Constrained Minkowski Sums

A framework for solving subsequence problems efficiently

Friedrich Eisenbrand

Joint work with Thorsten Bernholt and Thomas Hofmeister

Subsequence Problems

Subsequence problem:

Given n numbers $a_1, \ldots, a_n \in \mathbb{R}$, find an interval $[i, j] \subseteq \{1, \ldots, n\}$ satisfying certain constraints such that a certain weight function is maximized.

Examples:

- Length-constrained heaviest segment: Given a length bound L, find interval $[i, j]$ with length at least L such that $\sum_{k=i}^{j} a_k/(j - i + 1)$ is maximal.

  (Lin et al., JCSS 2002)

- DNA copy number data analysis: Find an interval $[i, j]$ such that $|\sum_{k=i}^{j} a_k|/\sqrt{j - i + 1}$ is as large as possible.

  (Lipson et al., RECOMB 2005)

Geometric Interpretation

- Interval $I = [i, j]$ has length $\ell(I) = j - i + 1$ and weight $s(I) = \sum_{k=i}^{j} a_k$
- Interpret Interval I as point $p_I = (\ell(I), s(I))$ in the plane

Task:

- Length-constrained heaviest segment: Find point, maximizing function $f(\ell, s) = s/\ell$
- DNA copy number data analysis: Find point, maximizing $f(\ell, s) = s^2/\ell$

In both cases, f is quasi-convex

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In both cases, f is quasi-convex

Recall: $\phi : \mathbb{R}^n \to \mathbb{R}$ is quasi-convex if for all x, x' in domain of ϕ:

$$\text{for all } 0 \leqslant \lambda \leqslant 1 : \quad \phi(\lambda x + (1 - \lambda)x') \leqslant \max\{\phi(x), \phi(x')\}$$

Maximum is attained at extreme point of domain of ϕ

The link to Minkowski sums

Recall: Minkowski sum of P and Q is

$$P \oplus Q = \{p + q \mid p \in P,\, q \in Q\}$$

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- Given numbers $a_1, \ldots, a_n$ consider points $P = \{p_i = (i, a_1 + \cdots + a_i) \mid i = 1, \ldots, n\}$ and $Q = \{q_i = (-i, -a_1 - \cdots - a_i) \mid i = 1, \ldots, n\}$
- If $j > i$, then $q_i + p_j = (j - i, \sum_{k=i+1}^{j} a_k)$
- This is p_I for interval $I = [i + 1, j]$
- Not all points $q_i + p_j$, $p \in P$, $q \in Q$ are points of intervals.
- Intervals are points in Minkowski sum of P and Q which satisfy constraint $x_1 \geqslant 1$.

The geometric approach to subsequence problems

- A quasi-convex function attains its maximum at extreme point
- Compute superset of vertices of

$$\operatorname{conv}\left(\left(P \oplus Q\right) \cap \left(x_1 \geqslant 1\right)\right)$$

- Return optimal vertex

Constrained Minkowski Sums

- $Ax \geqslant b$ a set of k constraints
- $P, Q \subseteq \mathbb{R}^2$ finite point-sets
- Constrained Minkowski sum of P and Q and $Ax \geqslant b$:

$$(P \oplus Q)_{Ax \geqslant b} = \{p + q \mid p \in P,\, q \in Q,\, A(p+q) \geqslant b\}$$

Main result of this talk:

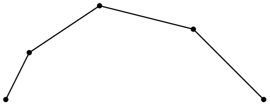
- Given $P \subseteq \mathbb{R}^2$ and $Q \subseteq \mathbb{R}^2$ with $|P| + |Q| = N$, one can compute a set $R \subseteq (P \oplus Q)_{Ax \geqslant b}$ containing all vertices of $(P \oplus Q)_{Ax \geqslant b}$ in time $O(N \log N)$ if k is fixed.
- A quasi-convex function can be maximized over $(P \oplus Q)_{Ax \geqslant b}$ in time $O(N \log N)$ if k is fixed.

A lower bound

- Set disjointness: Given sets of real numbers $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$, determine whether $A \cap B = \emptyset$.
- Lower bound of $\Omega(n \log n)$ in algebraic decision-tree model (Ben-Or 1983)
- Let $P = \{(0, -a) \mid a \in A\}$ and $Q = \{(0, b) \mid b \in B\}$
- A and B are not disjoint if and only if $(0, 0) \in P \oplus Q$
- if and only if $\min\{x_2 \mid x \in (P \oplus Q)_{x_2 \geqslant 0}\} = 0$

Maximizing a linear function over $(P \oplus Q)_{a^T x \geqslant b}$ requires $\Omega(N \log N)$ operations.

Computing convex hulls



- Sort points of $P \subseteq \mathbb{R}^2$ w.r.t. some linear function
- Sweep from "right to left", thereby updating the upper convex hull
- Running time to compute upper hull is $O(N)$ after sorting and $O(N \log N)$ together with sorting

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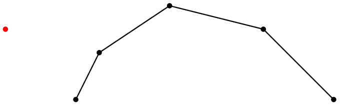

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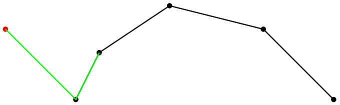

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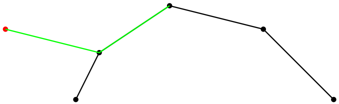

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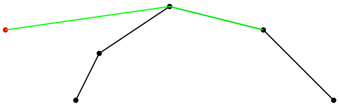

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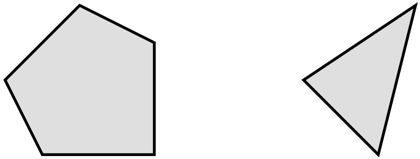
Computing convex hulls of Minkowski sums



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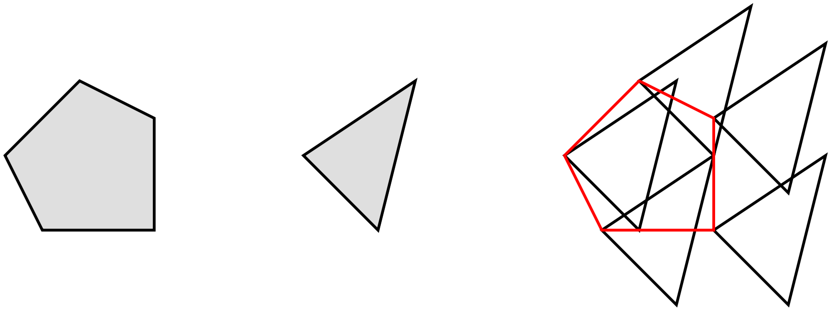

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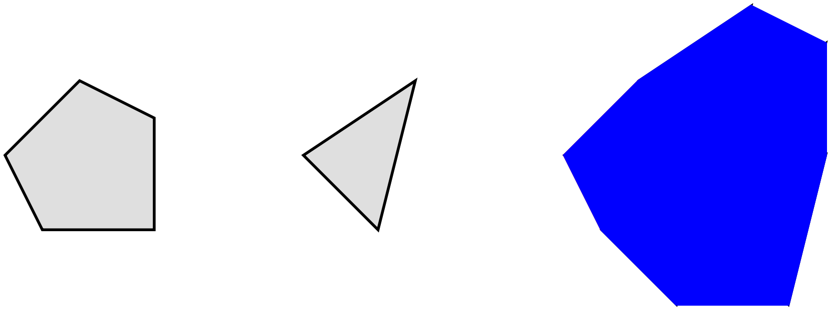

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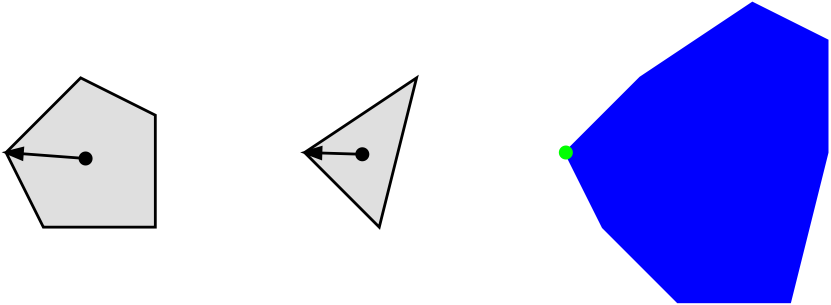

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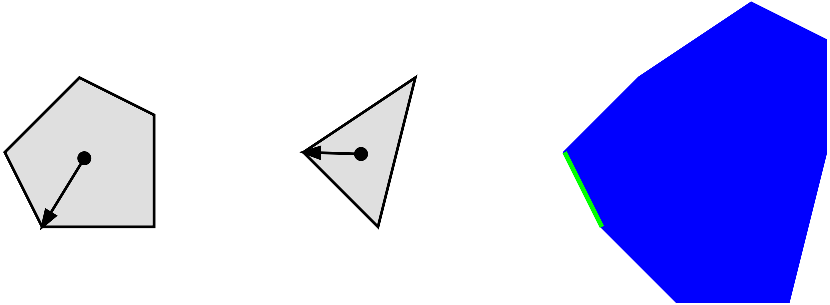

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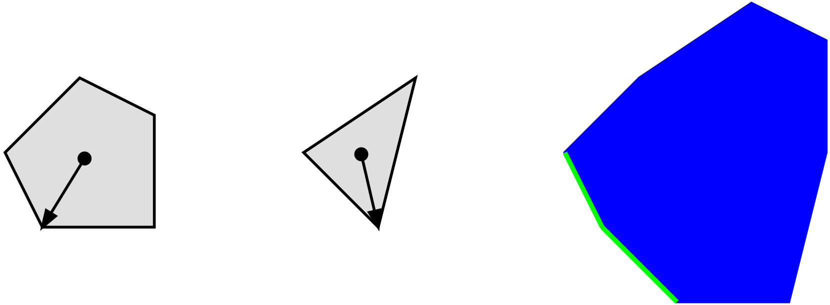

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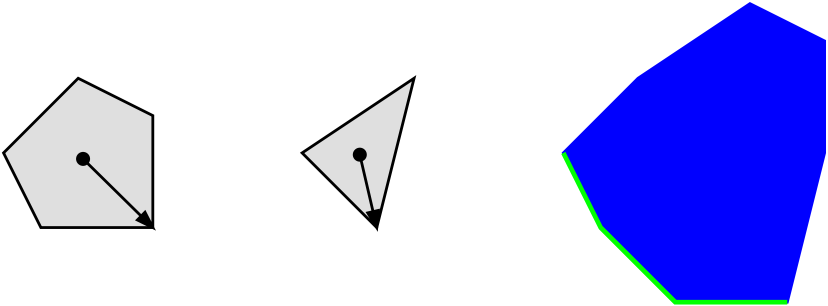

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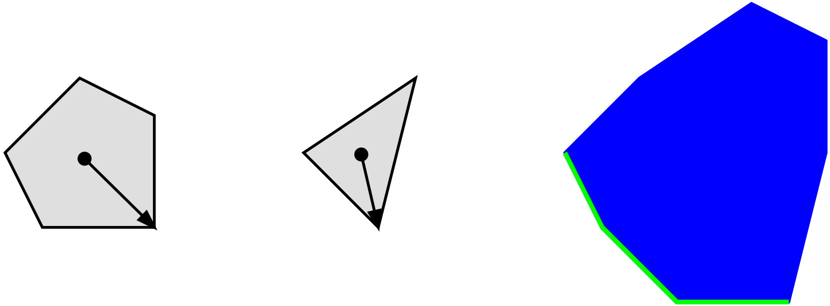
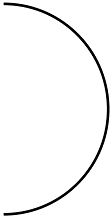


The appearance of $p \in P$ (or $q \in Q$) in a vertex $p + q$ of $\mathrm{conv}(P \oplus Q)$ is consecutive

Computing convex hulls of Minkowski Sums

- $\text{conv}(P \oplus Q)$ has at most $|P| + |Q|$ vertices
- If P and Q are sorted w.r.t. some linear function, then $\text{conv}(P \oplus Q)$ can be computed in linear time

One constraint: Number of vertices, lower bound



- Start with half circle with positive x_1-values

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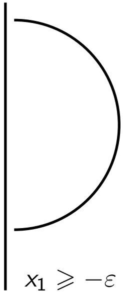

- Start with half circle with positive x_1-values
- Constraint is $x_1 \geqslant -\varepsilon$ for small $\varepsilon > 0$

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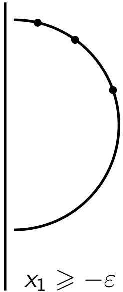

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- Distribute n points on upper half of half-circle

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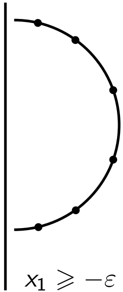
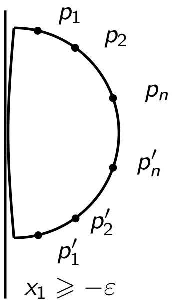


- Start with half circle with positive x_1-values
- Constraint is $x_1 \geqslant -\varepsilon$ for small $\varepsilon > 0$
- Distribute n points on upper half of half-circle
- Mirror points on the lower part

One constraint: Number of vertices, lower bound



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- Constraint is $x_1 \geqslant -\varepsilon$ for small $\varepsilon > 0$
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- Bend line-segment which is closing circle slightly to the left

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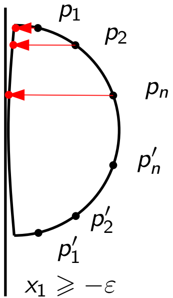

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- Bend line-segment which is closing circle slightly to the left
- q_i is vector orthogonal to x_2-axis such that $q_i + p_i$ and $q_i + p_i'$ are on curve.

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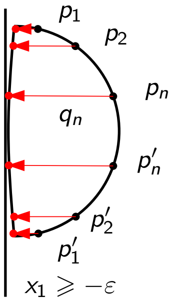

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- q_i is vector orthogonal to x_2-axis such that $q_i + p_i$ and $q_i + p'_i$ are on curve.
- $(\{p_1, \ldots, p_n, p'_1, \ldots, p'_n\} \oplus \{0, q_1, \ldots, q_n\})_{x_1 \geqslant -\varepsilon}$ has $4 \cdot n$ vertices $= |P| + 2 \cdot |Q| - 2$

One constraint: Number of vertices, tight upper bound

- Given: $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_m\}$ sorted w.r.t. first coordinate
- Constraint: $x_1 \geqslant 0$
- $J(i)$: smallest index j such that $p_i + q_j$ satisfies $x_1 \geqslant 0$.

Set $(P \oplus Q)_{x_1 \geqslant 0}$ is union of

$$\begin{array}{rcl} \{p_{\mathbf{1}}, \ldots, p_n\} & \oplus & \{q_{J(\mathbf{1})}, \ldots, q_m\} \\ \{p_{\mathbf{2}}, \ldots, p_n\} & \oplus & \{q_{J(\mathbf{2})}, \ldots, q_m\} \\ & \ldots & \\ \{p_{\mathbf{n}}\} & \oplus & \{q_{J(\mathbf{n})}, \ldots, q_m\} \end{array}$$

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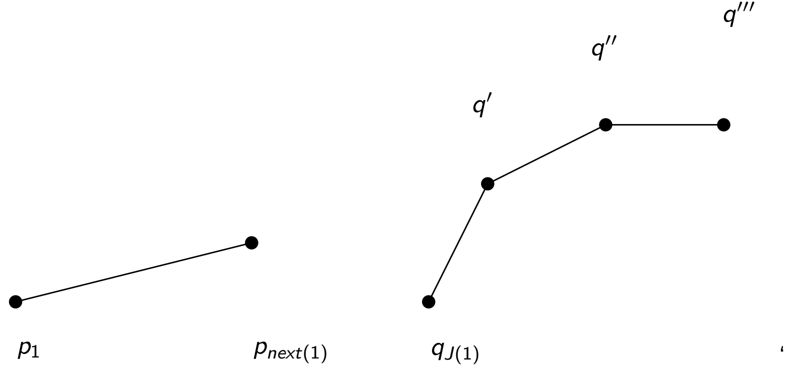
\ldots & & \\
\{p_{\mathbf{n}}\} & \oplus & \{q_{J(\mathbf{n})}, \ldots, q_m\}
\end{array}
$$

$J(i)$ is decreasing sequence

One constraint: Number of vertices, tight upper bound

Consider convex hull of

$$\{p_1, \ldots, p_n\} \oplus \{q_{J(1)}, \ldots, q_m\}$$



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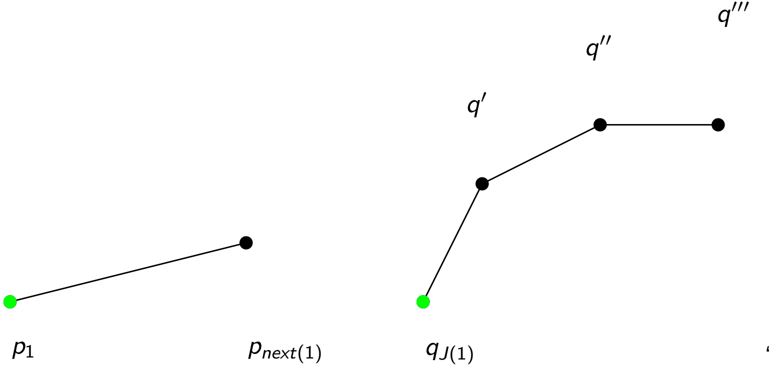

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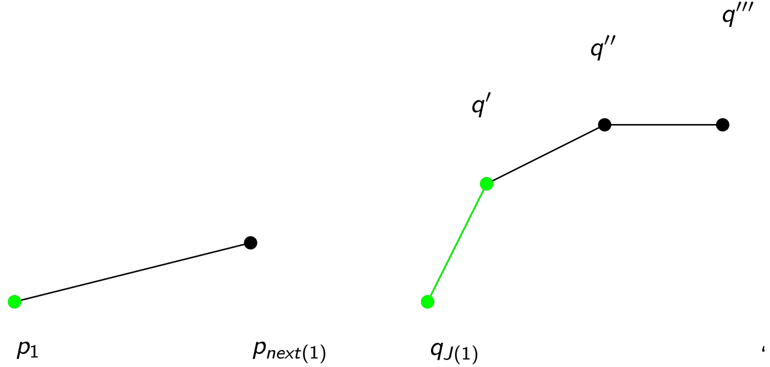

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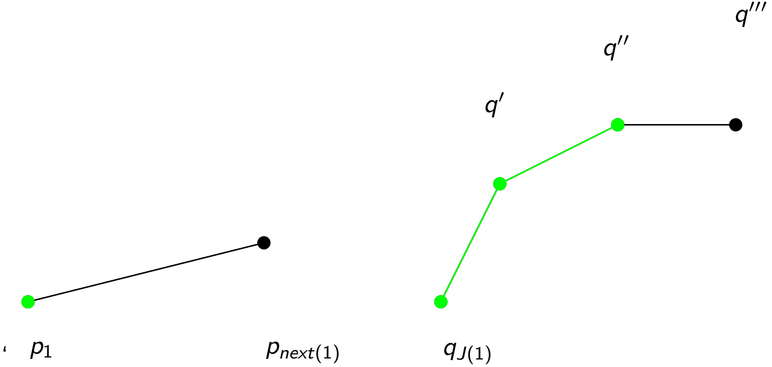

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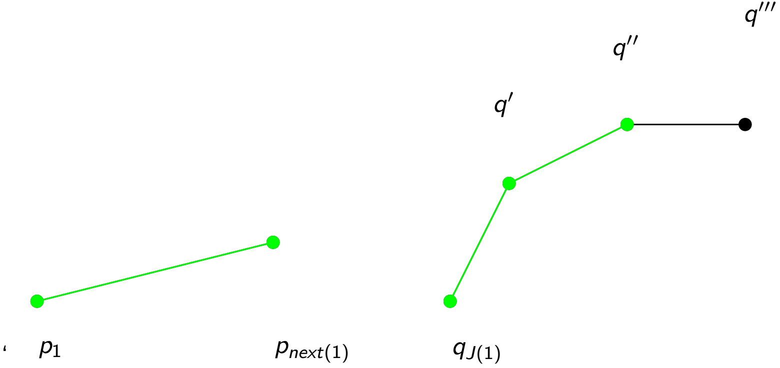

There is no vertex of the form $p + q'$ with $p \neq p_1$

One constraint: Number of vertices, tight upper bound

- If i-th block $\{p_{\mathbf{i}}, \ldots, p_n\} \oplus \{q_{J(\mathbf{i})}, \ldots, q_m\}$ creates $k_i + 2$ vertices involving p_i, then k_i points can be removed from Q.
- Yields bound $\min\{2 \cdot n + m, n + 2 \cdot m\} - 2$

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Number of vertices of $\text{conv}(U \oplus V)_{a^T x \geqslant \beta}$ is bounded by

$$\min\{2 \cdot |P| + |Q|, |P| + 2 \cdot |Q|\} - 2.$$

There are $P \subseteq \mathbb{R}^2$ and $Q \subseteq \mathbb{R}^2$ such that $\text{conv}(P \oplus Q)_{x_1 \geqslant 0}$ has $|P| + 2 \cdot |Q| - 2$ points.

One constraint: A linear algorithm

Given $P, Q \subseteq \mathbb{R}^2$ and a constraint $a^T x \geqslant \beta$ where P and Q are sorted w.r.t. $a^T x$, one can compute a $R \subseteq (P \oplus Q)_{a^T x \geqslant \beta}$ containing all vertices of $\operatorname{conv}\left((P \oplus Q)_{a^T x \geqslant \beta}\right)$ in linear time.

Two constraints

- Consider $(P \oplus Q)_{Ax \geqslant b}$ where $Ax \geqslant b$ consists of $a_1^T x \geqslant 0$ and $a_2^T x \geqslant b_2$
- Suppose that P and Q are sorted w.r.t. $a_1^T x$ and $a_2^T x$
- Notice that each subset of P and Q can be sorted w.r.t. $a_1^T x$ and $a_2^T x$ in time $O(N)$

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Divide and conquer algorithm:

- For $\gamma \in \mathbb{R}$ define

$$P_L = \{p \in P \mid a_1^T p < -\gamma\} \quad \text{and} \quad P_R = \{p \in P \mid a_1^T p > -\gamma\}$$
$$Q_L = \{q \in Q \mid a_1^T q < \gamma\} \quad \text{and} \quad Q_R = \{q \in Q \mid a_1^T Q > \gamma\}$$

- If $p \in P_L$ and $q \in Q_L$ then $a_1^T (p + q) < 0$
- Thus $P_L \oplus Q_L$ can be discarded
- $P_R \oplus Q_R$ satisfies $a_1^T x \geqslant 0$.

Two constraints: The algorithm

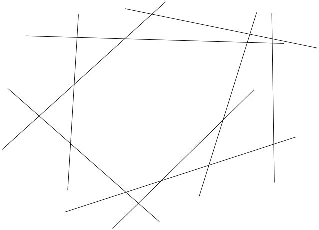
- Determine $\gamma \in \mathbb{R}$ such that $|P_L| + |Q_R| = N/2$
- Recursively compute sets containing vertices of $(P_L \oplus Q_R)_{Ax \geqslant b}$ and $(P_R \oplus Q_L)_{Ax \geqslant b}$
- Compute set containing vertices of $(P_R \oplus Q_R)_{a_2^T x \geqslant \beta}$

Running time:

$$\begin{aligned} T(N) &= 2T(n/2) + O(n) \\ &= O(n \log n) \end{aligned}$$

Given $P, Q \subseteq \mathbb{R}^2$ and two constraints $Ax \geqslant b$ one can compute a subset $R \subseteq (P \oplus Q)_{Ax \geqslant b}$ containing all vertices of $\operatorname{conv}((P \oplus Q)_{Ax \geqslant b})$ in time $O(n \log n)$.

A fixed nummber of constraints



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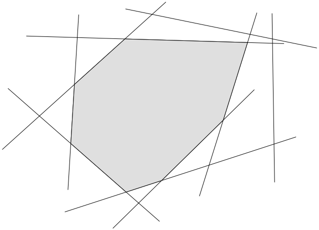

- Compute polygon defined by k constraints $O(k \log k)$

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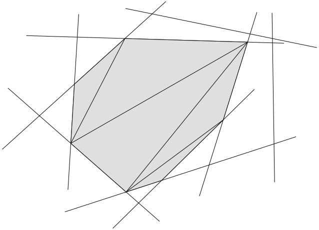

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- Triangulate polygon into k triangles

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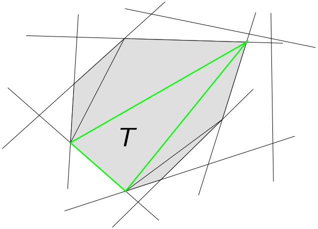

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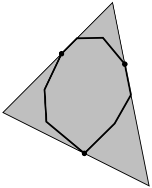
Remains to be shown:
A superset of vertices of $(P \oplus Q) \cap T$ for triangle T can be computed in time $O(N \log N)$

Triangles



- If each edge of T contains point of $P \oplus Q$, then vertices of three cones contain vertices of triangle

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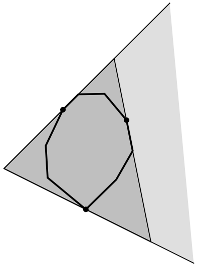

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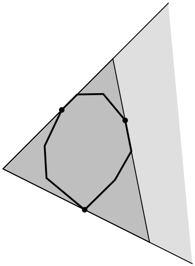
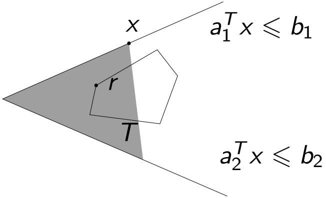


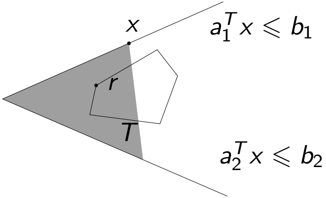
- If each edge of T contains point of $P \oplus Q$, then vertices of three cones contain vertices of triangle
- Goal: Transform T into triangle T' such that:
  - i) $(P \oplus Q) \cap T' = (P \oplus Q) \cap T$.
  - ii) Each edge of T' contains a point of $(P \oplus Q)$.

Triangle: Reducing to two constraints



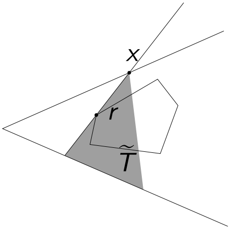
- Consider cone C

Triangle: Reducing to two constraints



- Consider cone C
- Compute a set $R \subseteq (P \oplus Q)$ containing all vertices of $(P \oplus Q) \cap C$

Triangle: Reducing to two constraints



- Consider cone C
- Compute a set $R \subseteq (P \oplus Q)$ containing all vertices of $(P \oplus Q) \cap C$
- Rotate "upper" constraint $a^T x \leqslant \beta$ inwards

Main result

- Given $P \subseteq \mathbb{R}^2$ and $Q \subseteq \mathbb{R}^2$ with $|P| + |Q| = N$, and $Ax \geqslant b$ with $A \in \mathbb{R}^{k \times 2}$ one can compute a set $R \subseteq (P \oplus Q)_{Ax \geqslant b}$ containing all vertices of $(P \oplus Q)_{Ax \geqslant b}$ in time $O(k \log k + k \cdot N \log N)$
- A quasi-convex function can be maximized over $(P \oplus Q)_{Ax \geqslant b}$ in time $O(k \log k + k \cdot N \log N)$

Subsequence problems

- Points corresponding to intervals have first component in $\{1, \ldots, n\}$
- Constraint $x_1 \geqslant 0$ can be "sorted in linear time"
- Length-constrained heaviest segment, DNA copy number data analysis and many other subsequence problems from computational biology can be computed in linear time

Open problems

- How many vertices has constrained Minkowski sum with two constraints.

  Our bound is $O(N \log N)$

- More generally: Let $S \subseteq P \oplus Q$ be arbitrary subset. How many vertices has $\text{conv}(S)$?

  $O(N^{3/2})$ is best upper bound.