omega_j)\tilde U_{ij} + \mathfrak{e}\mathcal{M}_{ij} + \mathfrak{e}\sum_k \mathcal{M}_{ik}U_{kj} - \mathfrak{e}U_{ij}\mathcal{M}_{jj} - \mathfrak{e}^2 U_{ij}\tilde\lambda_j$$

which is null by (135) and (141); (142) follows.

We finally prove (136). We claim that

$$|||\Phi(U)||| \le 4M_*\mathfrak{e} + \mathrm{ct}\,C\mathfrak{e}^2\,, \qquad \forall\, U \in B_{C\mathfrak{e}}\,. \tag{144}$$

Indeed $\Phi(U)$ is formed by four terms; we can estimate the square of the $|||\cdot|||$-norm of the first one by (135) having

$$\mathfrak{e}^2\sup_j j^2\sum_{i\neq j}\frac{|\mathcal{M}_{ij}|^2}{|\tilde\Omega_i - \tilde\Omega_j|^2} = \mathfrak{e}^2\sup_j\sum_{i\neq j}\frac{j^2}{i^2|\tilde\Omega_i - \tilde\Omega_j|^2}i^2|\mathcal{M}_{ij}|^2 \le 16M_*^2\mathfrak{e}^2$$

by (138) and (132). Since the other terms are analogous we get (144). Taking $C := 5M_*$ and $\mathfrak{e}_0$ small, by (144) we get $|||\Phi(U)||| \le C\mathfrak{e}$ for any $\mathfrak{e} \le \mathfrak{e}_0$.

We have only to prove that Φ is a contraction. Recalling (135), Φ is formed by four terms; we will consider only the fourth term since it is the only nonlinear one. For $U, W \in B_{C\mathfrak{e}}$ we have

$$\begin{aligned}
|||\Phi^{(4)}(U) - \Phi^{(4)}(W)|||^2 &= \mathfrak{e}^2\sup_j j^2\sum_{i\neq j}\frac{1}{|\tilde\Omega_i - \tilde\Omega_j|^2}\Big|U_{ij}\sum_k \mathcal{M}_{jk}U_{kj} - W_{ij}\sum_k \mathcal{M}_{jk}W_{kj}\Big|^2 \\
&\le 2\mathfrak{e}^2\sup_j j^2\sum_{i\neq j}\frac{1}{|\tilde\Omega_i - \tilde\Omega_j|^2}\bigg(\Big|(U_{ij} - W_{ij})\sum_k \mathcal{M}_{jk}U_{kj}\Big|^2 + \Big|W_{ij}\sum_k \mathcal{M}_{jk}(U_{kj} - W_{kj})\Big|^2\bigg) \\
&\le 2\mathfrak{e}^2M_*^2\big(|||U|||^2 + |||W|||^2\big)|||U - W|||^2 \le 4\mathfrak{e}^4C^2M_*^2\;|||U - W|||^2
\end{aligned}$$

by (131). Then (136) follows taking $\mathfrak{e}_0$ small enough. □

6.2 "Diagonalization" of $D^{(n)}$

In this subsection we will always consider $w_* \in \mathfrak{W}^+ \cap \mathfrak{R}^+$ with $\|w_*\|_{\bar a,\bar s,\alpha,\sigma} \le r_0$, $\varepsilon \in \Delta_n^\tau(w_*)$ and $|k| \le 4^n$.

We now apply Proposition 6.1 with $S := S_k$ defined in (128), $\mathfrak{e} := \varepsilon^2$ and $\mathcal{M} := \mathcal{M}_k(w_*;\varepsilon) \in \mathcal{L}^{0,1}$ defined in (89) (recall also (91)). So we find[21] operators[22] $V_k \in \mathcal{L}^{-1,-1} \cap \mathcal{L}^{0,0}$ with inverse $V_k^{-1} \in \mathcal{L}^{-1,-1} \cap \mathcal{L}^{0,0}$ (recall (133)) and diagonal operators $\tilde\Lambda_k \in \mathcal{L}^{0,1}$ with entries

$$(\tilde\Lambda_k)_{ij} = \delta_{ij}\varepsilon^2\big(\langle \mathcal{M}_k e_i, e_i\rangle + \varepsilon^2 \tilde\lambda_{ki}\big) \in \mathbb{R}\,, \qquad |\tilde\lambda_{ki}| \le \mathrm{ct}\,/i^2\,, \qquad \forall\,|i| > N\,, \tag{145}$$

such that

$$S_k V_k = V_k(\tilde\Omega + \tilde\Lambda_k)\,,. \tag{146}$$

We can say that for any k fixed,

$$\lambda_{ki} := \tilde\Omega_i + \tilde\lambda_{ki} = \tilde\Omega_i + \varepsilon^2\langle \mathcal{M}_k e_i, e_i\rangle + O(\varepsilon^4|i|^{-2}) = \mathrm{sign}(i)\sqrt{i^2+\mu} + O(\varepsilon\,|i|^{-1}) \tag{147}$$

are the eigenvalues of $S_k = \tilde\Omega + \varepsilon^2\mathcal{M}_k$. Note that

$$|j-i|/2 < |\lambda_{kj} - \lambda_{ki}| < 2|j-i|\,, \qquad i < \lambda_{ki} < i + \mu\,. \tag{148}$$

Lemma 6.2. *For every $k \in \mathbb{Z}$, $i > N$, we have $\lambda_{-k-i} = -\lambda_{ki}$ and $V_{-k} = \rho V_k \rho$.*

PROOF. Let λ_{ki} be the i–th eigenvalue of $\tilde\Omega + \varepsilon^2\mathcal{M}_k$ and λ_{-k-i} be the $-i$–th eigenvalue of $\tilde\Omega + \varepsilon^2\mathcal{M}_{-k}$. We conclude noting that, by Lemma 4.3, $\rho(\tilde\Omega + \varepsilon^2\mathcal{M}_k)\rho = -(\tilde\Omega + \varepsilon^2\mathcal{M}_{-k})$. ☐

The eigenvalues of D_k defined in (88) (with $a := \bar a$ and $s := \bar s$) are

$$\mu_{ki} = \mu_{ki}(w_*;\varepsilon) = \varepsilon k + \lambda_{ki} \in \mathbb{R} \quad \text{defined for} \quad |i| > N\,, \tag{149}$$

Recalling (72) we have that $|\mu_{ki}|$ can be small only if k and i have opposite signs. To work with positive integers we define

$$\tilde\mu_{ki} = \tilde\mu_{ki}(w_*;\varepsilon) := \lambda_{ki} - \varepsilon k\,, \qquad \forall\,k \ge 0\,,\ i > N\,, \tag{150}$$

so that

$$|\mu_{ki}| \ge |\tilde\mu_{|k||i|}|\,, \qquad \forall\,|k| \le 4^n\,,\ |i| > N \tag{151}$$

For every $k \ge 0$ let $i_0 = i_0(k) > N$ be the smallest integer such that

$$\min_{i>N} |\tilde\mu_{ki}| = |\tilde\mu_{ki_0}|\,. \tag{152}$$

We claim that

$$k\varepsilon \ge N \qquad \Longrightarrow \qquad i_0 - 1 \le k\varepsilon \le i_0 + \mu + 1\,. \tag{153}$$

Let us prove the first inequality. If $i_0 - 1 = N$, it is true. If $i_0 - 1 > N$ let us suppose by contradiction that $i_0 - 1 > k\varepsilon$; then by (148) we get $\mu_{ki_0} > i_0 - k\varepsilon > 1$, but $0 < \lambda_{ki_0} - \lambda_{k(i_0-1)} < 2$ so that $-1 < \mu_{k(i_0-1)} < \mu_{ki_0}$ contradicting the minimality of μ_{ki_0}. We now prove the second inequality in (153). Let us suppose by contradiction that $k\varepsilon > i_0 + \mu + 1$; then, again by (148), $\mu_{ki_0} < i_0 + \mu - k\varepsilon < -1$, but, since $0 < \lambda_{k(i_0+1)} - \lambda_{ki_0} < 2$, we have $\mu_{ki_0} < \mu_{k(i_0+1)} < 1$ contradicting the minimality of μ_{ki_0}. (153) follows.

Note that $\min\limits_{|k|\le 4^n,\ |i|>N} |\mu_{ki}| > 0$ (see (159) below) and that, by (149) and (147), $|\mu_{ki}| \sim |i|$ for $|i|$ large. Let us define the operator $U^{(n)} = U^{(n)}(w_*;\varepsilon)$ by

$$U^{(n)}\Big(\sum_{|k|\le 4^n} e^{\mathrm{i}kt} w_k\Big) := \sum_{|k|\le 4^n} e^{\mathrm{i}kt} U_k w_k\,, \quad \text{with} \quad U_k := \mathrm{diag}_{|i|>N}\,(\mu_{ki}/|\mu_{ki}|) \in \mathcal{L}^{-1/2,-1/2}\,.$$

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[21] Note that the constants C_0 and $\mathfrak{e}$ in Proposition 6.1 depend only on $M_* := \|\mathcal{M}\|_{0,1}$. Since in (91) the estimate on $\|\mathcal{M}_k\|_{0,1}$ is independent of k, w_* and ε, then all the constants below will be also independent of k, w_* and ε (and continuously depending on α,σ.).

[22] With abuse of notation we still denote by V_k the operators $V_k\,|_{\ell_*^{\bar a,\bar s}} \in \mathcal{L}^{0,0}$.

$U^{(n)} : W^{(n)}_{-1/2} \longrightarrow W^{(n)}_{-1/2}$ is invertible and $\|U^{(n)}\|_{-1/2,-1/2}$, $\|(U^{(n)})^{-1}\|_{-1/2,-1/2} = 1$.
Let

$$\mathcal{D}_k(w_*;\varepsilon) = \mathcal{D}_k := \mathrm{diag}_{|i|>N}\left(i^{-1}\mu_{ki}\right) \in \mathcal{L}^{0,-1}\,.$$

For any $p \in \mathbb{R}$ let[23]

$$|\mathcal{D}_k|^p := \mathrm{diag}_{|i|>N}\left(i^{-p}|\mu_{ki}|^p\right) \in \mathcal{L}^{p+s,s}\,, \qquad \forall\, s \in \mathbb{R}\,. \tag{154}$$

Let i_0 be such that (152) holds. It is simple to see that

$$\delta_{|k|} \;:=\; \Big(\min\big\{|\tilde\mu_{|k|1}|, |i_0-1||\tilde\mu_{|k|i_0}|\big\}\Big)^{-1/2} \tag{155}$$

$$\geq\; \mathrm{ct}\max_{|i|>N}|i|^{-1/2}|\mu_{ki}|^{-1/2} = \mathrm{ct}\,\Big\||\mathcal{D}_k|^{-1/2}\Big\|_{0,-1/2} = \mathrm{ct}\,\Big\||\mathcal{D}_k|^{-1/2}\Big\|_{-1/2,-1} \tag{156}$$

For $w := \sum_{|k|\leq 4^n} e^{\mathrm{i}kt}w_k$ let us define the operators

$$Z^{(n)}w := \sum_{|k|\leq 4^n} e^{\mathrm{i}kt}Z_k w_k\,, \quad \tilde Z^{(n)}w := \sum_{|k|\leq 4^n} e^{\mathrm{i}kt}\tilde Z_k w_k\,,$$

$$(Z^{(n)})^{-1}w := \sum_{|k|\leq 4^n} e^{\mathrm{i}kt}Z_k^{-1} w_k\,, \quad (\tilde Z^{(n)})^{-1}w := \sum_{|k|\leq 4^n} e^{\mathrm{i}kt}\tilde Z_k^{-1} w_k\,,$$

with

$$Z_k := V_k|\mathcal{D}_k|^{-1/2}\,, \quad \tilde Z_k := |\mathcal{D}_k|^{-1/2}V_k^{-1}\,, \quad Z_k^{-1} := |\mathcal{D}_k|^{1/2}V_k^{-1}\,, \quad \tilde Z_k^{-1} := V_k|\mathcal{D}_k|^{1/2}\,. \tag{157}$$

Note that $Z^{(n)} : W^{(n)}_{-1/2} \longrightarrow W^{(n)}$, $\tilde Z^{(n)} : W^{(n)}_{-1} \longrightarrow W^{(n)}_{-1/2}$, $(Z^{(n)})^{-1} : W^{(n)} \longrightarrow W^{(n)}_{-1/2}$ and $(\tilde Z^{(n)})^{-1} : W^{(n)}_{-1/2} \longrightarrow W^{(n)}_{-1}$. By definition we get that $(Z^{(n)})^{-1}Z^{(n)}$ and $(\tilde Z^{(n)})^{-1}\tilde Z^{(n)}$ coincide with the identity on $W^{(n)}_{-1/2}$ and $W^{(n)}_{-1}$ respectively. Noting that (88), (149) and (146) imply $D_kV_k = V_k\mathcal{D}_k$, we can state the following

Lemma 6.3 ("Diagonalization")**.** *Let $\varepsilon \in \Delta^\tau_n(w_*)$, with $\|w_*\|_{\bar a,\bar s,\alpha,\sigma} \leq r_0$. Then*

$$D^{(n)}(w_*;\varepsilon)Z^{(n)}(w_*;\varepsilon) = \big(\tilde Z^{(n)}(w_*;\varepsilon)\big)^{-1}U^{(n)}(w_*;\varepsilon) : W^{(n)}_{-1/2} \longrightarrow W^{(n)}_{-1}\,.$$

We also note that

$$U^{(n)}(\mathfrak{W}^\pm \cap \mathfrak{R}^\pm)\,,\; Z^{(n)}(\mathfrak{W}^\pm \cap \mathfrak{R}^\pm)\,,\; \tilde Z^{(n)}(\mathfrak{W}^\pm \cap \mathfrak{R}^\pm) \;\subseteq\; \mathfrak{W}^\mp \cap \mathfrak{R}^\mp\,. \tag{158}$$

Indeed by (149) and Lemma 6.2 we get $\mu_{-k-i} = -\mu_{ki}$ and $\rho U_k\rho = -U_{-k}$, $\rho Z_k\rho = -Z_{-k}$, etc.

We conclude this section with some estimates on $\tilde\mu_{ki}$, δ_k and the operators $Z^{(n)}, \tilde Z^{(n)}$.

Lemma 6.4. *Let $\varepsilon \in \Delta^\tau_n(w_*)$, with $\|w_*\|_{\bar a,\bar s,\alpha,\sigma} \leq r_0$, and $0 \leq k \leq 4^n$. Then*

$$|\tilde\mu_{ki_0}(w_*;\varepsilon)| \geq \begin{cases} \mathrm{ct}\,\dfrac{\gamma}{\varepsilon^{\tau-1}k^\tau}, & \textit{if } k\varepsilon > N, \\ \mathrm{ct}\,, & \textit{if } k\varepsilon \leq N; \end{cases} \tag{159}$$

$$|\tilde\mu_{k(N+1)}(w_*;\varepsilon)| \geq \begin{cases} \mathrm{ct}\,k\varepsilon, & \textit{if } k\varepsilon \geq N+\mu+2, \\ \mathrm{ct}\,\gamma\varepsilon, & \textit{if } 1/2 < k\varepsilon < N+\mu+2, \\ \mathrm{ct}\,, & \textit{if } k\varepsilon \leq N; \end{cases} \tag{160}$$

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[23] We drop the dependence on s in the notation of $|\mathcal{D}_k|^p$ since these operators have the same entries in any $\mathcal{L}^{p+s,s}$.

$$\delta_k(w_*;\varepsilon) \le \begin{cases} \mathrm{ct}\,\gamma^{-1/2}\varepsilon^{(\tau-2)/2}k^{(\tau-1)/2}\,, & \textit{if} \quad k\varepsilon > N \\ \\ \mathrm{ct}\ , & \textit{if} \quad k\varepsilon \le N\,; \end{cases} \tag{161}$$

and

$$\|Z^{(n)}(w_*;\varepsilon)\|_{\mathcal{L}(W^{(n)}_{-1/2},W^{(n)})}\,,\ \|\tilde Z^{(n)}(w_*;\varepsilon)\|_{\mathcal{L}(W^{(n)}_{-1},W^{(n)}_{-1/2})} \le \mathrm{ct}\,\gamma^{-1/2}\varepsilon^{(\tau-2)/2}4^{n(\tau-1)/2}\,. \tag{162}$$

PROOF. If $k\varepsilon \le N$ then $|\tilde\mu_{ki}| \ge i - N \ge 1$ by (148). If $k\varepsilon > N$ we get $i_0 \le k\varepsilon + 1 \le 2k\varepsilon$ by (153). By the first Melnikov condition introduced in Definition 5.3, the estimate (147) and $\tau < 2$, we have

$$|\tilde\mu_{ki_0}| \quad = \quad \left|\varepsilon k - \tilde\Omega_{i_0} - \varepsilon^2\langle \mathcal{M}_k e_{i_0}, e_{i_0}\rangle + O\left(\frac{\varepsilon^4}{i_0^2}\right)\right| \ge \frac{\gamma_n\varepsilon}{i_0^\tau} \ge \mathrm{ct}\,\frac{\gamma}{\varepsilon^{\tau-1}k^\tau}\,, \tag{163}$$

where γ_n has been defined in (101). (159) follows.

Let us prove (160). Let us assume $k\varepsilon \ge N + \mu + 2$ since the other cases directly follows by (159). Then, by (148) $|\mu_{k(N+1)}| \ge k\varepsilon - \lambda_{k(N+1)} \ge k\varepsilon - N - \mu - 1 \ge \mathrm{ct}\,k\varepsilon$. Therefore (160) follows.

We now prove (161). The case with $k\varepsilon \le N$ directly follows by (159) and (160). Let $k\varepsilon > N$. From (155), we have to show that

$$|\tilde\mu_{k(N+1)}| \ge \mathrm{ct}\,\gamma\varepsilon^{2-\tau}k^{1-\tau} \tag{164}$$

and

$$|i_0 - 1||\tilde\mu_{ki_0}| \ge \mathrm{ct}\,\gamma\varepsilon^{2-\tau}k^{1-\tau}\,. \tag{165}$$

(164) directly follows from (160). By (153) we have that $i_0 \approx k\varepsilon$, therefore by (159) $|i_0 - 1||\tilde\mu_{ki_0}| \ge \mathrm{ct}\,k\varepsilon\gamma\varepsilon^{1-\tau}k^{-\tau}$. (161) follows.

We have

$$\begin{aligned} \|Z^{(n)}\|^2_{W^{(n)}_{-1/2},W^{(n)}} \quad &= \quad \sup_{\|w\|_{W^{(n)}_{-1/2}}=1} \|Z^{(n)}w\|^2_{W^{(n)}} \\ &= \quad \sup_{\|w\|_{W^{(n)}_{-1/2}}=1} \sum_{|k|\le 4^n} e^{2\alpha_n|k|}|k|_*^{2\bar\sigma}|Z_k w_k|^2 \le \left(\mathrm{ct}\max_{|k|\le 4^n}\delta_{|k|}\right)^2 \end{aligned}$$

since $|Z_k w_k| \le \|V_k\|_{0,0}\|\||\mathcal{D}|^{-1/2}\|_{-1/2,0}|w_k|_{-1/2} \le \mathrm{ct}\,\delta_{|k|}|w_k|_{-1/2}$. An analogous estimate holds for $\|\tilde Z^{(n)}\|_{W^{(n)}_{-1},W^{(n)}_{-1/2}}$. (162) follows by (161). ☐

7 Off-diagonal term

In this section we deal with the operator $T^{(n)}$. We point out that $T^{(n)}$ is not a Töplitz operator. Define

$$\tilde T^{(n)} = \tilde T^{(n)}(w_*;\varepsilon) := \tilde Z^{(n)}T^{(n)}Z^{(n)} \tag{166}$$

namely

$$\big(\tilde T^{(n)}\big)_{k\ell} := |\mathcal{D}_k|^{-1/2}V_k^{-1}\Lambda_{k\ell}V_\ell\,|\mathcal{D}_\ell|^{-1/2}\,, \quad \text{for} \quad k \ne \ell \quad \text{and} \quad \big(\tilde T^{(n)}\big)_{kk} = 0\,. \tag{167}$$

Then we define

$$\Xi^{(n)} = \Xi^{(n)}(w_*;\varepsilon) := U^{(n)} + \varepsilon^2\tilde T^{(n)} : W^{(n)}_{-1/2} \longrightarrow W^{(n)}_{-1/2}\,.$$

We want to prove that, if ε is small, $\Xi^{(n)}$ is invertible for every n. We need the following *crucial* lemma on small divisors, that will be proved in Section 8.

Lemma 7.1 (Small divisors)**.** *Let* $1 < \tau < 2$. *Let* $\varepsilon \in \Delta_n^\tau(w_*)$, *with* $w_* \in \mathfrak{W}^+ \cap \mathfrak{R}^+$, $\|w_*\|_{\bar a,\bar s,\alpha,\sigma} \le r_0$, *and* $|k|, |\ell| \le 4^n$, $\ell \ne k$. *Then*

$$\big\||\mathcal{D}_k(w_*;\varepsilon)|^{-1/2}\big\|_{0,-1/2}\big\||\mathcal{D}_\ell(w_*;\varepsilon)|^{-1/2}\big\|_{-1/2,-1} \le \mathrm{ct}\,\gamma^{-1}\varepsilon^{-1}\,|\ell - k|^2$$

with ct > 0 *continuously depending on* α, σ.

Using Lemma 7.1 we give the following

Lemma 7.2. *Suppose that* $w_* \in W^{(n)} \cap \mathfrak{W}^+ \cap \mathfrak{R}^+$ *satisfies* $\|w_*\|_{\bar a, \bar s - 1, \alpha_n, \bar\sigma + 2} \le r_0$ *and* $\varepsilon \in \Delta_n^\tau(w_*)$. *Then we have* $\|\tilde T^{(n)}(w_*;\varepsilon)\|_{\mathcal{L}(W^{(n)}_{-1/2}, W^{(n)}_{-1/2})} \le C\,\gamma^{-1}\varepsilon^{-1}$ *with constant independent of* n.

PROOF. For $w \in W^{(n)}_{-1/2}$, we have, from (167) and Lemma 7.1,

$$\begin{aligned}
\big|\big(\tilde T^{(n)} w\big)_k\big|_{-1/2} &= \Big| \sum_{|\ell| \le 4^n, \ell \ne k} |\mathcal{D}_k|^{-1/2} V_k^{-1} \Lambda_{k\ell}\, V_\ell |\mathcal{D}_\ell\,|^{-1/2} w_\ell \Big|_{-1/2} \\
&\le \sum_{|\ell| \le 4^n, \ell \ne k} \big\||\mathcal{D}_k|^{-1/2}\big\|_{0,-1/2} \|V_k^{-1}\|_{0,0} \big\|\Lambda_{k\ell}\big\|_{-1,0} \|V_k\|_{-1,-1} \big\||\mathcal{D}_\ell|^{-1/2}\big\|_{-1/2,-1} |w_\ell|_{-1/2} \\
&\le \ \mathrm{ct} \sum_{|\ell| \le 4^n, \ell \ne k} \big\||\mathcal{D}_k|^{-1/2}\big\|_{0,-1/2} \big\||\mathcal{D}_\ell|^{-1/2}\big\|_{-1/2,-1} \big\|\Lambda_{k\ell}\big\|_{-1,0} |w_\ell|_{-1/2} \\
&\le \ \mathrm{ct} \sum_{|\ell| \le 4^n, \ell \ne k} \gamma^{-1}\varepsilon^{-1} |\ell - k|^2 \|\Lambda_{k\ell}\|_{-1,0} |w_\ell|_{-1/2}\,.
\end{aligned} \tag{168}$$

Hence, by an elementary inequality (see, e.g., Lemma A.2 in the Appendix with $m := 1$) and (168),

$$\begin{aligned}
\big\|\tilde T^{(n)} w\big\|^2_{W^{(n)}_{-1/2}} &= \sum_{|k| \le 4^n} e^{2k\alpha_n} k^{2\bar\sigma} \big|(\tilde T^{(n)} w)_k\big|^2_{-1/2} \\
&\le \frac{1}{\gamma^2\varepsilon^2} \sum_{|k| \le 4^n} e^{2k\alpha_n} k^{2\bar\sigma} \Big(\sum_{|\ell| \le 4^n, \ell \ne k} |\ell - k|^2 \|\Lambda_{k\ell}\|_{-1,0} |w_\ell|_{-1/2} \Big)^2 \\
&\le \mathrm{ct} \frac{1}{\gamma^2\varepsilon^2} \sum_{|k| \le 4^n} e^{2k\alpha_n} k^{2\bar\sigma} \sum_{|\ell| \le 4^n, \ell \ne k} \frac{|\ell - k|^{2\bar\sigma} \ell^{2\bar\sigma}}{k^{2\bar\sigma}} |k - \ell|^4 \|\Lambda_{k\ell}\|^2_{-1,0} |w_\ell|^2_{-1/2} \\
&\le \mathrm{ct} \frac{1}{\gamma^2\varepsilon^2} \sum_{|\ell| \le 4^n} e^{2\ell\alpha_n} \ell^{2\bar\sigma} |w_\ell|^2_{-1/2} \sum_{|k| \le 4^n, k \ne \ell} |\ell - k|^{2(\bar\sigma + 2)} e^{2|\ell - k|\alpha_n} \|\Lambda_{k\ell}\|^2_{-1,0} \\
&\le \ \mathrm{ct}\,\gamma^{-2}\varepsilon^{-2} \|w\|^2_{W^{(n)}_{-1/2}} \|\Lambda\|^2_{\mathcal{L}^{\alpha_n, \bar\sigma + 2}(\ell_*^{\bar a, \bar s - 1}, \ell_*^{\bar a, \bar s})} \le C^2 \gamma^{-2}\varepsilon^{-2} \|w\|^2_{W^{(n)}_{-1/2}}
\end{aligned}$$

using $k \le \ell + |k - \ell|$, Proposition 4.2 with $a := \bar a$, $s := \bar s - 1$, $\alpha := \alpha_n$, $\sigma := \bar\sigma + 2$ and taking the supremum of all the involved constants for $n \ge 1$ (recall that all the constants continuously depend on $\alpha = \alpha_n$.) ☐

PROOF OF LEMMA 5.4 By Lemma 7.2 and the invertibility of $U^{(n)}$, $\Xi^{(n)} : W^{(n)}_{-1/2} \longrightarrow W^{(n)}_{-1/2}$ is invertible for ε small enough (but independent of n) and $\|(\Xi^{(n)})^{-1}\|_{\mathcal{L}(W^{(n)}_{-1/2}, W^{(n)}_{-1/2})} \le \mathrm{ct}$. So we can define

$$\mathcal{G}^{(n)} := Z^{(n)} (\Xi^{(n)})^{-1} \tilde Z^{(n)} : W^{(n)}_{-1} \longrightarrow W^{(n)}\,. \tag{169}$$

Estimate (102) follows by (162). Recalling that $\mathcal{L}^{(n)} = D^{(n)} + \varepsilon^2 T^{(n)} : W^{(n)} \longrightarrow W^{(n)}_{-1}$, we get, by Lemma 6.3

$$\begin{aligned}
\mathcal{L}^{(n)} \mathcal{G}^{(n)} &= \Big(D^{(n)} + \varepsilon^2 T^{(n)}\Big)\Big(Z^{(n)} (\Xi^{(n)})^{-1} \tilde Z^{(n)}\Big) \\
&= D^{(n)} Z^{(n)} (\Xi^{(n)})^{-1} \tilde Z^{(n)} + \varepsilon^2 T^{(n)} Z^{(n)} (\Xi^{(n)})^{-1} \tilde Z^{(n)} \\
&= (\tilde Z^{(n)})^{-1} U^{(n)} (\Xi^{(n)})^{-1} \tilde Z^{(n)} + \varepsilon^2 (\tilde Z^{(n)})^{-1} \tilde Z^{(n)} T^{(n)} Z^{(n)} (\Xi^{(n)})^{-1} \tilde Z^{(n)} \\
&= (\tilde Z^{(n)})^{-1} \big(U^{(n)} + \varepsilon^2 \tilde Z^{(n)} T^{(n)} Z^{(n)}\big) (\Xi^{(n)})^{-1} \tilde Z^{(n)} \\
&= (\tilde Z^{(n)})^{-1} \Xi^{(n)} (\Xi^{(n)})^{-1} \tilde Z^{(n)} = (\tilde Z^{(n)})^{-1} \tilde Z^{(n)} = \mathbb{I}_{W^{(n)}_{-1}}\,.
\end{aligned}$$

By (169) and (158), it remains to prove that $(\Xi^{(n)})^{-1}(\mathfrak{W}^- \cap \mathfrak{R}^-) \subseteq \mathfrak{W}^+ \cap \mathfrak{R}^+$. Since $(\Xi^{(n)})^{-1} = \sum_{m \ge 0} (-1)^m \varepsilon^{2m} \big(U^{(n)} \tilde Z^{(n)} T^{(n)} Z^{(n)}\big)^m U^{(n)}$ we conclude by (100) and (158). ☐

8 Small divisors

The aim of this section is the proof of Lemma 7.1. In particular we will prove that

$$\delta_{k\ell} := \delta_k \delta_\ell \le \mathrm{ct}\, \gamma^{-1} \varepsilon^{-1} |\ell - k|^2 \,. \tag{170}$$

Then Lemma 7.1 follows by (155) and (170).

All the integers of this section are *positive*. Note that

$$\ell, k \in \mathbb{N}^+ \,,\ \ell \ne k \quad \Longrightarrow \quad \frac{\ell}{k} \le 2|\ell - k| \quad \text{and} \quad \frac{k}{\ell} \le 2|\ell - k| \,. \tag{171}$$

since

$$\frac{\ell}{k} \le \frac{|\ell - k| + k}{k} \le 1 + |\ell - k| \le 2|\ell - k| \,,$$

Let $c_1 := N + \mu + 2$. In the proof of (170) we distinguish four different cases.

FIRST CASE: $k\varepsilon \le c_1$ or $\ell\varepsilon \le c_1$.
Let $k\varepsilon \le c_1$. We have $\delta_k \le \mathrm{ct}\, \gamma^{-1/2} \varepsilon^{-1/2}$ by (161). If $\ell\varepsilon \le 2c_1$ (161) implies that $\delta_\ell \le \mathrm{ct}\, \gamma^{-1/2} \varepsilon^{-1/2}$ and $\delta_{k\ell} \le \mathrm{ct}\, \gamma^{-1} \varepsilon^{-1}$. If $\ell\varepsilon > 2c_1$, since $k\varepsilon \le c_1 \le \ell\varepsilon/2$ implies $\ell \le 2|\ell - k|$, then $\delta_{k\ell} \le \mathrm{ct}\, \gamma^{-1} \varepsilon^{(\tau-3)/2} \ell^{(\tau-1)/2}$, again by (161). The case $\ell\varepsilon \le c_1$ is analogous.

From now on we consider $k\varepsilon > c_1$ and $\ell\varepsilon > c_1$.

SECOND CASE: $|\ell - k| \ge \min\{\ell^\beta, k^\beta\}$, where $0 \le \beta \le 1$ will be chosen later in (182).
We claim that

$$2|\ell - k| \ge \max\{\ell^\beta, k^\beta\} \,. \tag{172}$$

To fix ideas consider $\ell > k$. By hypothesis $\ell - k \ge k^\beta$. Let us suppose by contradiction that $2(\ell - k) < \ell^\beta$, then $2k > 2\ell - \ell^\beta \ge \ell$ and $2(\ell - k) \ge 2k^\beta \ge \ell^\beta$, proving (172). Then by (161) and (172) we get

$$\delta_{k\ell} \le \mathrm{ct}\, \gamma^{-1} \varepsilon^{\tau-2} k^{(\tau-1)/2} \ell^{(\tau-1)/2} \le \mathrm{ct}\, \gamma^{-1} \varepsilon^{\tau-2} |\ell - k|^{(\tau-1)/\beta} \,. \tag{173}$$

Let $j_0 = j_0(\ell)$ be the smallest integer such that $\min_j |\tilde{\mu}_{\ell j}| = |\tilde{\mu}_{\ell j_0}|$.

THIRD CASE: $0 < |j_0 - i_0|/(4\varepsilon) < |\ell - k| < \min\{\ell^\beta, k^\beta\}$.
From the Melnikov condition[24]

$$|\ell\varepsilon - j_0 - k\varepsilon + i_0| = \big| |\ell - k| - |j_0 - i_0| \big| \ge \frac{\varepsilon\gamma}{|j_0 - i_0|^\tau} \ge \mathrm{ct}\, \frac{\gamma \varepsilon^{1-\tau}}{|\ell - k|^\tau} \,.$$

We have two cases:

$$|k\varepsilon - i_0| \ge \mathrm{ct}\, \frac{\gamma \varepsilon^{1-\tau}}{|\ell - k|^\tau} \ge \mathrm{ct}\, \frac{\gamma \varepsilon^{1-\tau}}{k^{\tau\beta}} \tag{174}$$

or

$$|\ell\varepsilon - j_0| \ge \mathrm{ct}\, \frac{\gamma \varepsilon^{1-\tau}}{|\ell - k|^\tau} \ge \mathrm{ct}\, \frac{\gamma \varepsilon^{1-\tau}}{\ell^{\tau\beta}} \,.$$

We will consider only the case (174), the other one being analogous. Then we claim that

$$\beta < (\tau - 1)/\tau \quad \Longrightarrow \quad \frac{1}{2}\, |k\varepsilon - i_0| \ge \frac{\mu}{i_0} \tag{175}$$

(if ε is small enough). In particular we show that for every $C > 0$, if ε is small enough, $\varepsilon^{\tau-1} k^{\tau\beta} \le i_0/(C\mu)$. Indeed $\tau\beta < 1$ and by (153) we get $\varepsilon^{\tau-1} k^{\tau\beta} \le \mathrm{ct}\, \varepsilon^{\tau-1-\tau\beta} i_0^{\tau\beta} \le i_0/(C\mu)$. By (147) and since $\sqrt{i_0^2 + \mu} - i_0 < \mu/(2i_0)$ we get (for ε small)

$$|\tilde{\mu}_{k i_0}| \ge |\varepsilon k - i_0| - \frac{\mu}{i_0} \ge \frac{1}{2} |\varepsilon k - i_0| \ge \mathrm{ct}\, \frac{\gamma \varepsilon^{1-\tau}}{|\ell - k|^\tau} \,, \tag{176}$$

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[24] Note that $\ell > k$ implies $j_0 \ge i_0$.

by (175). By (160), (155), (171) and $\tau \le 2$ we have that

$$|\tilde{\mu}_{k(N+1)}| \le |i_0 - 1|_* |\tilde{\mu}_{ki_0}| \quad \Longrightarrow \quad \delta_{k\ell} \le \mathrm{ct}\, \gamma^{-1/2} \varepsilon^{(\tau-3)/2} \ell^{(\tau-1)/2} k^{-1/2} \le \mathrm{ct}\, \gamma^{-1/2} \varepsilon^{(\tau-3)/2} |\ell - k|^{(\tau-1)/2} . \tag{177}$$

On the other hand, if $|\tilde{\mu}_{k(N+1)}| > |i_0 - 1||\tilde{\mu}_{ki_0}|$, then, by (153), (155) and (176), $\delta_k \le \mathrm{ct}\, \gamma^{-1/2} \varepsilon^{(\tau-2)/2} |\ell - k|^{\tau/2} k^{-1/2}$ so that

$$\begin{aligned} \delta_{k\ell} &\le \mathrm{ct}\, \gamma^{-1} \varepsilon^{\tau-2} \ell^{(\tau-1)/2} |\ell - k|^{\tau/2} k^{-1/2} \le \mathrm{ct}\, \gamma^{-1} \varepsilon^{\tau-2} |\ell - k|^{\tau-1/2} k^{\tau/2-1} \\ &\le \mathrm{ct}\, \gamma^{-1} \varepsilon^{\tau-2} |\ell - k|^{\tau-1/2-(2-\tau)/2\beta} , \end{aligned} \tag{178}$$

by (171) and $k > |\ell - k|^{1/\beta}$.

FOURTH CASE: $0 < |\ell - k| \le |j_0 - i_0|/(4\varepsilon)$ or $i_0 = j_0$.
If $0 < |\ell - k| \le |j_0 - i_0|/(4\varepsilon)$

$$|\tilde{\mu}_{\ell j_0} - \tilde{\mu}_{ki_0}| = |\ell\varepsilon - \lambda_{\ell j_0} - k\varepsilon + \lambda_{ki_0}| \ge |\lambda_{ki_0} - \lambda_{\ell j_0}| - |\ell - k|\varepsilon \ge \frac{1}{4}|j_0 - i_0| \ge \frac{1}{4} , \tag{179}$$

while if $i_0 = j_0$, by (147) we have

$$|\tilde{\mu}_{\ell j_0} - \tilde{\mu}_{ki_0}| = |\ell\varepsilon - \lambda_{\ell j_0} - k\varepsilon + \lambda_{ki_0}| \ge |\ell - k|\varepsilon - \mathrm{ct}\, \frac{\varepsilon^2}{i_0} \ge \frac{|\ell - k|\varepsilon}{2} \ge \frac{\varepsilon}{2} . \tag{180}$$

We note that (179) is a particular case of (180). By (180) we have two cases:

$$\text{(a)} \quad |\tilde{\mu}_{k,i_0}| \ge \frac{\varepsilon}{4} \qquad \text{or} \qquad \text{(b)} \quad |\tilde{\mu}_{\ell,j_0}| \ge \frac{\varepsilon}{4} .$$

If (a) holds then, recalling (177), we have to consider only the case when $|i_0 - 1||\tilde{\mu}_{ki_0}| < |\tilde{\mu}_{k(N+1)}|$. By (155) and (153) $\delta_k \le \mathrm{ct}\, \varepsilon^{-1} k^{-1/2}$. By (161), (171) and $k\varepsilon > c_1$,

$$\delta_{k\ell} \le \mathrm{ct}\, \gamma^{-1/2} \varepsilon^{(\tau-4)/2} \ell^{(\tau-1)/2} k^{-1/2} \le \mathrm{ct}\, \gamma^{-1/2} \varepsilon^{-1} |\ell - k|^{(\tau-1)/2} . \tag{181}$$

The case in which (b) holds is analogous.

Finally (170) follows by (173), (178) and (181), choosing

$$\beta := (\tau - 1)/2 . \tag{182}$$

□

9 Measure estimates

In this section we will give an estimate from below of the measure of the set $\mathcal{C}$ of admissible frequencies. We first need the following standard result on the measure of diophantine numbers.

Lemma 9.1. *Let*

$$D := \left\{ 0 < \varepsilon \le 1/2 , \ \text{s.t.} \ \left| \varepsilon\ell - j \right| > \frac{\varepsilon^{1+\delta}}{j^\tau} , \forall\, \ell, j \ge 1 \right\} , \qquad 1 < \tau < 2 , \tag{183}$$

then

$$\lim_{\varepsilon_* \to 0^+} \frac{\mathrm{meas}\left((0, \varepsilon_*] \cap D \right)}{\varepsilon_*} = 1 . \tag{184}$$

PROOF. Let $f_{\ell j}(\varepsilon) := \varepsilon^{-\delta}\ell - \varepsilon^{-1-\delta} j$ and $D_{\ell j} := \{0 < \varepsilon \le 1/2\,,\ \text{s.t.}\ \ |f_{\ell j}(\varepsilon)| \le j^{-\tau} =: d_j\}$. Note that $D_{\ell j} \subset [\frac{j}{2\ell}, \frac{3j}{2\ell}]$. For $\varepsilon \in D_{\ell j}$ we have

$$f'_{\ell j}(\varepsilon) \ge \frac{j}{2\varepsilon^{2+\delta}} \ge \text{ct}\,\frac{\ell^{2+\delta}}{j^{1+\delta}}$$

and

$$\text{meas}(D_{\ell j}) \le \left[\min_{\varepsilon \in [\frac{j}{2\ell}, \frac{3j}{2\ell}]} |f'_{\ell j}(\varepsilon)|\right]^{-1} 2d_j \le \frac{\text{ct}}{\ell^{2+\delta} j^{\tau-1-\delta}}\,. \tag{185}$$

Being, by definition of $D_{\ell j}$, $D_{\ell j} \cap (0, \varepsilon_*] = \emptyset$ if $\ell < j/2\varepsilon_*$, one has $(0,\varepsilon_*) \cap D^c = \cup_{j\ge 1} \cup_{\ell \ge j/2\varepsilon_*} D_{\ell j}$. Therefore by (185),

$$\text{meas}\big((0,\varepsilon_*) \cap D^c\big) \le \text{ct} \int_{1/2}^{\infty} \int_{j/4\varepsilon_*}^{\infty} \frac{1}{j^{\tau-1-\delta}\ell^{2+\delta}}\, d\ell\, dj = \text{ct}\,{\varepsilon_*}^{1+\delta}$$

□

Proposition 9.2. *Let* A_n *as in* (104) *and* $\mathcal{C}$ *as in (125). Then*

$$\lim_{\varepsilon_* \to 0^+} \frac{\text{meas}\,\big(\mathcal{C} \cap (0,\varepsilon_*)\big)}{\varepsilon_*} = 1\,. \tag{186}$$

PROOF. Let $f^n_{ki}(\varepsilon) := \varepsilon^{-\delta}k - \varepsilon^{-1-\delta}\tilde{\Omega}_i(\varepsilon) - \varepsilon^{1-\delta}\big(\mathcal{M}_k(w^{(n)}(\varepsilon);\varepsilon)\big)_{ii}$. For any $0 < \varepsilon_* < \varepsilon_0$ fixed let us define $\tilde{A}_n := (0,\varepsilon_*)$ and

$$\tilde{A}_n := \{\varepsilon \in \tilde{A}_{n-1}\,,\ \ \text{s.t.}\ \ |f^{n-1}_{ki}(\varepsilon)| > \tilde{\gamma}_n i^{-\tau}\,,\ \ \forall\, i > N\,,\ k \le 4^n\}$$

where $\tilde{\gamma}_n := 1 + 1/2^{n+1}$. Let also

$$E_n := \tilde{A}_n \setminus \tilde{A}_{n+1} = \{\varepsilon \in \tilde{A}_n\,,\ \ \text{s.t.}\ \ \exists\, i > N\,,\ k \le 4^{n+1}\,,\ \ \text{with}\ \ |f^n_{ki}(\varepsilon)| \le \tilde{\gamma}_{n+1} i^{-\tau}\}\,. \tag{187}$$

Recalling Definition 5.3, (101), (104), (183) we have $\mathcal{C} \cap (0,\varepsilon_*) \supseteq (\cap_n \tilde{A}_n) \cap D = (\cup_n E_n)^c \cap D$. By Lemma 9.1, we get that to prove (186) it is enough to show that

$${\varepsilon_*}^{-1}\text{meas}\,(\cup_n E_n) = {\varepsilon_*}^{-1} \sum_n \text{meas}\, E_n \longrightarrow 0\,, \qquad \text{when}\ \ \varepsilon_* \to 0\,. \tag{188}$$

Let us define also $F^n_{ki} := \{\varepsilon > 0\,,\ \ \text{s.t.}\ \ |f^n_{ki}(\varepsilon)| \le \tilde{\gamma}_{n+1} i^{-\tau}\}$. We note that

$$E_n \subseteq \cup_{k \le 4^{n+1}} \cup_{i \ge 1} \big(F^n_{ki} \cap (0,\varepsilon_*)\big)\,. \tag{189}$$

We claim that, for ε_* small,

$$F^n_{ki} \subset \left(\frac{i}{2k}, \frac{2i+\mu}{k}\right)\,. \tag{190}$$

Indeed $i < \omega_i < i + \mu$ (recall $\omega_i = \sqrt{i^2+\mu}$) and for $\varepsilon \le i/2k$ or $\varepsilon \ge (2i+\mu)/k$, $|f^n_{ki}(\varepsilon)| \ge \text{ct}\, i\varepsilon^{-1-\delta}$. By (190) it follows that

$$F^n_{ki} \cap (0,\varepsilon_*) = \emptyset \qquad \text{if}\ \ i \ge 2\varepsilon_* k \tag{191}$$

and

$$E_n = \emptyset \qquad \text{if}\ \ 2\varepsilon 4^{n+1} < 1\,,\ \ \text{namely}\ \ n < n_0(\varepsilon_*) := \frac{1}{\ln 4}\ln\frac{1}{2\varepsilon_*} - 1\,. \tag{192}$$

We claim that

$$\varepsilon \in \tilde{A}_n\,,\ \ k \le 4^n \qquad \Longrightarrow \qquad |f^n_{ki}(\varepsilon)| > \tilde{\gamma}_{n+1} i^{-\tau}\,, \qquad \forall\, i > N\,. \tag{193}$$

Indeed if $i \geq 2\varepsilon_* 4^n$ then $|f^n_{ki}(\varepsilon)| \geq \mathrm{ct}\, i\varepsilon^{-1-\delta}$; then let us consider $i < 2\varepsilon_* 4^n$. Since $\mathcal{M}_k = \Lambda_{kk}$ and $\Lambda = \varepsilon^{-1}(D\mathcal{N})$, by Proposition 2.11, (112) and (106) we get[25]

$$\begin{aligned} \|\mathcal{M}_k(w^{(n)};\varepsilon) - \mathcal{M}_k(w^{(n-1)};\varepsilon)\|_{\mathcal{L}(\ell_*^{\bar a,\bar s},\ell_*^{\bar a,\bar s+1})} &\leq \|\Lambda(w^{(n)};\varepsilon) - \Lambda(w^{(n-1)};\varepsilon)\|_{\mathcal{L}^{\alpha,\sigma}(\ell_*^{\bar a,\bar s},\ell_*^{\bar a,\bar s+1})} \\ &\leq \mathrm{ct}\,\varepsilon|h^{(n)}|_n \leq \mathrm{ct}\,\varepsilon^{\tau-\delta} e^{-\chi^n \alpha_0/8}\,. \end{aligned} \tag{194}$$

Moreover

$$\begin{aligned} |f^n_{ki}(\varepsilon) - f^{n-1}_{ki}(\varepsilon)| &= \varepsilon^{1-\delta}\big|\big(\mathcal{M}_k(w^{(n)};\varepsilon) - \mathcal{M}_k(w^{(n-1)};\varepsilon)\big)_{ii}\big| \\ &\overset{(194)}{\leq} \mathrm{ct}\,{\varepsilon_*}^{\tau+1-2\delta} e^{-\chi^n\alpha_0/8} \leq 2^{-n-2}(2\varepsilon_* 4^n)^{-\tau} \leq (\tilde\gamma_n - \tilde\gamma_{n+1}) i^{-\tau}\,, \end{aligned}$$

where we have used that $\mathrm{ct}\,{\varepsilon_*}^{2\tau+1-2\delta} 2^{n+2+2\tau} 4^{n\tau} e^{-\chi^n\alpha_0/8} \leq 1$ $\forall\, n$ (for ε small enough), and in the last inequality we have used that $\tilde\gamma_n - \tilde\gamma_{n+1} = 2^{-n-2}$ and $i < 2\varepsilon_* 4^n$. Since $\varepsilon \in \tilde A_n$, we get $|f^n_{ki}(\varepsilon)| \geq |f^{n-1}_{ki}(\varepsilon)| - |f^n_{ki}(\varepsilon) - f^{n-1}_{ki}(\varepsilon)| \geq \tilde\gamma_n i^{-\tau} - (\tilde\gamma_n - \tilde\gamma_{n+1}) i^{-\tau}$, proving (193).
By (191) and (193) we can rewrite (189) into

$$E_n \subseteq \cup_{4^n < k \leq 4^{n+1}} \cup_{i < 2\varepsilon_* k} \left(F^n_{ki} \cap (0,\varepsilon_*)\right). \tag{195}$$

Let us evaluate

$$\begin{aligned} \big(f^n_{ki}(\varepsilon)\big)' &= \varepsilon^{-2-\delta}\big((1+\delta)\omega_i - (BA^{-1}\omega)_i\big) + \varepsilon^{-1-\delta}\big((1+\delta)(BI_0)_i - \delta k\big) \\ &\quad -\varepsilon^{-\delta}(1-\delta)\big(\mathcal{M}_k(w^{(n)};\varepsilon)\big)_{ii} - \varepsilon^{1-\delta}\big(D\mathcal{M}_k(w^{(n)};\varepsilon)[(w^{(n)})'] + \partial_\varepsilon \mathcal{M}_k(w^{(n)};\varepsilon)\big)_{ii}\,. \end{aligned} \tag{196}$$

Noting that $D\mathcal{M}_k = (D\Lambda)_{kk}$, $\partial_\varepsilon \mathcal{M}_k = (\partial_\varepsilon \Lambda)_{kk}$ and $\Lambda = \varepsilon^{-1} D\mathcal{N}$, by Proposition 2.11, (112), (119) and (113), the dominant term (in ε) in (196) is the first one. Since $\omega_i \to \infty$ and $(BA^{-1}\omega)_i \to 0$ when $i \to \infty$, there exists $i_* \geq 1$ such that $\omega_i - (BA^{-1}\omega)_i \geq i/2$, $\forall\, i \geq i_*$ and we can chose $0 < \delta < \tau - 1$ such that

$$|(1+\delta)\omega_i - (BA^{-1}\omega)_i| > 0\,, \qquad \forall\, i < i_*\,. \tag{197}$$

By the above considerations, for $i \leq 2\varepsilon_* k$ and ε small, we get

$$\min_{\varepsilon \in (\frac{i}{2k},\frac{2i}{k})} |(f^n_{ki}(\varepsilon))'| \geq \begin{cases} \mathrm{ct}\, k^{2+\delta} i^{-1-\delta}\,, & \text{if} \quad i \geq i_* \\ \\ \mathrm{ct}\, k^{2+\delta} i^{-2-\delta}\,, & \text{if} \quad i < i_*\,. \end{cases} \tag{198}$$

and, therefore,

$$\mathrm{meas}\, F^n_{ki} \leq \begin{cases} \mathrm{ct}\, k^{-2-\delta} i^{1+\delta-\tau}\,, & \text{if} \quad i \geq i_* \\ \\ \mathrm{ct}\, k^{-2-\delta} i^{2+\delta-\tau}\,, & \text{if} \quad i < i_*\,. \end{cases} \tag{199}$$

By (199)

$$\sum_{i < 2\varepsilon_* k} \mathrm{meas}\,(F^n_{ki} \cap (0,\varepsilon_*)) = \sum_{i \leq i_*} + \sum_{i_* \leq i < 2\varepsilon_* k} \leq \frac{\mathrm{ct}}{k^{2+\delta}} + \frac{\mathrm{ct}\,{\varepsilon_*}^{2+\delta-\tau}}{k^\tau}\,.$$

Recalling (195)

$$\mathrm{meas}\, E_n \leq \sum_{k > 4^n} \sum_{i < 2\varepsilon_* k} \mathrm{meas}\,(F^n_{ki} \cap (0,\varepsilon_*)) \leq \frac{\mathrm{ct}}{4^{n(1+\delta)}} + \frac{\mathrm{ct}\,{\varepsilon_*}^{2+\delta-\tau}}{4^{n(\tau-1)}}\,.$$

Finally

$$\begin{aligned} \sum_{n \geq n_0(\varepsilon_*)} \mathrm{meas}\, E_n &\leq \sum_{n \geq n_0(\varepsilon_*)} \left(\frac{\mathrm{ct}}{4^{n(1+\delta)}} + \frac{\mathrm{ct}\,{\varepsilon_*}^{2+\delta-\tau}}{4^{n(\tau-1)}}\right) \\ &\leq \mathrm{ct}\, e^{-n_0(\varepsilon_*)(1+\delta)\ln 4} + \mathrm{ct}\,{\varepsilon_*}^{2+\delta-\tau} e^{-n_0(\varepsilon_*)(\tau-1)\ln 4} \leq \mathrm{ct}\,{\varepsilon_*}^{1+\delta}\,. \end{aligned}$$

By (192) we get (188). □

---

[25] Omitting the explicit dependence on ε of $w^{(n)}$ and $w^{(n+1)}$.

10 Proof of the theorem

By Corollary 5.6 we have that for every $\varepsilon \in \mathcal{C}$ (defined in (125) and satisfying (186)) there exists $w(\varepsilon) = \big(z(\varepsilon), \tilde{z}(\varepsilon)\big) \in \mathfrak{W}^+ \cap \mathfrak{R}^+$ solving (70). By Proposition 3.4 we can define $\mathfrak{u} := \big(J(w(\varepsilon);\varepsilon), \psi(w(\varepsilon);\varepsilon), w(\varepsilon)\big) \in H^{\bar{\alpha},\bar{\sigma}}_{\mathbb{R}^{2N}\times\ell^{\bar{a},\bar{s}}_*} \cap S$ (S defined in (59)) solving (54) with $\phi_0 = 0$. By (9), (11), (28), (29), (49), (53) (with $\phi_0 = 0$) we have a $2\pi/\varepsilon$–periodic solution of (1)

$$\begin{aligned} u(t,x) &= \sum_{i\geq 1} \sqrt{\frac{2}{\pi\omega_i}} q_i(t) \sin ix = \sum_{i\geq 1} \sqrt{\frac{1}{\pi\omega_i}} \big(\mathrm{z}_i(t) + \tilde{\mathrm{z}}_i(t)\big) \sin ix = \ldots \\ &= \sqrt{\varepsilon} \sum_{i\leq N} \sqrt{\frac{2I_{0i}(\varepsilon)}{\pi\omega_i}} \cos\big(\tilde{\omega}_i(\varepsilon)t\big) \sin ix + O(\varepsilon^{3/2}) \end{aligned}$$

We have that $u(t,x)$ is real and even in t since $q_i(t) = (\mathrm{z}_i(t) + \tilde{\mathrm{z}}_i(t))/\sqrt{2}$ are real and even $\forall\, i \geq 1$ since $\tilde{\mathrm{z}}_i(t) = \mathrm{z}_i(-t) = \overline{\mathrm{z}_i(t)}$.

Nothing remains but to prove the estimate on the minimal period. This is the object of the next subsection.

10.1 Minimal period

Lemma 10.1. *Let $\varsigma > 1$. Then there exists $\Upsilon_\varsigma \subset \mathbb{R}^+$, with $\mathrm{meas}\big(\mathbb{R}^+ \setminus \Upsilon_\varsigma\big) = 0$ such that for all $\mu \in \Upsilon_\varsigma$ the vector $\bar{\omega}_\mu := \big(\sqrt{1+\mu}, \sqrt{4+\mu}\,\big)$ is diophantine, namely there exists $c(\mu) > 0$ such that*

$$\big|\bar{\omega}_\mu \cdot h\big| > \frac{c(\mu)}{|h|^\varsigma}, \qquad \forall h \in \mathbb{Z}^2 \setminus \{0\}.$$

PROOF. Fix $n > 0$. Define, for all $\bar{\gamma} > 0$ and $h \in \mathbb{Z}^2 \setminus \{0\}$,

$$A^{\bar{\gamma},n}_h = A^{\bar{\gamma},n}_{h_1,h_2} := \left\{\mu \in (0,n] \text{ s.t. } \big|\bar{\omega}_\mu \cdot h\big| \leq \frac{\bar{\gamma}}{|h|^\varsigma}\right\}$$

and

$$\Upsilon^{\bar{\gamma},n} := \left\{\mu \in (0,n] \text{ s.t. } \big|\bar{\omega}_\mu \cdot h\big| > \frac{\bar{\gamma}}{|h|^\varsigma}, \quad \forall h \in \mathbb{Z}^2 \setminus \{0\}\right\}.$$

We note that if $A^{\bar{\gamma},n}_{h_1,h_2} \neq \emptyset$, then $\tilde{c}|h_1| \leq |h_2| \leq \bar{c}|h_1|$, $\bar{c} < 1$. Moreover

$$\Upsilon^{\bar{\gamma},n} = (0,n] \setminus \bigcup_{h\in\mathbb{Z}^2\setminus\{0\}} A^{\bar{\gamma},n}_h. \tag{200}$$

Let $\Upsilon^n := \cup_{\bar{\gamma}>0} \Upsilon^{\bar{\gamma},n}$; we claim that

$$\mathrm{meas}\Big((0,n] \setminus \Upsilon^n\Big) = 0. \tag{201}$$

Note that $\mathrm{meas}(A^{\bar{\gamma},n}_h) \leq \mathrm{ct}\,\bar{\gamma}|h|^{-\varsigma-1}$. In fact, we can write

$$|h \cdot \bar{\omega}_\mu| = \Big|h_1\sqrt{1+\mu} + h_2\sqrt{4+\mu}\Big| = \frac{|h_1^2(1+\mu) + h_2^2(4+\mu)|}{|h_1|\sqrt{1+\mu} + |h_2|\sqrt{4+\mu}|}.$$

Hence $\mu \in A^{\bar{\gamma},n}_h$ implies that $|f(\mu)| := |h_1^2 - 4h_2^2 + \mu(h_1^2 - h_2^2)| \leq \mathrm{ct}\,\bar{\gamma}|h|^{1-\varsigma}$. Moreover, being $|f'(\mu)| = |h_1^2 - h_2^2| \approx |h_1|^2 \approx |h|^2$, we get $\mathrm{meas}(A^{\bar{\gamma},n}_h) < \mathrm{ct}\,\bar{\gamma}|h|^{1-\varsigma}|h|^{-2} = \mathrm{ct}\,\bar{\gamma}|h|^{-\varsigma-1}$. Therefore we have that

$$\mathrm{meas} \bigcup_{h\in\mathbb{Z}^2\setminus\{0\}} (A^{\bar{\gamma},n}_h) \quad \leq \quad \sum_{h\in\mathbb{Z}^2\setminus\{0\}} \mathrm{meas}(A^{\bar{\gamma},n}_h) \approx \int_1^\infty \int_{\tilde{c}\,h_1}^{\bar{c}\,h_1} \frac{\bar{\gamma}}{(h_1+h_2)^{\varsigma+1}}\, dh_1 dh_2$$

$$= \quad \bar{\gamma} \int_1^\infty \int_{\tilde{c}}^{\bar{c}} \frac{h_1}{(1+w)^{\varsigma+1} h_1^{\varsigma+1}} \, dw dh_1 = \bar{\gamma} \int_1^\infty \frac{1}{h_1^\varsigma} \, dh_1 \int_{\tilde{c}}^{\bar{c}} \frac{1}{(1+w)^{\varsigma+1}} \, dw = \mathrm{ct}\, \bar{\gamma} \, .$$

By (200) $\operatorname{meas}\big((0,n] \setminus \Upsilon^{\bar{\gamma},n}\big) = O(\bar{\gamma})$ and (201) follows. Finally let $\Upsilon_\varsigma := \bigcup_{n>0} \Upsilon^n$. $\square$

Lemma 10.2. *Fix* $\rho < 1/2$. *There exists* $\Upsilon = \Upsilon(\rho) \subset \mathbb{R}^+$, *with* $\operatorname{meas}\big(\mathbb{R}^+ \setminus \Upsilon\big) = 0$ *such that, if* $\mu \in \Upsilon$, $N \geq 2$ *and* $T^{\min}$ *is the minimal period of a* 2π*-periodic orbit of* $\mathcal{H}$ *of the form (53), then* $T^{\min} \geq \mathrm{ct}\, \varepsilon^{1-\rho}$.

PROOF. We know that $\phi(2\pi) = 2\pi\kappa$ (recall that ψ is 2π–periodic and odd) with $\kappa \in \mathbb{Z}^N$ defined in (48). Denoting by $T_\phi^{\min} \leq T^{\min}$ the minimal period of $\phi(t)$, we have that there exist $n \in \mathbb{N}^+$ such that $nT_\phi^{\min} = 2\pi$ and $\tilde{k} \in \mathbb{Z}^N$ such that $\phi(T_\phi^{\min}) = 2\pi\tilde{k}$, verifying $n\tilde{k} = \kappa$. We get that

$$T^{\min} \geq T_\phi^{\min} = 2\pi/n \, . \tag{202}$$

By (48) we have $2\pi n\tilde{k} = 2\pi\kappa = 2\pi\omega/\varepsilon + O(1)$. Consider the vector $\bar{\omega}_\mu := (\omega_1, \omega_2)$. Then $2\pi n(\tilde{k}_1, \tilde{k}_2) = 2\pi\bar{\omega}_\mu/\varepsilon + O(1)$ and taking $h = (h_1, h_2) = (\tilde{k}_2, -\tilde{k}_1)$, we get $0 = 2\pi n(\tilde{k}_1, \tilde{k}_2) \cdot h = 2\pi\varepsilon^{-1}\bar{\omega}_\mu \cdot h + O(|h|)$. For $\mu \in \Upsilon(\rho) := \Upsilon_{\frac{1}{\rho}-1}$ defined in Lemma 10.1, we have $|\bar{\omega}_\mu \cdot h| \geq c(\mu)|h|^{-\varsigma}$, where $\varsigma := -1 + 1/\rho > 1$, from which it follows that $O(|h|) = |2\pi\varepsilon^{-1}\bar{\omega}_\mu \cdot h| \geq \mathrm{ct}\, \varepsilon^{-1}|h|^{-\varsigma}$ and hence $|h|^{\varsigma+1} \geq \mathrm{ct}\,/\varepsilon$, namely $|h| \geq \mathrm{ct}\, \varepsilon^{-1/(\varsigma+1)} = \mathrm{ct}\, \varepsilon^{-\rho}$. From (48) we get $n = |\kappa||\tilde{k}|^{-1} = |\kappa||h|^{-1} \leq \mathrm{ct}\, \varepsilon^{\rho-1}$ and we conclude by (202). $\square$

Remark 10.3. *We note that* $\mu = 1$ *belongs to the set* Υ *above, since the vector* $(\sqrt{2}, \sqrt{5})$ *is diophantine for every* $\varsigma > 1$.

A Technical lemmata

Here we prove some lemmata enunciated in Section 2.5, to which we refer for notations. We first need a simple result

Lemma A.1. *Suppose that the sequence* $\tilde{f}_k \in E$ *satisfy* $\sum_k e^{2\alpha|k|}|k|_*^{2\sigma}\|\tilde{f}_k\|_E^2 < \infty$, *then the series* $\sum_k \tilde{f}_k e^{\mathrm{i}kt}$ *uniformly converges on* $\mathbb{T}_\alpha$ *to a certain* $f(t)$. *Moreover* $\tilde{f}_k = f_k$ *with* f_k *defined in (39) and, therefore,* $f \in H_E^{\alpha,\sigma}$.

PROOF. The sequence of analytic functions $S_K(t) := \sum_{|k|\leq K} \tilde{f}_k e^{\mathrm{i}kt}$ uniformly converges on $\overline{\mathbb{T}_\alpha}$ since

$$\sup_{t\in\overline{\mathbb{T}_\alpha}} \|S_K(t)\|_E \leq \sum_k \|\tilde{f}_k\|_E e^{\alpha|k|} \leq \left(\sum_k |k|_*^{-2\sigma}\right)^{1/2} \left(\sum_k \|f_k\|_E^2 e^{2\alpha|k|}|k|_*^{2\sigma}\right)^2 < \infty \qquad (\sigma > 1/2) \tag{A.1}$$

therefore its limit $f(t)$ is continuous on $\overline{\mathbb{T}_\alpha}$ and analytic on $\mathbb{T}_\alpha$ (recall e.g. Theorem 2 on pg. 137 of [PT]). $\square$

PROOF OF PROPOSITION 2.5 Let $\tilde{f}(t) := \sum_k f_k e^{\mathrm{i}kt}$. Since $f \in H_E^{\alpha,\sigma}$, Lemma A.1 implies that $\tilde{f}$ is analytic on $\mathbb{T}_\alpha$. We want to prove that $f(t) = \tilde{f}(t)$, $\forall\, t \in \mathbb{R}$. If $\{e_j\}_{j\in\mathbb{N}}$ denotes an orthonormal basis of E, it is enough to show that $\langle f(t), e_j\rangle_E =: f^{(j)}(t) = \tilde{f}^{(j)}(t) := \langle \tilde{f}(t), e_j\rangle_E$, $\forall\, j \in \mathbb{N}\,, t \in \mathbb{R}$. For any $j \in \mathbb{N}$, $f^{(j)}$ is continuous (and L^2) and coincides almost everywhere with its Fourier series: $f^{(j)}(t) = \sum_k f_k^{(j)} e^{\mathrm{i}kt}$ a.e. $t \in \mathbb{R}$, where $f_k^{(j)} := {\textstyle\int\!\!\!\!-}_0^{2\pi} e^{-\mathrm{i}kt} f^{(j)}(t) dt$. We have $f_k^{(j)} = \langle f_k, e_j\rangle_E$ with f_k defined in (39). Moreover $\tilde{f}^{(j)}(t) = \langle \sum_k f_k e^{\mathrm{i}kt}, e_j\rangle_E = \sum_k \langle f_k, e_j\rangle_E e^{\mathrm{i}kt} = \sum_k f_k^{(j)} e^{\mathrm{i}kt}$, $\forall\, t \in \mathbb{R}$. Therefore $f^{(j)}(t) = \tilde{f}^{(j)}(t)$ a.e. $t \in \mathbb{R}$. Since both $f^{(j)}$ and $\tilde{f}^{(j)}$ are continuous, they must coincide. $\square$

PROOF OF PROPOSITION 2.6 Let us show that $\langle f, f\rangle_{H_E^{\alpha,\sigma}} = 0$ implies $f = 0$; indeed we have $f_k = 0$ $\forall\, k \in \mathbb{Z}$, and $f = 0$ by (41). We now prove that $H_E^{\alpha,\sigma}$ is complete. Let $\{f^{(n)}\}_{n\in\mathbb{N}}$ a Cauchy sequence in $H_E^{\alpha,\sigma}$. Then, $\forall\, k \in \mathbb{Z}$, $\{f_k^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in E; let $\tilde{f}_k \in E$ be its limit. It is simple to see that $\tilde{f}_k$ satisfy the hypothesis of Lemma A.1, hence $f(t) := \sum_k \tilde{f}_k e^{\mathrm{i}kt} \in H_E^{\alpha,\sigma}$ and $f_k = \tilde{f}_k$. Finally $f^{(n)}$ converges to f in $H_E^{\alpha,\sigma}$. The uniform estimate on f follows by (A.1). $\square$

Lemma A.2. *Let* $\sigma > 1/2$, $m \in \mathbb{N}^+$ *and* $x_k \geq 0$ *for every* $k = (k_1, \dots, k_m) \in \mathbb{Z}^m$. *Then*

$$\left(\sum_{k\in\mathbb{Z}^m} x_k\right)^2 \leq B_{2\sigma}^m \sum_{k\in\mathbb{Z}^m} (\gamma_k^\sigma x_k)^2 \,, \tag{A.2}$$

where

$$\gamma_k := |j - k_1 - \ldots - k_m|_* |k_1|_* \cdots |k_m|_* / |j|_* \,, \quad \textit{with} \quad j \in \mathbb{Z} \,, \quad \textit{and} \quad B_\varsigma := 2^\varsigma \sum_{n\in\mathbb{Z}} |n|_*^{-\varsigma} \,. \tag{A.3}$$

PROOF. Note that $\sigma > 1/2$ implies $B_{2\sigma} < \infty$. By the Cauchy–Schwarz inequality we get

$$\left(\sum_{k\in\mathbb{Z}^m} x_k\right)^2 = \left(\sum_{k\in\mathbb{Z}^m} \gamma_k^{-\sigma}\gamma_k^\sigma x_k\right)^2 \leq \sum_{k\in\mathbb{Z}^m} \gamma_k^{-2\sigma} \sum_{k\in\mathbb{Z}^m} (\gamma_k^\sigma x_k)^2$$

and (A.2) follows if we show that

$$\sum_{k\in\mathbb{Z}^m} \gamma_k^{-\varsigma} \leq B_\varsigma^m \,, \quad \text{with} \quad \varsigma := 2\sigma > 1 \,. \tag{A.4}$$

We prove (A.4) by induction over m. We first show (A.4) for $m = 1$. Using the elementary inequality

$$(a + b)^\varsigma \leq 2^{\varsigma-1}(a^\varsigma + b^\varsigma) \,, \quad a, b \geq 0 \,, \ \varsigma \geq 1 \,, \tag{A.5}$$

we get

$$\sum_{k\in\mathbb{Z}} \gamma_k^{-\varsigma} = \sum_{k\in\mathbb{Z}} \left(\frac{|j|_*}{|j-k|_*|k|_*}\right)^\varsigma \leq \sum_{k\in\mathbb{Z}} \left(\frac{|j-k|_* + |k|_*}{|j-k|_*|k|_*}\right)^\varsigma \leq 2^{\varsigma-1} \sum_{k\in\mathbb{Z}} \frac{|j-k|_*^\varsigma + |k|_*^\varsigma}{|j-k|_*^\varsigma |k|_*^\varsigma} = B_\varsigma \,.$$

We now suppose (A.4) true for $m-1$ and prove it for m. Using (A.5) we have $|j|_*^\varsigma \leq (|j - k_m|_* + |k_m|_*)^\varsigma \leq 2^{\varsigma-1}(|j-k_m|_*^\varsigma + |k_m|_*^\varsigma)$ and therefore

$$\sum_{k\in\mathbb{Z}^m} \gamma_k^{-\varsigma} \leq 2^{\varsigma-1} \sum_{k\in\mathbb{Z}^m} \left(\frac{|j-k_m|^\varsigma}{|j - k_1 - \ldots - k_m|_*^\varsigma |k_1|_*^\varsigma \cdots |k_m|_*^\varsigma} + \frac{1}{|j - k_1 - \ldots - k_m|_*^\varsigma |k_1|_*^\varsigma \cdots |k_{m-1}|_*^\varsigma}\right) . \tag{A.6}$$

Let $j_0 := j - k_m$ then using (A.4) with $m-1$ (inductive hypothesis) we have

$$\begin{aligned} &\sum_{k\in\mathbb{Z}^m} \frac{|j-k_m|^\varsigma}{|j - k_1 - \ldots - k_m|_*^\varsigma |k_1|_*^\varsigma \cdots |k_m|_*^\varsigma} \\ &= \sum_{k_m} \frac{1}{|k_m|_*^\varsigma} \left(\sum_{k_1,\dots,k_{m-1}} \frac{|j_0|^\varsigma}{|j_0 - k_1 - \ldots - k_{m-1}|_*^\varsigma |k_1|_*^\varsigma \cdots |k_{m-1}|_*^\varsigma}\right) \leq \sum_{k_m} \frac{1}{|k_m|_*^\varsigma}\left(B_\varsigma^{m-1}\right) = 2^{-\varsigma} B_\varsigma^m \,. \end{aligned} \tag{A.7}$$

On the other hand

$$\begin{aligned} &\sum_{k\in\mathbb{Z}^m} \frac{1}{|j - k_1 - \ldots - k_m|_*^\varsigma |k_1|_*^\varsigma \cdots |k_{m-1}|_*^\varsigma} \\ &\quad = \sum_{k_1,\dots,k_{m-1}} \frac{1}{|k_1|_*^\varsigma \cdots |k_{m-1}|_*^\varsigma} \sum_{k_m} \frac{1}{|j - k_1 - \ldots - k_m|_*^\varsigma} = (2^{-\varsigma} B_\varsigma)^m \,. \end{aligned} \tag{A.8}$$

By (A.6),(A.7),(A.8), we get $\sum_{k\in\mathbb{Z}^m} \gamma_k^{-\varsigma} \leq 2^{\varsigma-1}(2^{-\varsigma}B_\varsigma^m + 2^{-\varsigma m}B_\varsigma^m) \leq B_\varsigma^m$. □

From now on, we will denote by $\| \cdot \|_{\mathrm{op}}$ the standard operatorial norm.

Lemma A.3. *Let be $L_n : E \times \cdots \times E(n \text{ times}) \longrightarrow F$ a linear and continuous operator. Let be $h^{(1)}, \dots, h^{(n)} \in H_E^{\alpha,\sigma}$, with $\alpha > 0$, $\sigma > 1/2$ and $g(t) := L_n[h^{(1)}(t), \cdots, h^{(n)}(t)]$. Then $g \in H_F^{\alpha,\sigma}$ and, in particular,*

$$\|g\|_{H_F^{\alpha,\sigma}} \le B_{2\sigma}^{\frac{n-1}{2}} \|L_n\|_{\mathrm{op}} \|h^{(1)}\|_{H_E^{\alpha,\sigma}} \dots \|h^{(n)}\|_{H_E^{\alpha,\sigma}} ,$$

where $B_{2\sigma}$ is defined in (A.3) and $\|L_n\|_{\mathrm{op}} := \sup_{\|h^{(i)}\|_E = 1\,, 1 \le i \le n} \|L_n[h^{(1)}, \dots, h^{(n)}]\|_F$.

PROOF. Let $h^{(j)}(t) = \sum_k e^{\mathrm{i}kt} h_k^{(j)}$, $j = 1, \dots, n$. We have $g(t) = L_n\Big[\sum_{k_1} e^{\mathrm{i}k_1 t} h_{k_1}^{(1)}, \dots, \sum_{k_n} e^{\mathrm{i}k_n t} h_{k_n}^{(n)}\Big] = \sum_j e^{\mathrm{i}jt} g_j \,, L_n[h_{k_1}^{(1)}, \dots, h_{k_{n-1}}^{(n-1)}, h_{j-k_1-\dots-k_{n-1}}^{(n)}]$, with $g_j = \sum_{k_1,\dots,k_{n-1}} L_n[h_{k_1}^{(1)}, \dots, h_{k_{n-1}}^{(n-1)}, h_{j-k_1-\dots-k_{n-1}}^{(n)}]$ and, from the triangular inequality,

$$\begin{aligned} \|g_j\|_F &\le \sum_{k_1,\dots,k_{n-1}} \|L_n[h_{k_1}^{(1)}, \dots, h_{k_{n-1}}^{(n-1)}, h_{j-k_1-\dots-k_{n-1}}^{(n)}]\|_F \\ &\le \|L_n\|_{\mathrm{op}} \sum_{k_1,\dots,k_{n-1}} \|h_{k_1}^{(1)}\|_E \dots \|h_{j-k_1-\dots-k_{n-1}}^{(n)}\|_E \,. \end{aligned} \tag{A.9}$$

For every $k := (k_1, \dots, k_{n-1}) \in \mathbb{Z}^{n-1}$ let us define $x_k = x_k^{(j)} := \|h_{k_1}^{(1)}\|_E \cdots \|h_{k_{n-1}}^{(n-1)}\|_E \|h_{j-k_1-\dots-k_{n-1}}^{(n)}\|_E$ By (A.9), we get $\|g\|^2_{H_F^{\alpha,\sigma}} = \sum_j e^{2\alpha|j|} |j|_*^{2\sigma} \|g_j\|_F^2 \le \|L_n\|^2_{\mathrm{op}} \sum_j e^{2\alpha|j|} |j|_*^{2\sigma} \left(\sum_{k \in \mathbb{Z}^{n-1}} x_k^{(j)}\right)^2$.
Using Lemma A.2 with $m = n-1$ and $x_k = x_k^{(j)}$ defined above, we get

$$\|g\|^2_{H_F^{\alpha,\sigma}} \le B_{2\sigma}^{n-1} \|L_n\|^2_{\mathrm{op}} \sum_j e^{2\alpha|j|} |j|_*^{2\sigma} \sum_{k \in \mathbb{Z}^{n-1}} \left(\gamma_k^\sigma x_k^{(j)}\right)^2 . \tag{A.10}$$

Being

$$\begin{aligned} \sum_{k \in \mathbb{Z}^{n-1}} \left(\gamma_k^\sigma x_k\right)^2 = {} & \frac{1}{|j|_*^{2\sigma}} \sum_{k_1} \left(|k_1|_*^{2\sigma} \|h_{k_1}^{(1)}\|_E^2\right) \cdot \sum_{k_2} \left(|k_2|_*^{2\sigma} \|h_{k_2}^{(2)}\|_E^2\right) \cdots \\ & \cdots \sum_{k_{n-1}} \left(|k_{n-1}|_*^{2\sigma} \|h_{k_{n-1}}^{(n-1)}\|_E^2\right) |j - k_1 - \dots - k_{n-1}|_*^{2\sigma} \|h_{j-k_1-\dots-k_{n-1}}^{(n)}\|_E^2 \,, \end{aligned}$$

from (A.10) it follows that

$$\begin{aligned} \|g\|^2_{H_F^{\alpha,\sigma}} \le {} & \|L_n\|^2_{\mathrm{op}} B_{2\sigma}^{n-1} \sum_{k_1} \left(|k_1|_*^{2\sigma} \|h_{k_1}^{(1)}\|_E^2\right) \cdots \sum_{k_{n-1}} \left(|k_{n-1}|_*^{2\sigma} \|h_{k_{n-1}}^{(n-1)}\|_E^2\right) \\ & \cdot \sum_j \left(|j - k_1 - \dots - k_{n-1}|_*^{2\sigma} \|h_{j-k_1-\dots-k_{n-1}}^{(n)}\|_E^2\right) e^{2\alpha|j|} \\ \le {} & \|L_n\|^2_{\mathrm{op}} B_{2\sigma}^{n-1} \left(\sum_{k_1} e^{2\alpha|k_1|} |k_1|_*^{2\sigma} \|h_{k_1}^{(1)}\|_E^2\right) \cdots \left(\sum_{k_{n-1}} e^{2\alpha|k_{n-1}|} |k_{n-1}|_*^{2\sigma} \|h_{k_{n-1}}^{(n-1)}\|_E^2\right) \\ & \cdot \left(\sum_j e^{2\alpha|j-k_1-\dots-k_{n-1}|} |j - k_1 - \dots - k_{n-1}|_*^{2\sigma} \|h_{j-k_1-\dots-k_{n-1}}^{(n)}\|_E^2\right) \\ = {} & \|L_n\|^2_{\mathrm{op}} B_{2\sigma}^{n-1} \|h^{(1)}\|^2_{H_E^{\alpha,\sigma}} \cdots \|h^{(n)}\|^2_{H_E^{\alpha,\sigma}} \,, \end{aligned}$$

using that $e^{2\alpha|j|} \le e^{2\alpha|k_1|} \dots e^{2\alpha|j-k_1-\dots-k_{n-1}|}$. □

Lemma A.4. *Let be $f : E \longrightarrow F$ analytic for $\|x\|_E < r_0$, $f(x) = \sum_{n \ge n_0} \frac{1}{n!} d^n f(0)[x, \dots, x]$. Then $\|d^n f(0)\|_{\mathrm{op}} \le M_{r_1} \left(n/r_1\right)^n \ \forall\, 0 < r_1 < r_0$, where $M_{r_1} := \sup_{\|x\|_E \le r_1} \|f(x)\|_F$.*

PROOF. Consider the map $d^n f(0)$ defined by[26]

$$d^n f(0)[h_1, \ldots, h^{(n)}] = \frac{1}{(2\pi \mathrm{i})^n} \int_{|\zeta_1|=\epsilon} \cdots \int_{|\zeta_n|=\epsilon} \frac{f(\zeta_1 h_1 + \ldots + \zeta_n h^{(n)})}{\zeta_1^2 \cdots \zeta_n^2}\, d\zeta_1 \cdots d\zeta_n\,.$$

Defining $\epsilon := r_1/n$, we have, if $\|h_i\|_E = 1$, for all $1 \le i \le n$, $\|\zeta_1 h_1 + \ldots + \zeta_n h^{(n)}\|_E \le \epsilon(\|h_1\|_E + \ldots + \|h^{(n)}\|_E) = \epsilon n = r_1 < r_0$. Therefore $\|d^n f(0)\|_{\rm op} := \sup_{\|h_i\|_E=1\,,1\le i\le n} \|d^n f(0)[h_1, \ldots, h^{(n)}]\|_F \le M_{r_1}/\epsilon^n = M_{r_1}\,(n/r_1)^n$. ☐

Remark A.5. *If $h, \tilde h \in H_E^{\alpha,\sigma}$ with $L_n = d^n f(0)$, then, if $g_n := d^n f(0)[h, \ldots, h, \tilde h]$, one has $g_n(t) := L_n f(0)[h(t), \ldots, h(t), \tilde h(t)]$, from which it follows that, using Lemma A.3,*

$$\|g_n\|_{H_F^{\alpha,\sigma}} \le n^\sigma B_{2\sigma}^{\frac{n-1}{2}} \|d^n f(0)\|_{\rm op} \|h\|^{n-1}_{H_E^{\alpha,\sigma}} \|\tilde h\|_{H_E^{\alpha,\sigma}}\,.$$

PROOF OF PROPOSITION 2.7 Let us take $0 < r_1 < r_0$ such that $M := \sup_{\|x\|_E \le r_1} \|f(x)\|_F < \infty$. For $h \in H_E^{\alpha,\sigma}$, by Lemma A.3 with $L_n := d^n f(0)$,

$$\begin{aligned}
\|f(h)\|_{H_F^{\alpha,\sigma}} &= \Big\| \sum_{n\ge n_0} \frac{1}{n!} d^n f(0)[h, \ldots, h] \Big\|_{H_F^{\alpha,\sigma}} \le \sum_{n \ge n_0} \frac{1}{n!} \|d^n f(0)[h, \ldots, h]\|_{H_F^{\alpha,\sigma}} \\
&\le \sum_{n\ge n_0} \frac{1}{n!} n^\sigma B_{2\sigma}^{\frac{n-1}{2}} \|d^n f(0)\|_{\rm op} \|h\|^n_{H_E^{\alpha,\sigma}} \le M \sum_{n\ge n_0} \Big(\frac{1}{n!} n^\sigma B_{2\sigma}^{\frac{n-1}{2}} \frac{n^n}{r_1^n} \Big) \|h\|^n_{H_E^{\alpha,\sigma}}\,,
\end{aligned}$$

where, in the last inequality, we used Lemma A.4. Then we take $1/c(\sigma) := 2 \max_{n\ge1} \Big(n^\sigma B_{2\sigma}^{\frac{n-1}{2}} n^n/n! \Big)^{1/n}$ and $r := c(\sigma) r_1$.

Now, let us prove that f_* is continuously differentiable. We have $Df_* : H_E^{\alpha,\sigma} \longrightarrow \mathcal{L}\big(H_E^{\alpha,\sigma}, H_F^{\alpha,\sigma}\big)$. Noting that $f' : E \longrightarrow \mathcal{L}(E,F)$, one has $Df_*(h) = f' \circ h$, namely for every $\tilde h \in H_E^{\alpha,\sigma}$, $\big(Df_*(h)\big)[\tilde h] \in H_F^{\alpha,\sigma}$, is defined by $\Big(\big(Df_*(h)\big)[\tilde h]\Big)(t) := \big(f'(h(t))\big)[\tilde h(t)]$. We have $\big(f'(x)\big)[\tilde x] := \sum_{n\ge n_0} \frac{1}{n!}\, n\, d^n f(0)[x, \ldots, x, \tilde x]$ and hence, using the symmetry of $d^n f(0)$, $\big(f'(x)\big)[\tilde x] = \frac{d}{ds}\big|_{s=0} f(x + s\tilde x) := \sum_{n\ge n_0} \frac{1}{(n-1)!} d^n f(0)[x, \ldots, x, \tilde x]\,.$ Denoting $g_n(t) := d^n f(0)[h(t), \ldots, h(t), \tilde h(t)]$, we obtain that $Df_*(h)$ is bounded, indeed

$$\begin{aligned}
\big\|Df_*(h)\big\|_{\mathcal{L}(H_E^{\alpha,\sigma}, H_F^{\alpha,\sigma})} &= \sup_{\|\tilde h\|_{H_E^{\alpha,\sigma}}=1} \Big\| (Df_*(h))[\tilde h] \Big\|_{H_F^{\alpha,\sigma}} = \sup_{\|\tilde h\|_{H_E^{\alpha,\sigma}}=1} \Big\| \sum_{n\ge n_0} \frac{1}{(n-1)!} g_n \Big\|_{H_F^{\alpha,\sigma}} \\
&\le \sup_{\|\tilde h\|_{H_E^{\alpha,\sigma}}=1} \sum_{n\ge n_0} \frac{n^\sigma B_{2\sigma}^{\frac{n-1}{2}}}{(n-1)!} \|d^n f(0)\|_{\rm op} \|h\|^{n-1}_{H_E^{\alpha,\sigma}} \|\tilde h\|_{H_E^{\alpha,\sigma}} \\
&\le \sum_{n\ge n_0} \frac{n^\sigma B_{2\sigma}^{\frac{n-1}{2}}}{(n-1)!} M \Big(\frac{n}{r_1}\Big)^n \|h\|^{n-1}_{H_E^{\alpha,\sigma}} \le {\rm ct} \sum_{n\ge n_0-1} \Big(\frac{\|h\|_{H_E^{\alpha,\sigma}}}{2r} \Big)^n\,,
\end{aligned}$$

where we used Lemma A.4 for $0 < r < r_1$ sufficiently small and Remark A.5. The proof that Df_* is continuous is similar and we omit it. ☐

Lemma A.6. *Let us suppose that $\tilde L_{k\ell} \in \mathcal{L}(E,F)$, with $k, \ell \in \mathbb{Z}$, satisfy $C := \sup_\ell \sum_k e^{2\alpha|k-\ell|} |k - \ell|_*^{2\sigma} \|\tilde L_{k\ell}\|^2_{\mathcal{L}(E,F)} < \infty$, then it is well defined $L \in \mathcal{L}(H_E^{\alpha,\sigma}, H_F^{\alpha,\sigma})$ by*

$$\forall\, f \in H_E^{\alpha,\sigma}\,, \qquad \big(L[f]\big)(t) := \sum_k e^{\mathrm{i}kt} \sum_\ell \tilde L_{k\ell}[f_\ell] \ \in\ H_F^{\alpha,\sigma}\,, \tag{A.11}$$

with f_ℓ defined in (39). Moreover $\tilde L_{k\ell} = L_{k\ell}$ defined in (42); hence $L \in \mathcal{L}^{\alpha,\sigma}(E,F)$.

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[26] See [PT] pg. 136, 137.

PROOF. For every $k \in \mathbb{Z}$, $\Big\{ \sum_{|\ell| \leq n} \tilde{L}_{k\ell}[f_\ell] \Big\}_{n \in \mathbb{N}}$ is a Cauchy sequence in F. Let us denote its limit by $\tilde{f}_k := \sum_\ell \tilde{L}_{k\ell}[f_\ell] \in F$. To see that $L[f] \in H_F^{\alpha,\sigma}$, we apply Lemma A.1 (with $E = F$). Indeed $\tilde{f}_k$ satisfy

$$\sum_k e^{2\alpha|k|} |k|_*^{2\sigma} \|\tilde{f}_k\|_F^2 \leq \sum_k e^{2\alpha|k|} |k|_*^{2\sigma} \Big(\sum_\ell \|\tilde{L}_{k\ell}\|_{\mathcal{L}(E,F)} \|x_\ell\|_E \Big)^2$$
$$\leq \mathrm{ct} \sum_k e^{2\alpha|k|} |k|_*^{2\sigma} \sum_\ell \frac{|k-\ell|_*^{2\sigma} |\ell|_*^{2\sigma}}{|k|_*^{2\sigma}} \|L_{k\ell}\|^2 \|x_\ell\|_E^2 \leq \mathrm{ct}\ C \, \|x\|^2_{H_E^{\alpha,\sigma}} < \infty \,, \tag{A.12}$$

where we have used Lemma A.2 with $m = 1$. It is obvious that L is linear in f. Then L is a linear operator from $H_E^{\alpha,\sigma}$ to $H_F^{\alpha,\sigma}$; we now show that it is continuous:

$$\|L\|_{\mathcal{L}(H_E^{\alpha,\sigma}, H_F^{\alpha,\sigma})} := \sup_{\|x\|_{H_E^{\alpha,\sigma}} = 1} \|L[x]\|^2_{H_F^{\alpha,\sigma}} \leq \sup_{\|x\|_{H_E^{\alpha,\sigma}} = 1} \sum_k e^{2\alpha|k|} |k|_*^{2\sigma} \|\tilde{f}_k\|_F^2 \leq \mathrm{ct}\ C < \infty \,, \tag{A.13}$$

using (A.12). For every $x \in E$ we have $L[e^{\mathrm{i}\ell t} x] = \sum_n e^{\mathrm{i}nt} \tilde{L}_{n\ell}[x]$, where the series converges in $H_F^{\alpha,\sigma}$; since, by Proposition 2.6, this last series uniformly converges, we get $L_{k\ell}[x] = \fint_0^{2\pi} e^{-\mathrm{i}kt} \sum_n e^{\mathrm{i}nt} \tilde{L}_{n\ell}[x] \, dt$ $= \sum_n \tilde{L}_{n\ell}[x] \fint_0^{2\pi} e^{\mathrm{i}(n-k)t} dt = \tilde{L}_{k\ell}[x]$. Since $L_{k\ell} = \tilde{L}_{k\ell}$, we have that $L \in \mathcal{L}^{\alpha,\sigma}(E,F)$. ☐

PROOF OF PROPOSITION 2.10. By Lemma A.6, it is well defined $\tilde{L} \in \mathcal{L}^{\alpha,\sigma}(E,F)$ by $\Big(\tilde{L}[f]\Big)(t) : = \sum_k e^{\mathrm{i}kt} \sum_\ell L_{k\ell}[f_\ell] \,.\ \forall\, f \in H_E^{\alpha,\sigma}$. We want to prove that $\tilde{L} = L$. Recalling (41), it is enough to show that for every $f \in H_E^{\alpha,\sigma}$ and $k \in \mathbb{Z}$, we have $\big(\tilde{L}[f]\big)_k = \big(L[f]\big)_k$. We get $\big(L[f]\big)_k := \fint_0^{2\pi} e^{-\mathrm{i}kt} \big(L[f]\big)(t) \, dt =$ $\fint_0^{2\pi} e^{-\mathrm{i}kt} L\Big[\sum_\ell e^{\mathrm{i}\ell t} f_\ell \Big] \, dt = \sum_\ell \fint_0^{2\pi} e^{-\mathrm{i}kt} L\Big[e^{\mathrm{i}\ell t} f_\ell \Big] \, dt = \sum_\ell L_{k\ell}[f_\ell] =: \big(\tilde{L}[f]\big)_k$, where in the third equality we have exchanged the summation with the integral since $\sum_\ell L\big[e^{\mathrm{i}\ell t} f_\ell\big]$ uniformly converges on $[0, 2\pi]$. Indeed by the Cauchy–Schwarz inequality

$$\sum_\ell \sup_{t \in [0,2\pi]} \|L\big[e^{\mathrm{i}\ell t} f_\ell\big]\|_F = \sum_\ell \|L\big[f_\ell\big]\|_F \leq \|L\|_{\mathcal{L}^{\alpha,\sigma}(E,F)} \sum_\ell \|f_\ell\|_E \leq \mathrm{ct}\, \|L\|_{\mathcal{L}^{\alpha,\sigma}(E,F)} \|f\|_{H_E^{\alpha,\sigma}} < \infty \,.$$

$$\begin{aligned} \sum_\ell \sup_{t \in [0,2\pi]} \|L\big[e^{\mathrm{i}\ell t} f_\ell\big]\|_F &= \sum_\ell \|L\big[f_\ell\big]\|_F \leq \|L\|_{\mathcal{L}^{\alpha,\sigma}(E,F)} \sum_\ell \|f_\ell\|_E \\ &\leq \mathrm{ct}\, \|L\|_{\mathcal{L}^{\alpha,\sigma}(E,F)} \|f\|_{H_E^{\alpha,\sigma}} < \infty \,. \end{aligned}$$

☐

PROOF OF PROPOSITION 2.11. For brevity we set $|\!|\!| \cdot |\!|\!| := \| \cdot \|_{\mathcal{L}^{\alpha,\sigma}(E,F)}$ and $\| \cdot \| := \| \cdot \|_{\mathcal{L}(E,F)}$. We first prove that $|\!|\!| L |\!|\!| = 0$ implies $L = 0$. Indeed we have $L_{k\ell} = 0$ for every $k\ell \in \mathbb{Z}$ and, by (44), we get $L = 0$. We now show the triangular inequality. Let $L^{(1)}$, $L^{(2)}$ be linear operators of $\mathcal{L}^{\alpha,\sigma}(E,F)$. Then

$$\begin{aligned} |\!|\!| L^{(1)} + L^{(2)} |\!|\!|^2 &\leq \sup_\ell \sum_k e^{2\alpha|k-\ell|} |k-\ell|_*^{2\sigma} \Big(\|L^{(1)}_{k\ell}\|^2 + \|L^{(2)}_{k\ell}\|^2 + 2 \|L^{(1)}_{k\ell}\| \, \|L^{(2)}_{k\ell}\| \Big) \\ &\leq |\!|\!| L^{(1)} |\!|\!|^2 + |\!|\!| L^{(2)} |\!|\!|^2 + 2 |\!|\!| L^{(1)} |\!|\!| \, |\!|\!| L^{(2)} |\!|\!| = \Big(|\!|\!| L^{(1)} |\!|\!| + |\!|\!| L^{(2)} |\!|\!| \Big)^2 \,, \end{aligned}$$

where in the last inequality we have used the Cauchy-Schwarz inequality. Let us show that $\mathcal{L}^{\alpha,\sigma}(E,F)$ endowed with the norm $|\!|\!| \cdot |\!|\!|$ is complete. Let be $L^{(n)}$ a Cauchy sequence, namely for all $\epsilon > 0$ there exists $N > 0$ such that, for all $m > n \geq N$, it results that $|\!|\!| L^{(n)} - L^{(m)} |\!|\!| \leq \epsilon$. It follows that for all $k\ell$, $L^{(n)}_{k\ell}$ is a Cauchy sequence in $\mathcal{L}(E,F)$; then $L^{(n)}_{k\ell} \to \tilde{L}_{k\ell}$ for a suitable $\tilde{L}_{k\ell} \in \mathcal{L}(E,F)$. We want to show that $\tilde{L}_{k\ell}$ satisfy the hypothesis of Lemma A.6. Indeed for every $M_1, M_2 > 0$ we get $\sup_{|\ell| \leq M_2} \sum_{|k| \leq M_1} e^{2\alpha|k-\ell|} |k-\ell|_*^{2\sigma} \|L^{(n)}_{k\ell} - L^{(m)}_{k\ell}\|^2 \leq |\!|\!| L^{(n)} - L^{(m)} |\!|\!|^2 \leq \epsilon^2$; taking the limit for $m \to \infty$, and then the sup on M_1 and M_2 we obtain

$$\sup_\ell \sum_k e^{2\alpha|k-\ell|} |k-\ell|_*^{2\sigma} \|L^{(n)}_{k\ell} - \tilde{L}_{k\ell}\|^2 \leq \epsilon^2 \tag{A.14}$$

and $\sup_\ell \sum_k e^{2\alpha|k-\ell|}|k-\ell|_*^{2\sigma}\|\tilde{L}_{k\ell}\|^2 \le 2\epsilon^2 + 2\|L^{(n)}\| < \infty$. Therefore by Lemma A.6, we can define $L \in \mathcal{L}^{\alpha,\sigma}(E,F)$ as in (A.11). By (A.14), $L^{(n)} \to L$ in $\mathcal{L}^{\alpha,\sigma}(E,F)$.

The estimate $\|\cdot\|_{\mathcal{L}(H_E^{\alpha,\sigma},H_F^{\alpha,\sigma})} \le c\,\|\cdot\|_{\mathcal{L}^{\alpha,\sigma}(E,F)}$ follows by (A.13). Finally, for $x \in E$, let $f^x(t) := L[e^{\mathrm{i}kt}x]$; $f^x \in H_F^{\alpha,\sigma}$ and $L_{kk}[x] = f_k^x$ by (42). Since

$$|L_{kk}[x]|_F \le e^{-\alpha|k|}|k|_*^\sigma \|f^x\|_{H_F^{\alpha,\sigma}} \le e^{-\alpha|k|}|k|_*^\sigma \|L\|_{\mathcal{L}(H_E^{\alpha,\sigma},H_F^{\alpha,\sigma})}\|e^{\mathrm{i}kt}x\|_{H_E^{\alpha,\sigma}} \le \|L\|_{\mathcal{L}(H_E^{\alpha,\sigma},H_F^{\alpha,\sigma})}|x|_E\,,$$

taking the supremum for $|x|_E = 1$ we get the final estimate. □

Proof of Proposition 2.12 We know that $L := L^{(2)} \circ L^{(1)} \in \mathcal{L}(H_E^{\alpha,\sigma}, H_G^{\alpha,\sigma})$. Consider $L_{k\ell}^{(1)} \in \mathcal{L}(E,F)$ and $L_{k\ell}^{(2)} \in \mathcal{L}(F,G)$ defined in (42); we claim that

$$L_{k\ell} = \sum_n L_{kn}^{(2)} \circ L_{n\ell}^{(1)} \;\in\; \mathcal{L}(E,G)\,. \tag{A.15}$$

Indeed by definition,

$$\forall\, x \in E\,, \qquad\qquad L_{k\ell}[x] = \fint_0^{2\pi} e^{-\mathrm{i}kt} L[e^{\mathrm{i}\ell t}x]\,dt \tag{A.16}$$

and it is simple to see that

$$L[e^{\mathrm{i}\ell t}x] = \sum_n \sum_m e^{\mathrm{i}mt}\Big(L_{mn}^{(2)} \circ L_{n\ell}^{(1)}\Big)[x]\,, \tag{A.17}$$

using the representation formula (44) for $L^{(1)}$ and $L^{(2)}$. Since, by the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\sum_n \sum_m \sup_{0\le t\le T}\Big\|e^{\mathrm{i}mt}\Big(L_{mn}^{(2)} \circ L_{n\ell}^{(1)}\Big)[x]\Big\|_G &= \sum_n \sum_m \Big\|\Big(L_{mn}^{(2)} \circ L_{n\ell}^{(1)}\Big)[x]\Big\|_G \\ \le \|x\|_E \sum_n \|L_{n\ell}^{(1)}\|_{\mathcal{L}(E,F)} \sum_m \|L_{mn}^{(2)}\|_{\mathcal{L}(F,G)} &\le \|x\|_E \|L^{(1)}\|_{\mathcal{L}^{\alpha,\sigma}(E,F)}\|L^{(2)}\|_{\mathcal{L}^{\alpha,\sigma}(F,G)}\,,\end{aligned}$$

the two–dimensional series in (A.17) uniformly converges and can be exchanged with the integral in (A.16), proving (A.15). Finally let $c_{nk\ell} := \big\|L_{kn}^{(2)}\big\|_{\mathcal{L}(F,G)}\,\big\|L_{n\ell}^{(1)}\big\|_{\mathcal{L}(E,F)}$

$$\begin{aligned}\|L\|_{\mathcal{L}^{\alpha,\sigma}(E,G)}^2 \;&:=\; \sup_\ell \sum_k e^{2\alpha|\ell-k|}|\ell-k|_*^{2\sigma}\Big\|\sum_n L_{kn}^{(2)} \circ L_{n\ell}^{(1)}\Big\|_{\mathcal{L}(E,G)}^2 \\ &\le\; \sup_\ell \sum_k e^{2\alpha|\ell-k|}\left(\sum_n |\ell-k|_*^\sigma c_{nk\ell}\right)^2 \\ &\le\; \sup_\ell \sum_k e^{2\alpha|\ell-k|}\left(\sum_n 2^{\sigma-1}\Big(|k-n|_*^\sigma + |n-\ell|_*^\sigma\Big)c_{nk\ell}\right)^2 \\ &\le\; 2^{2\sigma-1}\sup_\ell \sum_k e^{2\alpha|\ell-k|}\left(\Big(\sum_n |k-n|_*^\sigma c_{nk\ell}\Big)^2 + \Big(\sum_n |n-\ell|_*^\sigma c_{nk\ell}\Big)^2\right) \\ &\le\; B_{2\sigma}\sup_\ell \sum_k e^{2\alpha|\ell-k|}\sum_n |k-n|_*^{2\sigma}|n-\ell|_*^{2\sigma}c_{nk\ell}^2 \\ &\le\; B_{2\sigma}\sup_\ell \sum_n e^{2\alpha|n-\ell|}|n-\ell|_*^{2\sigma}\big\|L_{n\ell}^{(1)}\big\|_{\mathcal{L}(E,F)}^2 \sum_k e^{2\alpha|k-n|}|k-n|_*^{2\sigma}\big\|L_{kn}^{(2)}\big\|_{\mathcal{L}(F,G)}^2 \\ &\le\; B_{2\sigma}\|L^{(1)}\|_{\mathcal{L}^{\alpha,\sigma}(E,F)}\|L^{(2)}\|_{\mathcal{L}^{\alpha,\sigma}(F,G)}\,,\end{aligned}$$

where $B_{2\sigma}$ was defined in (A.3) and we have used (A.5) in the second inequality and the Cauchy–Schwarz inequality in the fourth inequality. □

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