Carathéodory Bounds for Integer Cones

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Joint work with G. Shmonin

Carathéodory's Theorem

$X \subseteq \mathbb{R}^n$:

$\operatorname{cone}(X) = \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \geqslant 0; x_1, \ldots, x_t \in X; \lambda_1, \ldots, \lambda_t \geqslant 0\}$

Theorem. $X \subseteq \mathbb{R}^n$ *and* $b \in \operatorname{cone}(X)$ *then there exists* $\widetilde{X} \subseteq X$ *with* $|\widetilde{X}| \leqslant n$ *and* $b \in \operatorname{cone}(\widetilde{X})$.

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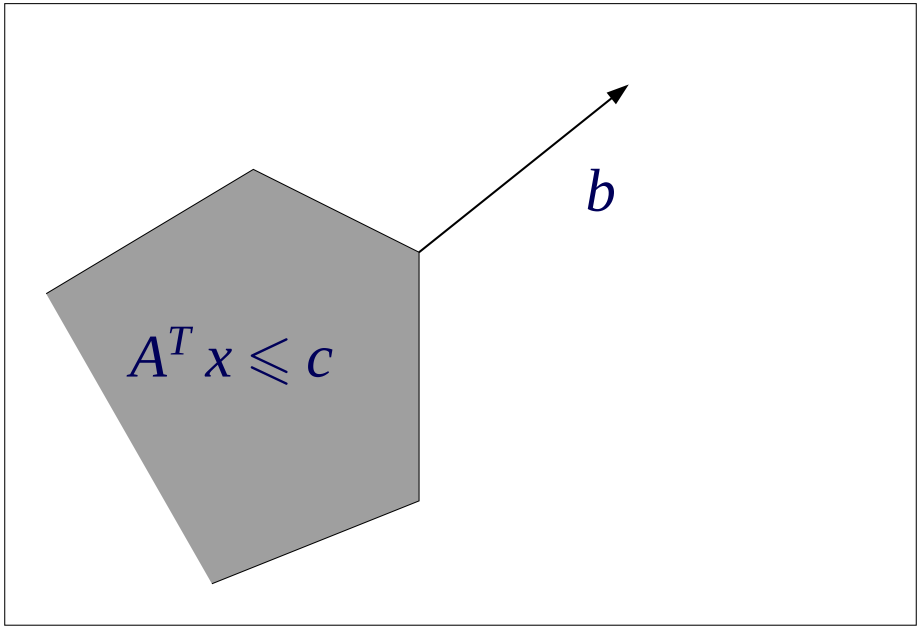
Theorem. *$X \subseteq \mathbb{R}^n$ and $b \in \mathrm{cone}(X)$ then there exists $\widetilde{X} \subseteq X$ with $|\widetilde{X}| \leqslant n$ and $b \in \mathrm{cone}(\widetilde{X})$.*

Proof.

- $b = \sum_{i=1}^{t} \lambda_i \cdot x_i$ where $x_1, \ldots, x_t$ linear dependent and $\lambda_i > 0$ for $i = 1, \ldots, t$.
- $\exists \mu \in \mathbb{R}^t \setminus \{0\}$ s.t. $\sum_{i=1}^{t} \mu_i \cdot x_i = 0$ with at least one positive entry.
- $b = \sum_{i=1}^{t} (\lambda_i - \varepsilon \mu_i) \cdot x_i$
- Choose $\varepsilon = \min\{\lambda_i / \mu_i \mid \mu_i > 0\}$.
- For index j where min is attained one has $(\lambda_j - \varepsilon \mu_j) = 0$.

$\square$

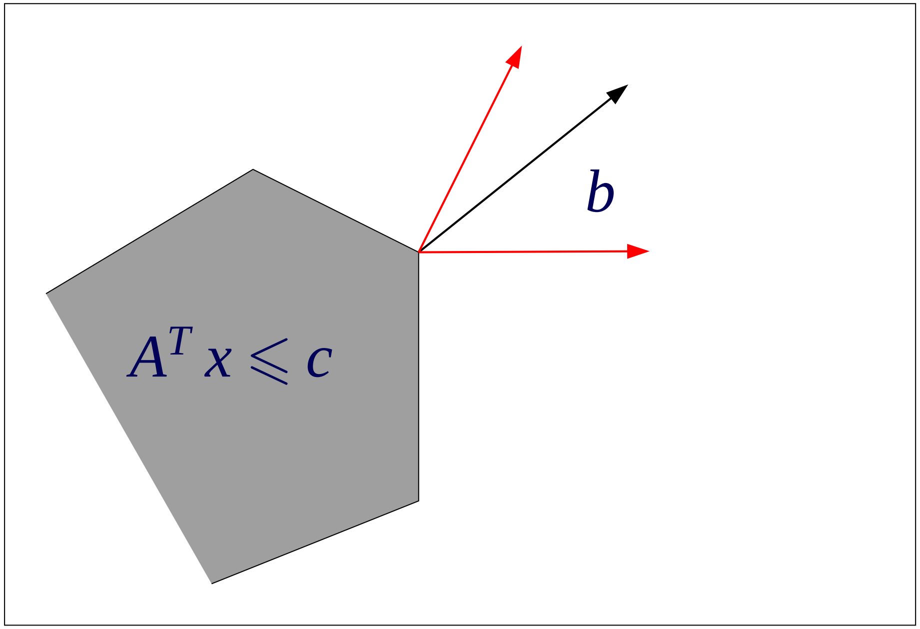
Linear Programming



Carathéodory + Complementary Slackness:

- $\min\{c^T x \mid Ax = b, x \geqslant 0\}, A \in \mathbb{R}^{d \times n}$ feasible and bounded has optimal solution with at most d nonzero entries.

Linear Programming



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Cutting Stock or Bin Packing

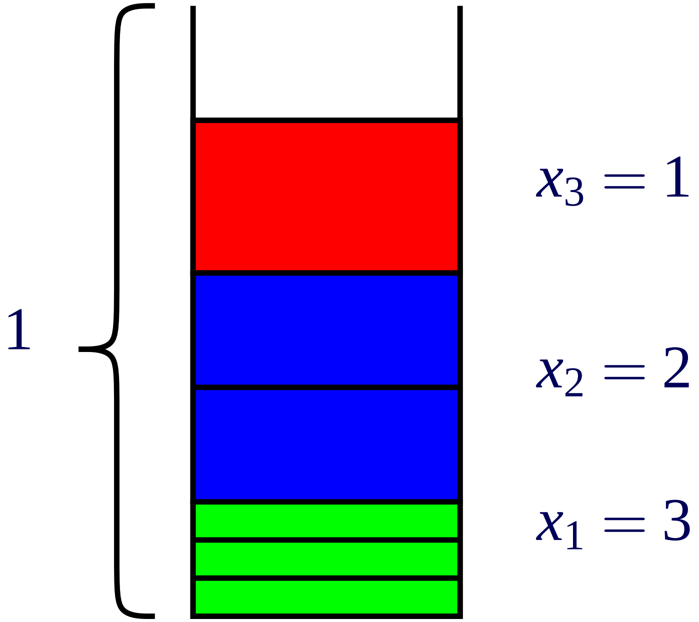
Input:

- $a \in \mathbb{Q}^d_{>0}$: $a(i) < 1$ size of items of type i
- $b \in \mathbb{N}^d$: $b(i)$ number of items of type i

Task:

- Compute minimum number of bins needed to pack all items

Integer Programming Formulation



- Pattern: Integer solution to knapsack constraint

$$a^T x \leqslant 1, x \geqslant 0$$

- $\mathcal{P}$: Set of all patterns
- Gilmore Gomory formulation:

$$\begin{array}{ll} \min & \mathbf{1}^T \lambda \\ \text{s.t} & \sum_{x \in \mathcal{P}} \lambda_x x = b, \\ & \lambda \geqslant 0 \text{ integral}, \end{array}$$

(Gilmore & Gomory 1961)

Problems and results related to cutting stock

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- Is Cutting Stock in NP **?** (Marcotte 1986)
- Problem polynomial if $d = 2$ (McCormick, Smallwood & Spieksma 2001)
- If d is fixed, does there exist an optimal solution with a constant number of patterns **?**
- If d fixed, is problem solvable in polynomial time **?**
- Is $OPT \leqslant \lceil LP \rceil + 1$ **?** MIRUP property
- $OPT \leqslant LP + O(\log^2 d)$ (Karmakar & Karp 1980)

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- Is Cutting Stock in NP **?** (Marcotte 1986) **Yes!**
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- If d is fixed, does there exist an optimal solution with a constant number of patterns **?** **Yes!**
- If d fixed, is problem solvable in polynomial time **?** **Open!**
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- $OPT \leqslant LP + O(\log^2 d)$ (Karmakar & Karp 1980)

Integer Cones

$X \subseteq \mathbb{Z}^n$:
$\mathrm{int\_cone}(X) = \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \geqslant 0; x_1, \ldots, x_t \in X; \lambda_1, \ldots, \lambda_t \in \mathbb{N}\}$

$X \subseteq \mathbb{Z}^n$ and $b \in \mathrm{int\_cone}(X)$, what is size of smallest subset $\widetilde{X} \subseteq X$ such that $b \in \mathrm{int\_cone}(\widetilde{X})$?

Integer Analogues of Carathéodory's Theorem

$X \subseteq \mathbb{Z}^d$ an integral Hilbert basis if $\mathrm{cone}(X) \cap \mathbb{Z}^d = \mathrm{int\_cone}(X)$

X Hilbert basis and $\mathrm{cone}(X)$ pointed:

- $|\widetilde{X}| \leqslant 2d - 1$ (Cook, Fonlupt & Schrijver 1986)
- $|\widetilde{X}| \leqslant 2d - 2$ (Sebő 1990)
- $|\widetilde{X}| \leqslant d$ disproved (Bruns, Gubeladze, Henk, Martin, Weismantel 1999)
- What is the largest ε such that $|\widetilde{X}| \leqslant (2 - \varepsilon)d$?

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This talk

$X \subset \mathbb{Z}^d$ and $b \in \mathrm{int\_cone}(X)$.

(i) If all vectors in X are nonnegative, then $|\widetilde{X}| \leqslant \mathrm{size}(b)$.

(ii) If $M = \max_{x \in X} \|x\|_\infty$, then $|\widetilde{X}| \leqslant 2d\log(4dM)$.

(iii) If X is closed under convex combinations then $|\widetilde{X}| \leqslant 2^d$

(i): X nonnegative

- $b = \sum_{x \in X} \lambda_x x$ where $\lambda_x > 0$ for all $x \in X$.
- If $2^{|X|} > \prod_{i=1}^{d}(b_i + 1)$, there exist two disjoint subsets $A, B \subseteq X$, $A \neq B$, with
$$\sum_{x \in A} x = \sum_{x \in B} x.$$
- Suppose that $A \neq \emptyset$ and set $\lambda = \min\{\lambda_x : x \in A\}$.

Rewrite

$$\begin{aligned}
\sum_{x\in X}\lambda_x x &= \sum_{x\in X\setminus A}\lambda_x x+\sum_{x\in A}\lambda_x x \\
&= \sum_{x\in X\setminus A}\lambda_x x+\sum_{x\in A}(\lambda_x-\lambda)x+\lambda\sum_{x\in A}x \\
&= \sum_{x\in X\setminus A}\lambda_x x+\sum_{x\in A}(\lambda_x-\lambda)x+\lambda\sum_{x\in B}x \\
&= \sum_{x\in X}\mu_x x,
\end{aligned}$$

- $\mu_x=\lambda_x$ if $x\in X\setminus(A\cup B)$, $\mu_x=\lambda_x+\lambda$ if $x\in B$ and $\mu_x=\lambda_x-\lambda$ if $x\in A$
- At least one $\mu_x, x\in A$ is zero
- $|X|$ minimal, then $2^{|X|}\leqslant\prod_{i=1}^{d}(b_i+1)\implies |X|\leqslant\mathrm{size}(b)$

(ii): X arbitrary

- $n = |X|$.
- Suppose $n > d \log(2n \max_{x \in X} \|x\|_\infty + 1)$.
- $\widetilde{X} \subseteq X \implies \|\sum_{x \in \widetilde{X}} x\|_\infty$ is bounded by $n \max_{x \in X} \|x\|_\infty$
- Number of different vectors representable as sum of vectors of subset $\widetilde{X}$ of X is bounded by

  $$(2\, n \max_{x \in X} \|x\|_\infty + 1)^d$$

  .
- By our assumption: $2^n > (2n \max_{x \in X} \|x\|_\infty + 1)^d$.
- There exist two subsets $A, B \subseteq X$, $A \neq B$, with $\sum_{x \in A} x = \sum_{x \in B} x$ and proceed as in the previous proof.
- $|\widetilde{X}| \leqslant 2\, d \log(4\, d\, M)$

(iii): X closed under convex combinations

- $\mathrm{conv}(X) \cap \mathbb{Z}^d = X$
- $b = \sum_{x \in X} \lambda_x x$, potential of representation is

$$\sum_{x \in X} \lambda_x \left\| \binom{1}{x} \right\|$$

- Suppose representation $b = \sum_{x \in X} \lambda_x x$ has minimal potential.
- Suppose $2^d + 1$ of λ_x strictly positive
- There exist x_1 and x_2 with $x_1 \equiv x_2 \pmod 2$
- X is closed under convex combinations $\Longrightarrow 1/2(x_1 + x_2) \in X$.

(iii): X closed under convex combinations

- $\lambda_{x_1}x_1 + \lambda_{x_2}x_2 = (\lambda_{x_1} - \lambda_{x_2})x_1 + 2\lambda_{x_2}(1/2(x_1 + x_2))$
- Since $\binom{1}{x_1}$ and $\binom{1}{x_2}$ are not co-linear, we have

$$\begin{aligned} (\lambda_{x_1} - \lambda_{x_2})\|\tbinom{1}{x_1}\| &+ 2\lambda_{x_2}\|\tbinom{1}{1/2(x_1+x_2)}\| \\ &= (\lambda_{x_1} - \lambda_{x_2})\|\tbinom{1}{x_1}\| + \lambda_{x_2}\|\tbinom{1}{x_1} + \tbinom{1}{x_2}\| \\ &< (\lambda_{x_1} - \lambda_{x_2})\|\tbinom{1}{x_1}\| + \lambda_{x_2}\left(\|\tbinom{1}{x_1}\| + \|\tbinom{1}{x_2}\|\right) \\ &= \lambda_{x_1}\|\tbinom{1}{x_1}\| + \lambda_{x_2}\|\tbinom{1}{x_2}\|. \end{aligned}$$

- Contradiction to minimality of potential.

Cutting Stock

Theorem. *Let* $a, b \in \mathbb{Z}^d_{\geqslant 0}$ *and* $M \in \mathbb{Z}_{>0}$ *be an instance of the cutting stock problem. Then there exists an optimal solution which uses at most* $\min\{2\,\mathrm{size}(b), 2\,d\,\log(4\,d\,M), 2^d\}$ *patterns.*

Integer Programming

Theorem. *Given IP* $\min\{c^T y \mid Ay = b, y \geqslant 0, y \text{ integer}\}$, *where* $A \in \mathbb{Z}^{d\times n}$ *and* $c \in \mathbb{Z}^n$ *with with optimal value* γ. *There exists optimal solution* $y^* \in \mathbb{Z}^m_{\geqslant 0}$ *which satisfies*

(i) *The number of nonzero components of* y^* *is at most* $\mathrm{size}(b)$ + $\mathrm{size}(\gamma)$, *if* A *and* c *are nonnegative.*

(ii) *The number of nonzero components of* y^* *is at most* $2\,(d+1)\,(\log(d+1)+s+2)$, *where* s *is the largest size of a coefficient of* A *and* c.

Cutting Stock: Two big open problems

$$\begin{array}{ll} \min & \mathbf{1}^T\lambda \\ \text{s.t} & \sum_{x\in\mathcal{P}}\lambda_x x = b, \\ & \lambda \geqslant 0 \text{ integral,} \end{array}$$

- Is $OPT \leqslant \lceil LP \rceil + 1$? MIRUP property
  Proved up to $d \leqslant 7$ (Sebő & Shmonin 2006),
  $OPT \leqslant \lceil LP \rceil + O(\log d)^2$ (Karmakar & Karp 1980)
- If d is constant, can the problem be solved in polynomial time ?