Carathéodory Bounds for Integer Cones

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Joint work with G. Shmonin
**Carathéodory’s Theorem**

\[ X \subseteq \mathbb{R}^n: \]
\[ \text{cone}(X) = \{ \lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \geq 0; x_1, \ldots, x_t \in X; \lambda_1, \ldots, \lambda_t \geq 0 \} \]

**Theorem.** *X \subseteq \mathbb{R}^n* and \( b \in \text{cone}(X) \) *then there exists* \( \tilde{X} \subseteq X \) *with* \( |\tilde{X}| \leq n \) *and* \( b \in \text{cone}(\tilde{X}) \).
Carathéodory’s Theorem

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**Theorem.** $X \subseteq \mathbb{R}^n$ and $b \in \text{cone}(X)$ then there exists $\tilde{X} \subseteq X$ with $|\tilde{X}| \leq n$ and $b \in \text{cone}(\tilde{X})$.

**Proof.**

- $b = \sum_{i=1}^{t} \lambda_i \cdot x_i$ where $x_1, \ldots, x_t$ linear dependent and $\lambda_i > 0$ for $i = 1, \ldots, t$.
- $\exists \mu \in \mathbb{R}^t \setminus \{0\}$ s.t. $\sum_{i=1}^{t} \mu_i \cdot x_i = 0$ with at least one positive entry.
- $b = \sum_{i=1}^{t} (\lambda_i - \varepsilon \mu_i) \cdot x_i$
- Choose $\varepsilon = \min \{ \lambda_i / \mu_i \mid \mu_i > 0 \}$.
- For index $j$ where min is attained one has $(\lambda_j - \varepsilon \mu_j) = 0$. 

$\square$
Linear Programming

Carathéodory + Complementary Slackness:

- \( \min \{ c^T x \mid Ax = b, x \geq 0 \} \), \( A \in \mathbb{R}^{d \times n} \) feasible and bounded has optimal solution with at most \( d \) nonzero entries.
Linear Programming

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Cutting Stock or Bin Packing

Input:

- \( a \in \mathbb{Q}_{>0}^d : a(i) < 1 \) size of items of type \( i \)
- \( b \in \mathbb{N}^d : b(i) \) number of items of type \( i \)

Task:

- Compute minimum number of bins needed to pack all items
Pattern: Integer solution to knapsack constraint

\[ a^T x \leq 1, \ x \geq 0 \]

\( \mathcal{P} \): Set of all patterns

Gilmore Gomory formulation:

\[
\begin{align*}
\min & \quad 1^T \lambda \\
\text{s.t} & \quad \sum_{x \in \mathcal{P}} \lambda_x x = b, \\
& \quad \lambda \geq 0 \text{ integral},
\end{align*}
\]

(Gilmore & Gomory 1961)
Problems and results related to cutting stock

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- Is Cutting Stock in NP? (Marcotte 1986)
- Problem polynomial if \( d = 2 \) (McCormick, Smallwood & Spieksma 2001)
- If \( d \) is fixed, does there exist an optimal solution with a constant number of patterns?
- If \( d \) fixed, is problem solvable in polynomial time?
- Is \( OPT \leq \lceil LP \rceil + 1 \) ? MIRUP property
- \( OPT \leq LP + O(\log^2 d) \) (Karmakar & Karp 1980)
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- Is Cutting Stock in NP? (Marcotte 1986) **Yes!**
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- If \( d \) is fixed, does there exist an optimal solution with a constant number of patterns? **Yes!**
- If \( d \) fixed, is problem solvable in polynomial time? **Open!**
- Is \( OPT \leq \lceil LP \rceil + 1 \) ? **MIRUP property** **Open!**
- \( OPT \leq LP + O(\log^2 d) \) (Karmakar & Karp 1980)
Integer Cones

$X \subseteq \mathbb{Z}^n$:
\[
\text{int}_{\text{cone}}(X) = \{ \lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \geq 0; x_1, \ldots, x_t \in X; \lambda_1, \ldots, \lambda_t \in \mathbb{N} \}
\]

$X \subseteq \mathbb{Z}^n$ and $b \in \text{int}_{\text{cone}}(X)$, what is size of smallest subset $\tilde{X} \subseteq X$ such that $b \in \text{int}_{\text{cone}}(\tilde{X})$?
Integer Analogues of Carathéodory’s Theorem

$X \subseteq \mathbb{Z}^d$ an integral Hilbert basis if $\text{cone}(X) \cap \mathbb{Z}^d = \text{int}_\text{cone}(X)$

$X$ Hilbert basis and $\text{cone}(X)$ pointed:

- $|\tilde{X}| \leq 2d - 1$ (Cook, Fonlupt & Schrijver 1986)
- $|\tilde{X}| \leq 2d - 2$ (Sebő 1990)
- $|\tilde{X}| \leq d$ disproved (Bruns, Gubeladze, Henk, Martin, Weismantel 1999)
- What is the largest $\varepsilon$ such that $|\tilde{X}| \leq (2 - \varepsilon)d$?
Integer Analogues of Carathéodory’s Theorem

\[ X \subseteq \mathbb{Z}^d \text{ an integral Hilbert basis if } \text{cone}(X) \cap \mathbb{Z}^d = \text{int}_\text{cone}(X) \]

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- What is the largest \( \varepsilon \) such that \( |\tilde{X}| \leq (2 - \varepsilon) d \)? Open!
This talk

\( X \subset \mathbb{Z}^d \) and \( b \in \text{int}_{\text{cone}}(X) \).

(i) If all vectors in \( X \) are nonnegative, then \( |\tilde{X}| \leq \text{size}(b) \).

(ii) If \( M = \max_{x \in X} \|x\|_{\infty} \), then \( |\tilde{X}| \leq 2d \log(4dM) \).

(iii) If \( X \) is closed under convex combinations then \( |\tilde{X}| \leq 2^d \).
(i): $X$ nonnegative

- $b = \sum_{x \in X} \lambda_x x$ where $\lambda_x > 0$ for all $x \in X$.
- If $2^{|X|} > \prod_{i=1}^{d} (b_i + 1)$, there exist two disjoint subsets $A, B \subseteq X$, $A \neq B$, with
  $$\sum_{x \in A} x = \sum_{x \in B} x.$$  
- Suppose that $A \neq \emptyset$ and set $\lambda = \min\{\lambda_x : x \in A\}$. 
Rewrite

\[
\sum_{x \in X} \lambda_x x = \sum_{x \in X \setminus A} \lambda_x x + \sum_{x \in A} \lambda_x x
\]

\[
= \sum_{x \in X \setminus A} \lambda_x x + \sum_{x \in A} (\lambda_x - \lambda)x + \lambda \sum_{x \in A} x
\]

\[
= \sum_{x \in X \setminus A} \lambda_x x + \sum_{x \in A} (\lambda_x - \lambda)x + \lambda \sum_{x \in B} x
\]

\[
= \sum_{x \in X} \mu_x x,
\]

- \( \mu_x = \lambda_x \) if \( x \in X \setminus (A \cup B) \), \( \mu_x = \lambda_x + \lambda \) if \( x \in B \) and \( \mu_x = \lambda_x - \lambda \) if \( x \in A \)
- At least one \( \mu_x, x \in A \) is zero
- \( |X| \) minimal, then \( 2^{|X|} \leq \prod_{i=1}^{d} (b_i + 1) \implies |X| \leq \text{size}(b) \)
(ii): $X$ arbitrary

- $n = |X|$.
- Suppose $n > d \log(2n \max_{x \in X} \|x\|_\infty + 1)$.
- $\tilde{X} \subseteq X \implies \|\sum_{x \in \tilde{X}} x\|_\infty$ is bounded by $n \max_{x \in X} \|x\|_\infty$.
- Number of different vectors representable as sum of vectors of subset $\tilde{X}$ of $X$ is bounded by
  \[
  (2n \max_{x \in X} \|x\|_\infty + 1)^d
  \]
- By our assumption: $2^n > (2n \max_{x \in X} \|x\|_\infty + 1)^d$.
- There exist two subsets $A, B \subseteq X$, $A \neq B$, with $\sum_{x \in A} x = \sum_{x \in B} x$ and proceed as in the previous proof.
- $|\tilde{X}| \leq 2d \log(4dM)$
(iii): $X$ closed under convex combinations

- $\text{conv}(X) \cap \mathbb{Z}^d = X$
- $b = \sum_{x \in X} \lambda_x x$, potential of representation is
  \[ \sum_{x \in X} \lambda_x \| (1_x) \| \]
- Suppose representation $b = \sum_{x \in X} \lambda_x x$ has minimal potential.
- Suppose $2^d + 1$ of $\lambda_x$ strictly positive
- There exist $x_1$ and $x_2$ with $x_1 \equiv x_2 \pmod{2}$
- $X$ is closed under convex combinations $\implies 1/2(x_1 + x_2) \in X$. 
(iii): $X$ closed under convex combinations

- $\lambda x_1 x_1 + \lambda x_2 x_2 = (\lambda x_1 - \lambda x_2)x_1 + 2\lambda x_2 (1/2(x_1 + x_2))$

- Since $(\frac{1}{x_1})$ and $(\frac{1}{x_2})$ are not co-linear, we have

  $$(\lambda x_1 - \lambda x_2)\|\left(\frac{1}{x_1}\right)\| + 2\lambda x_2 \|\left(\frac{1}{1/2(x_1+x_2)}\right)\|$$

  $$= (\lambda x_1 - \lambda x_2)\|\left(\frac{1}{x_1}\right)\| + \lambda x_2 \|\left(\frac{1}{x_1}\right)\| + \|\left(\frac{1}{x_2}\right)\|$$

  $$< (\lambda x_1 - \lambda x_2)\|\left(\frac{1}{x_1}\right)\| + \lambda x_2 \left(\|\left(\frac{1}{x_1}\right)\| + \|\left(\frac{1}{x_2}\right)\|\right)$$

  $$= \lambda x_1 \|\left(\frac{1}{x_1}\right)\| + \lambda x_2 \|\left(\frac{1}{x_2}\right)\|.$$

- **Contradiction** to minimality of potential.
Theorem. Let $a, b \in \mathbb{Z}_{\geq 0}^d$ and $M \in \mathbb{Z}_{>0}$ be an instance of the cutting stock problem. Then there exists an optimal solution which uses at most $\min\{2 \cdot \text{size}(b), 2d \log(4dM), 2^d\}$ patterns.
Theorem. Given IP \( \min \{ c^T y \mid A y = b, y \geq 0, y \text{ integer} \} \), where \( A \in \mathbb{Z}^{d \times n} \) and \( c \in \mathbb{Z}^n \) with with optimal value \( \gamma \). There exists optimal solution \( y^* \in \mathbb{Z}^m_{\geq 0} \) which satisfies

(i) The number of nonzero components of \( y^* \) is at most \( \text{size}(b) + \text{size}(\gamma) \), if \( A \) and \( c \) are nonnegative.

(ii) The number of nonzero components of \( y^* \) is at most \( 2(d + 1)(\log(d + 1) + s + 2) \), where \( s \) is the largest size of a coefficient of \( A \) and \( c \).
Cutting Stock: Two big open problems

\[
\begin{align*}
\text{min} & \quad 1^T \lambda \\
\text{s.t} & \quad \sum_{x \in P} \lambda_x x = b, \\
& \quad \lambda \geq 0 \text{ integral},
\end{align*}
\]

- Is \(OPT \leq \lceil LP \rceil + 1\)? **MIRUP property**
  Proved up to \(d \leq 7\) (Sebő & Shmonin 2006),
  \(OPT \leq \lceil LP \rceil + O(\log d)^2\) (Karmakar & Karp 1980)

- If \(d\) is constant, can the problem be solved in **polynomial time**?