Derived categories, resolutions, and Brown representability

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Abstract. These notes are based on a series of five lectures given during the summer school "Interactions between Homotopy Theory and Algebra" held at the University of Chicago in 2004.

Contents

Introduction

Derived categories were introduced in the 1960s by Grothendieck and his school. A derived category is defined for any abelian category and the idea is quite simple. One takes as objects all complexes, and the usual maps between complexes are modified by inverting all maps which induce an isomorphism in cohomology.

This definition is easily stated but a better description of the maps is needed. To this end some extra structure is introduced. A derived category carries a triangulated structure which complements the abelian structure of the underlying category of complexes.

The next step is to replace a complex by an injective or projective resolution, assuming that the underlying abelian category provides enough injective or projective objects. The construction of such resolutions requires some machinery. Here, we use the Brown representability theorem which characterizes the representable functors on a triangulated category. The proof of the Brown representability theorem is based on an embedding of a triangulated category into some abelian category. This illustrates the interplay between abelian and triangulated structures.

The classical context for doing homological algebra is the module category over some associative algebra. Here, we work more generally over differential graded algebras or, even more generally, over differential graded categories. Again, we prove that complexes in such module categories can be replaced by projective or injective resolutions.

There is a good reason for studying the derived category of a differential graded category. It turns out that every triangulated category which arises from algebraic constructions embeds into such a derived category. This leads to the notion of an algebraic triangulated category which can be defined axiomatically.

These notes are based on a series of five lectures given during a summer school in Chicago in 2004. The prerequisites for these lectures are quite modest, consisting of the basic notions from homological algebra and some experience with examples. The latter should compensate for the fact that specific examples are left out in order to keep the exposition concise and self-contained.

Students and colleagues in Paderborn helped with numerous questions and suggestions to prepare these notes; it is a pleasure to thank all of them. Also, Apostolos Beligiannis and Srikanth Iyengar provided many helpful comments. In addition, I am grateful to the organizers of the summer school: Lucho Avramov, Dan Christensen, Bill Dwyer, Mike Mandell, and most prominently, Brooke Shipley. Last but not least, I would like to thank the participants of the summer school for their interest and enthusiasm.

1. Derived categories

The derived category $\mathbf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ provides a framework for studying the homological properties of $\mathcal{A}$. The main idea is to replace objects in $\mathcal{A}$ by complexes, and to invert maps between complexes if they induce an isomorphism in cohomology. The actual construction of the derived category proceeds in several steps which is reflected by the following sequence of functors.

$$\mathcal{A} \longrightarrow \mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{K}(\mathcal{A}) \longrightarrow \mathbf{D}(\mathcal{A})$$

1.1. Additive and abelian categories. A category $\mathcal{A}$ is *additive* if every finite family of objects has a product, each set $\operatorname{Hom}_{\mathcal{A}}(A,B)$ is an abelian group, and the composition maps

$$\operatorname{Hom}_{\mathcal{A}}(A,B) \times \operatorname{Hom}_{\mathcal{A}}(B,C) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(A,C)$$

sending a pair (ϕ,ψ) to $\psi\circ\phi$ are bilinear.

An additive category $\mathcal{A}$ is *abelian*, if every map $\phi\colon A\to B$ has a kernel and a cokernel, and if the canonical factorization

$$\begin{array}{ccccccc} \operatorname{Ker}\phi & \xrightarrow{\phi'} & A & \xrightarrow{\phi} & B & \xrightarrow{\phi''} & \operatorname{Coker}\phi \\ & & \downarrow & & \uparrow & & \\ & & \operatorname{Coker}\phi' & \xrightarrow{\bar{\phi}} & \operatorname{Ker}\phi'' & & \end{array}$$

of ϕ induces an isomorphism $\bar{\phi}$.

Example. The category $\operatorname{Mod}\Lambda$ of right modules over an associative ring Λ is an abelian category.

1.2. Categories of complexes. Let $\mathcal{A}$ be an additive category. A *complex* in $\mathcal{A}$ is a sequence of maps

$$\cdots \longrightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \longrightarrow \cdots$$

such that $d^n \circ d^{n-1} = 0$ for all $n \in \mathbb{Z}$. We denote by $\mathbf{C}(\mathcal{A})$ the category of complexes, where a map $\phi\colon X \to Y$ between complexes consists of maps $\phi^n\colon X^n \to Y^n$ with $d_Y^n \circ \phi^n = \phi^{n+1} \circ d_X^n$ for all $n \in \mathbb{Z}$.

A map $\phi\colon X \to Y$ is *null-homotopic* if there are maps $\rho^n\colon X^n \to Y^{n-1}$ such that $\phi^n = d_Y^{n-1} \circ \rho^n + \rho^{n+1} \circ d_X^n$ for all $n \in \mathbb{Z}$. The null-homotopic maps form an *ideal* $\mathcal{I}$ in $\mathbf{C}(\mathcal{A})$, that is, for each pair X, Y of complexes a subgroup

$$\mathcal{I}(X, Y) \subseteq \operatorname{Hom}_{\mathbf{C}(\mathcal{A})}(X, Y)$$

such that any composition $\psi \circ \phi$ of maps in $\mathbf{C}(\mathcal{A})$ belongs to $\mathcal{I}$ if ϕ or ψ belongs to $\mathcal{I}$. The *homotopy category* $\mathbf{K}(\mathcal{A})$ is the quotient of $\mathbf{C}(\mathcal{A})$ with respect to this ideal. Thus

$$\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(X, Y) = \operatorname{Hom}_{\mathbf{C}(\mathcal{A})}(X, Y)/\mathcal{I}(X, Y)$$

for every pair of complexes X, Y.

Now suppose that $\mathcal{A}$ is abelian. Then one defines for a complex X and each $n \in \mathbb{Z}$ the *cohomology*

$$H^n X = \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}.$$

A map $\phi\colon X \to Y$ between complexes induces a map $H^n\phi\colon H^n X \to H^n Y$ in each degree, and ϕ is a *quasi-isomorphism* if $H^n\phi$ is an isomorphism for all $n \in \mathbb{Z}$. Note that two maps $\phi, \psi\colon X \to Y$ induce the same map $H^n\phi = H^n\psi$, if $\phi - \psi$ is null-homotopic.

The *derived category* $\mathbf{D}(\mathcal{A})$ of $\mathcal{A}$ is obtained from $\mathbf{K}(\mathcal{A})$ by formally inverting all quasi-isomorphisms. To be precise, one defines

$$\mathbf{D}(\mathcal{A}) = \mathbf{K}(\mathcal{A})[S^{-1}]$$

as the localization of $\mathbf{K}(\mathcal{A})$ with respect to the class S of all quasi-isomorphisms.

1.3. Localization. Let $\mathcal{C}$ be a category and S be a class of maps in $\mathcal{C}$. The *localization* of $\mathcal{C}$ with respect to S is a category $\mathcal{C}[S^{-1}]$, together with a functor $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ such that

(L1) $Q\sigma$ is an isomorphism for all σ in S, and
(L2) any functor $F\colon \mathcal{C} \to \mathcal{D}$ such that $F\sigma$ is an isomorphism for all σ in S factors uniquely through Q.

Ignoring set-theoretic problems, one can show that such a localization always exists. Let us give an explicit construction for $\mathcal{C}[S^{-1}]$. To this end recall, that S is a *multiplicative system* if it satisfies the following conditions.

(MS1) If σ, τ are composable maps in S, then $\tau \circ \sigma$ is in S. The identity map id_X is in S for all X in $\mathcal{C}$.

(MS2) Let $\sigma\colon X \to Y$ be in S. Then every pair of maps $Y' \to Y$ and $X \to X''$ in $\mathcal{C}$ can be completed to a pair of commutative diagrams

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle \sigma'} & & \downarrow{\scriptstyle \sigma} \\ Y' & \longrightarrow & Y \end{array} \qquad\qquad \begin{array}{ccc} X & \longrightarrow & X'' \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma''} \\ Y & \longrightarrow & Y'' \end{array}$$

such that σ' and σ'' are in S.

(MS3) Let $\alpha, \beta\colon X \to Y$ be maps in $\mathcal{C}$. Then there is some $\sigma\colon X' \to X$ in S with $\alpha \circ \sigma = \beta \circ \sigma$ if and only if there is some $\tau\colon Y \to Y'$ in S with $\tau \circ \alpha = \tau \circ \beta$.

Assuming that S is a multiplicative system, one obtains the following description of $\mathcal{C}[S^{-1}]$. The objects are those of $\mathcal{C}$. Given objects X and Y, the maps $X \to Y$ in $\mathcal{C}[S^{-1}]$ are equivalence classes of diagrams

$$X \xrightarrow{\ \alpha\ } Y' \xleftarrow{\ \sigma\ } Y$$

with σ in S, where two pairs (α_1, σ_1) and (α_2, σ_2) are equivalent if there exists a commutative diagram

$$\begin{array}{ccccc} & & Y_1 & & \\ & {\scriptstyle \alpha_1}\nearrow & \downarrow & \nwarrow{\scriptstyle \sigma_1} & \\ X & \xrightarrow{\ \alpha_3\ } & Y_3 & \xleftarrow{\ \sigma_3\ } & Y \\ & {\scriptstyle \alpha_2}\searrow & \uparrow & \swarrow{\scriptstyle \sigma_2} & \\ & & Y_2 & & \end{array}$$

with σ_3 in S. The composition of two pairs (α, σ) and (β, τ) is by definition the pair $(\beta' \circ \alpha, \sigma' \circ \tau)$ where σ' and β' are obtained from condition (MS2) as in the following commutative diagram.

$$\begin{array}{ccccccc} & & & Z'' & & & \\ & & {\scriptstyle \beta'}\nearrow & & \nwarrow{\scriptstyle \sigma'} & & \\ & Y' & & & & Z' & \\ {\scriptstyle \alpha}\nearrow & & \nwarrow{\scriptstyle \sigma} & & {\scriptstyle \beta}\nearrow & & \nwarrow{\scriptstyle \tau} \\ X & & & Y & & & Z \end{array}$$

The universal functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$ sends a map $\alpha\colon X \to Y$ to the pair (α, id_Y). A pair (α, σ) is called a *fraction* because it is identified with $\sigma^{-1} \circ \alpha$ in $\mathcal{C}[S^{-1}]$.

Example. The quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$ form a multiplicative system; see (3.1).

1.4. An alternative definition. Let $\mathcal{A}$ be an abelian category. We denote by S the class of quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$ and by T the class of quasi-isomorphisms in $\mathbf{C}(\mathcal{A})$. The derived category of $\mathcal{A}$ is by definition the localization of $\mathbf{K}(\mathcal{A})$ with respect S. Alternatively, one could take the localization of $\mathbf{C}(\mathcal{A})$ with respect T.

Lemma. *The canonical functor* $P\colon \mathbf{C}(\mathcal{A}) \to \mathbf{K}(A)$ *induces a unique isomorphism* $\bar{P}$ *making the following diagram of functors commutative.*

$$\begin{array}{ccc} \mathbf{C}(\mathcal{A}) & \xrightarrow{Q_T} & \mathbf{C}(\mathcal{A})[T^{-1}] \\ \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle \bar{P}} \\ \mathbf{K}(\mathcal{A}) & \xrightarrow{Q_S} & \mathbf{K}(\mathcal{A})[S^{-1}] \end{array}$$

Proof. We have $T = P^{-1}(S)$. Thus $Q_S \circ P$ inverts all maps in T and induces therefore a unique functor $\bar{P}$ making the above diagram commutative. On the other hand, Q_T factors through P via a functor $F\colon \mathbf{K}(\mathcal{A}) \to \mathbf{C}(\mathcal{A})[T^{-1}]$. This follows from the fact that for two maps $\phi, \psi\colon X \to Y$ in $\mathbf{C}(\mathcal{A})$, we have $Q_T\phi = Q_T\psi$ if $\phi - \psi$ is null-homotopic; see [**11**, Lemma III.4.3]. The functor F inverts all maps in S and factors therefore through Q_S via a functor $\bar{F}\colon \mathbf{K}(\mathcal{A})[S^{-1}] \to \mathbf{C}(\mathcal{A})[T^{-1}]$. Clearly, $Q_T \circ \bar{F} \circ \bar{P} = Q_T$ and $Q_S \circ \bar{P} \circ \bar{F} = Q_S$. This implies $\bar{F} \circ \bar{P} = \mathrm{Id}$ and $\bar{P} \circ \bar{F} = \mathrm{Id}$. □

1.5. Extension groups. Let $\mathcal{A}$ be an abelian category. An object A in $\mathcal{A}$ is identified with the complex

$$\cdots \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow \cdots$$

concentrated in degree zero. Given any complex X in $\mathcal{A}$, we denote by ΣX or $X[1]$ the shifted complex with

$$(\Sigma X)^n = X^{n+1} \quad \text{and} \quad d^n_{\Sigma X} = -d^{n+1}_X.$$

Next we describe the maps in $\mathbf{D}(\mathcal{A})$ for certain complexes.

Lemma. *Let* Y *be a complex in* $\mathcal{A}$ *such that each* Y^n *is injective and* $Y^n = 0$ *for* $n \ll 0$.

(1) *Every quasi-isomorphism* $\sigma\colon Y \to Y'$ *has a left inverse* $\sigma'\colon Y' \to Y$ *such that* $\sigma' \circ \sigma = \mathrm{id}_Y$ *in* $\mathbf{K}(\mathcal{A})$.
(2) *Given a complex* X*, the map* $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X,Y) \to \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X,Y)$ *is bijective.*

Proof. For (1), see (2.5). To show (2), let $X \xrightarrow{\alpha} Y' \xleftarrow{\sigma} Y$ be a diagram representing a map $X \to Y$ in $\mathbf{D}(\mathcal{A})$. Then σ is a quasi-isomorphism and has therefore a left inverse σ' in $\mathbf{K}(\mathcal{A})$, by (1). Thus (α, σ) is equivalent to $(\sigma' \circ \alpha, \mathrm{id}_Y)$, which belongs to the image of the canonical map $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X,Y) \to \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X,Y)$.

Now let α_1 and α_2 be maps $X \to Y$ such that $(\alpha_1, \mathrm{id}_Y)$ and $(\alpha_2, \mathrm{id}_Y)$ are equivalent. Then there is a quasi-isomorphism $\sigma\colon Y \to Y'$ with $\sigma \circ \alpha_1 = \sigma \circ \alpha_2$. The map σ has a left inverse in $\mathbf{K}(\mathcal{A})$, by (1). Thus $\alpha_1 = \alpha_2$. □

Example. Let I be an injective object in $\mathcal{A}$. Then

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(-, I) \cong \mathrm{Hom}_{\mathcal{A}}(H^0-, I).$$

A motivation for introducing the derived category is the fact that the maps in $\mathbf{D}(\mathcal{A})$ provide all homological information on objects in $\mathcal{A}$. This becomes clear from the following description of the derived functors $\mathrm{Ext}^n_{\mathcal{A}}(-,-)$. Here, we use the convention that $\mathrm{Ext}^n_{\mathcal{A}}(-,-)$ vanishes for $n < 0$.

LEMMA. *For all objects A, B in $\mathcal{A}$ and $n \in \mathbb{Z}$, there is a canonical isomorphism*

$$\operatorname{Ext}^n_{\mathcal{A}}(A,B) \longrightarrow \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(A, \Sigma^n B).$$

PROOF. We give a proof in case $\mathcal{A}$ has enough injective objects; otherwise see [**29**, III.3.2]. Choose for B an injective resolution $\mathbf{i}B$. Thus we have a quasi-isomorphism $B \to \mathbf{i}B$ which induces the following isomorphism.

$$\begin{aligned}\operatorname{Ext}^n_{\mathcal{A}}(A,B) = H^n \operatorname{Hom}_{\mathcal{A}}(A, \mathbf{i}B) &\cong \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(A, \Sigma^n \mathbf{i}B)\\ \cong \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(A, \Sigma^n \mathbf{i}B) &\cong \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(A, \Sigma^n B)\end{aligned}$$

□

Note as a consequence that the canonical functor $\mathcal{A} \to \mathbf{D}(\mathcal{A})$ which sends an object to the corresponding complex concentrated in degree zero is fully faithful. In fact, it identifies $\mathcal{A}$ with the full subcategory of complexes X in $\mathbf{D}(\mathcal{A})$ such that $H^n X = 0$ for all $n \neq 0$.

1.6. Hereditary categories. Let $\mathcal{A}$ be a *hereditary* abelian category, that is, $\operatorname{Ext}^2_{\mathcal{A}}(-,-)$ vanishes. In this case, there is an explicit description of all objects and maps in $\mathbf{D}(\mathcal{A})$. Every complex X is completely determined by its cohomology because there is an isomorphism between X and

$$\cdots \longrightarrow H^{n-1}X \xrightarrow{0} H^n X \xrightarrow{0} H^{n+1}X \longrightarrow \cdots .$$

To construct this isomorphism, note that the vanishing of $\operatorname{Ext}^2_{\mathcal{A}}(H^n X, -)$ implies the existence of a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^{n-1} & \longrightarrow & E^n & \longrightarrow & H^n X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \operatorname{Im} d^{n-1} & \longrightarrow & \operatorname{Ker} d^n & \longrightarrow & H^n X & \longrightarrow & 0 \end{array}$$

with exact rows. We obtain the following commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^n X & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & X^{n-1} & \longrightarrow & E^n & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \| & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & X^{n-2} & \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & X^{n+1} & \longrightarrow & \cdots \end{array}$$

and the vertical maps induce cohomology isomorphism in degree n. Thus we have in $\mathbf{D}(\mathcal{A})$

$$\coprod_{n \in \mathbb{Z}} \Sigma^{-n}(H^n X) \cong X \cong \prod_{n \in \mathbb{Z}} \Sigma^{-n}(H^n X).$$

Using the description of $\operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(A,B)$ for objects A, B in $\mathcal{A}$, we see that each map $X \to Y$ in $\mathbf{D}(\mathcal{A})$ is given by a family of maps in $\operatorname{Hom}_{\mathcal{A}}(H^n X, H^n Y)$ and a family of extensions in $\operatorname{Ext}^1_{\mathcal{A}}(H^n X, H^{n-1} Y)$, with $n \in \mathbb{Z}$. Thus

$$\mathbf{D}(\mathcal{A}) = \bigsqcup_{n \in \mathbb{Z}} \Sigma^n \mathcal{A}$$

with non-zero maps $\Sigma^i\mathcal{A} \to \Sigma^j\mathcal{A}$ only if $j - i \in \{0, 1\}$.

Example. The category of abelian groups is hereditary. More generally, a module category is hereditary if and only if every submodule of a projective module is projective.

1.7. Bounded derived categories. Let $\mathcal{A}$ be an additive category. Consider the following full subcategories of $\mathbf{C}(\mathcal{A})$.

$$\begin{aligned} \mathbf{C}^-(\mathcal{A}) &= \{X \in \mathbf{C}(\mathcal{A}) \mid X^n = 0 \text{ for } n \gg 0\} \\ \mathbf{C}^+(\mathcal{A}) &= \{X \in \mathbf{C}(\mathcal{A}) \mid X^n = 0 \text{ for } n \ll 0\} \\ \mathbf{C}^b(\mathcal{A}) &= \{X \in \mathbf{C}(\mathcal{A}) \mid X^n = 0 \text{ for } |n| \gg 0\} \end{aligned}$$

For $* \in \{-, +, b\}$, let the homotopy category $\mathbf{K}^*(\mathcal{A})$ be the quotient of $\mathbf{C}^*(\mathcal{A})$ modulo null-homotopic maps, and let the derived category $\mathbf{D}^*(\mathcal{A})$ be the localization with respect to all quasi-isomorphisms.

Lemma. *For each* $* \in \{-, +, b\}$*, the inclusion* $\mathbf{C}^*(\mathcal{A}) \to \mathbf{C}(\mathcal{A})$ *induces fully faithful functors* $\mathbf{K}^*(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ *and* $\mathbf{D}^*(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$.

Proof. See [**29**, III.1.2.3]. □

1.8. Notes. Derived categories were introduced by Grothendieck and his school in the 1960s; see Verdier's (posthumously published) thèse [**29**]. The calculus of fractions which appears in the description of the localization of a category was developed by Gabriel and Zisman [**10**]. For a modern treatment of derived categories and related material, see [**11, 15, 30**].

2. Triangulated categories

The derived category $\mathbf{D}(\mathcal{A})$ of an abelian category is an additive category. There is some additional structure which complements the abelian structure of $\mathbf{C}(\mathcal{A})$. The axiomatization of this structure leads to the notion of a triangulated category.

2.1. The axioms. Let $\mathcal{T}$ be an additive category with an equivalence $\Sigma\colon \mathcal{T} \to \mathcal{T}$. A *triangle* in $\mathcal{T}$ is a sequence (α, β, γ) of maps

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X,$$

and a morphism between two triangles (α, β, γ) and $(\alpha', \beta', \gamma')$ is a triple (ϕ_1, ϕ_2, ϕ_3) of maps in $\mathcal{T}$ making the following diagram commutative.

$$\begin{array}{ccccccc} X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & \Sigma X \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \Sigma\phi_1} \\ X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z' & \xrightarrow{\gamma'} & \Sigma X' \end{array}$$

The category $\mathcal{T}$ is called *triangulated* if it is equipped with a class of distinguished triangles (called *exact triangles*) satisfying the following conditions.

(TR1) A triangle isomorphic to an exact triangle is exact. For each object X, the triangle $0 \to X \xrightarrow{\mathrm{id}} X \to 0$ is exact. Each map α fits into an exact triangle (α, β, γ).

(TR2) A triangle (α, β, γ) is exact if and only if $(\beta, \gamma, -\Sigma\alpha)$ is exact.

(TR3) Given two exact triangles (α, β, γ) and $(\alpha', \beta', \gamma')$, each pair of maps ϕ_1 and ϕ_2 satisfying $\phi_2 \circ \alpha = \alpha' \circ \phi_1$ can be completed to a morphism

$$\begin{array}{ccccccc} X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & \Sigma X \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \Sigma\phi_1} \\ X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z' & \xrightarrow{\gamma'} & \Sigma X' \end{array}$$

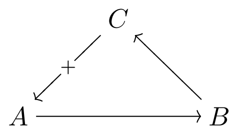
of triangles.

(TR4) Given exact triangles $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$, and $(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_1 = \beta_1 \circ \alpha_1$, there exists an exact triangle $(\delta_1, \delta_2, \delta_3)$ making the following diagram commutative.

$$\begin{array}{ccccccc} X & \xrightarrow{\alpha_1} & Y & \xrightarrow{\alpha_2} & U & \xrightarrow{\alpha_3} & \Sigma X \\ \| & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \delta_1} & & \| \\ X & \xrightarrow{\gamma_1} & Z & \xrightarrow{\gamma_2} & V & \xrightarrow{\gamma_3} & \Sigma X \\ & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \Sigma\alpha_1} \\ & & W & = & W & \xrightarrow{\beta_3} & \Sigma Y \\ & & \downarrow{\scriptstyle \beta_3} & & \downarrow{\scriptstyle \delta_3} & & \\ & & \Sigma Y & \xrightarrow{\Sigma\alpha_2} & \Sigma U & & \end{array}$$

The category $\mathcal{T}$ is called *pre-triangulated* if the axioms (TR1) – (TR3) are satisfied.

2.2. The octahedral axiom. Let $\mathcal{T}$ be a pre-triangulated category. The axiom (TR4) is known as *octahedral axiom* because the four exact triangles can be arranged in a diagram having the shape of an octahedron. The exact triangles $A \to B \to C \to \Sigma A$ are represented by faces of the form



and the other four faces are commutative triangles.

Let us give a more intuitive formulation of the octahedral axiom which is based on the notion of a homotopy cartesian square. Call a commutative square

$$\begin{array}{ccc} X & \xrightarrow{\alpha'} & Y' \\ \downarrow{\scriptstyle \alpha''} & & \downarrow{\scriptstyle \beta'} \\ Y'' & \xrightarrow{\beta''} & Z \end{array}$$

homotopy cartesian if there exists an exact triangle

$$X \xrightarrow{\left[\begin{smallmatrix}\alpha' \\ \alpha''\end{smallmatrix}\right]} Y' \amalg Y'' \xrightarrow{[\beta' \; -\beta'']} Z \xrightarrow{\gamma} \Sigma X.$$

The map γ is called a *differential* of the homotopy cartesian square. Note that a differential of the homotopy cartesian square changes its sign if the square is flipped along the main diagonal.

(TR4') Every pair of maps $X \to Y$ and $X \to X'$ can be completed to a morphism

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \| & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & \Sigma X' \end{array}$$

between exact triangles such that the left hand square is homotopy cartesian and the composite $Y' \to Z \to \Sigma X$ is a differential.

We can think of a homotopy cartesian square as the triangulated analogue of a pull-back and push-out square in an abelian category. Recall that a square

$$\begin{array}{ccc} X & \xrightarrow{\alpha'} & Y' \\ {\scriptstyle \alpha''}\downarrow & & \downarrow{\scriptstyle \beta'} \\ Y'' & \xrightarrow{\beta''} & Z \end{array}$$

is a pull-back and a push-out if and only if the sequence

$$0 \longrightarrow X \xrightarrow{\left[\begin{smallmatrix}\alpha' \\ \alpha''\end{smallmatrix}\right]} Y' \amalg Y'' \xrightarrow{[\beta' \; -\beta'']} Z \longrightarrow 0$$

is exact. The axiom (TR4') is the triangulated analogue of the fact that parallel maps in a pull-back square have isomorphic kernels, whereas parallel maps in a push-out square have isomorphic cokernels.

PROPOSITION. *The axioms (TR4) and (TR4') are equivalent for any pre-triangulated category.*

The proof can be found in the appendix.

2.3. Cohomological functors. Let $\mathcal{T}$ be a pre-triangulated category. Note that the axioms of a pre-triangulated category are symmetric in the sense that the opposite category $\mathcal{T}^{\mathrm{op}}$ carries a canonical pre-triangulated structure.

Given an abelian category $\mathcal{A}$, a functor $\mathcal{T} \to \mathcal{A}$ is called *cohomological* if it sends each exact triangle in $\mathcal{T}$ to an exact sequence in $\mathcal{A}$.

LEMMA. *For each object X in $\mathcal{T}$, the representable functors*

$$\mathrm{Hom}_{\mathcal{T}}(X,-)\colon \mathcal{T} \longrightarrow \mathrm{Ab} \quad \textit{and} \quad \mathrm{Hom}_{\mathcal{T}}(-,X)\colon \mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{Ab}$$

into the category Ab *of abelian groups are cohomological functors.*

PROOF. Fix an exact triangle $U \xrightarrow{\alpha} V \xrightarrow{\beta} W \xrightarrow{\gamma} \Sigma U$. We need to show the exactness of the induced sequence

$$\mathrm{Hom}_{\mathcal{T}}(X,U) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(X,V) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(X,W) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(X,\Sigma U).$$

It is sufficient to check exactness at one place, say $\mathrm{Hom}_{\mathcal{T}}(X, V)$, by (TR2). To this end fix a map $\phi\colon X \to V$ and consider the following diagram.

$$\begin{array}{ccccccc} X & \xrightarrow{\mathrm{id}} & X & \longrightarrow & 0 & \longrightarrow & \Sigma X \\ & & \downarrow{\scriptstyle \phi} & & & & \\ U & \xrightarrow{\alpha} & V & \xrightarrow{\beta} & W & \xrightarrow{\gamma} & \Sigma U \end{array}$$

If ϕ factors through α, then (TR3) implies the existence of a map $0 \to W$ making the diagram commutative. Thus $\beta \circ \phi = 0$. Now assume $\beta \circ \phi = 0$. Applying (TR2) and (TR3), we find a map $X \to U$ making the diagram commutative. Thus ϕ factors through α. □

2.4. Uniqueness of exact triangles. Let $\mathcal{T}$ be a pre-triangulated category. Given a map $\alpha\colon X \to Y$ in $\mathcal{T}$ and two exact triangles $\Delta = (\alpha, \beta, \gamma)$ and $\Delta' = (\alpha, \beta', \gamma')$ which complete α, there exists a comparison map $(\mathrm{id}_X, \mathrm{id}_Y, \phi)$ between Δ and Δ', by (TR3). The map ϕ is an isomorphism, by the following lemma, but it need not to be unique.

Lemma. *Let (ϕ_1, ϕ_2, ϕ_3) be a morphism between exact triangles. If two maps from $\{\phi_1, \phi_2, \phi_3\}$ are isomorphisms, then also the third.*

Proof. Use lemma (2.3) and apply the 5-lemma. □

The third object Z in an exact triangle $X \to Y \to Z \to \Sigma X$ is called the *cone* or the *cofiber* of the map $X \to Y$.

2.5. $\mathbf{K}(\mathcal{A})$ is triangulated. Let $\mathcal{A}$ be an additive category and let $\mathbf{K}(\mathcal{A})$ be the homotopy category of complexes. Consider the equivalence $\Sigma\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ which takes a complex to its shifted complex. Given a map $\alpha\colon X \to Y$ of complexes, the *mapping cone* is the complex Z with $Z^n = X^{n+1} \amalg Y^n$ and differential $\left[\begin{smallmatrix} -d_X^{n+1} & 0 \\ \alpha^{n+1} & d_Y^n \end{smallmatrix}\right]$. The mapping cone fits into a *mapping cone sequence*

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

which is defined in degree n by the following sequence.

$$X^n \xrightarrow{\alpha^n} Y^n \xrightarrow{\left[\begin{smallmatrix} 0 \\ \mathrm{id} \end{smallmatrix}\right]} X^{n+1} \amalg Y^n \xrightarrow{[-\mathrm{id}\ 0]} X^{n+1}$$

By definition, a triangle in $\mathbf{K}(\mathcal{A})$ is *exact* if it is isomorphic to a mapping cone sequence as above. It is easy to verify the axioms (TR1) – (TR4); see [**29**, II.1.3.2] or (7.4). Thus $\mathbf{K}(\mathcal{A})$ is a triangulated category.

Any mapping cone sequence of complexes induces a long exact sequence when one passes to its cohomology. To make this precise, we identify $H^i(\Sigma^j X) = H^{i+j} X$ for every complex X and all i, j.

Lemma. *Let $\mathcal{A}$ be an abelian category. An exact triangle*

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

in $\mathbf{K}(\mathcal{A})$ induces the following long exact sequence.

$$\cdots \longrightarrow H^{n-1} Z \xrightarrow{H^{n-1}\gamma} H^n X \xrightarrow{H^n \alpha} H^n Y \xrightarrow{H^n \beta} H^n Z \xrightarrow{H^n \gamma} H^{n+1} X \longrightarrow \cdots$$

Proof. We may assume that the triangle is a mapping cone sequence as above. The short exact sequence $0 \to Y \to Z \to \Sigma X \to 0$ of complexes induces a long exact sequence

$$\cdots \longrightarrow H^{n-1}(\Sigma X) \xrightarrow{\delta^{n-1}} H^n Y \xrightarrow{H^n \beta} H^n Z \xrightarrow{H^n \gamma} H^n(\Sigma X) \xrightarrow{\delta^n} H^{n+1} Y \longrightarrow \cdots$$

with connecting morphism δ^*. This follows from the Snake Lemma. Now observe that $\delta^n = H^{n+1}\alpha$. □

Remark. The triangulated structure of $\mathbf{K}(\mathcal{A})$ allows a quick proof of the following fact. Given a complex Y such that each Y^n is injective and $Y^n = 0$ for $n \ll 0$, every quasi-isomorphism $\beta \colon Y \to Z$ has a left inverse β' such that $\beta' \circ \beta = \mathrm{id}_Y$ in $\mathbf{K}(\mathcal{A})$. In order to construct β', complete β to an exact triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$. Then α is null-homotopic since $H^n X = 0$ for all $n \in \mathbb{Z}$. Thus id_Y factors through β, by lemma (2.3).

2.6. Notes. Triangulated categories were introduced independently in algebraic geometry by Verdier in his thèse [**29**], and in algebraic topology by Puppe [**26**]. The basic properties of triangulated categories can be found in [**29**]. The reformulation (TR4') of the octahedral axiom (TR4) which is given here is a variation of a reformulation due to Dlab, Parshall, and Scott [**25**]. There is an equivalent formulation of (TR4) due to May which displays the exact triangles in a diagram having the shape of a braid [**20**]. Another reformulation is discussed in Neeman's book [**24**], which also contains a discussion of homotopy cartesian squares. We refer to [**15, 24**] for a modern treatment of triangulated categories.

3. Localization of triangulated categories

The triangulated structure of $\mathbf{K}(\mathcal{A})$ induces a triangulated structure of $\mathbf{D}(\mathcal{A})$ via the localization functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$. This follows from the fact that the quasi-isomorphisms form a class of maps in $\mathbf{K}(\mathcal{A})$ which is compatible with the triangulation.

3.1. Quasi-isomorphisms. Let $\mathcal{T}$ be a triangulated category and S be a class of maps which is a multiplicative system. Then S is *compatible with the triangulation* if

(MS4) given σ in S, the map $\Sigma^n \sigma$ belongs to S for all $n \in \mathbb{Z}$, and
(MS5) given a map (ϕ_1, ϕ_2, ϕ_3) between exact triangles with ϕ_1 and ϕ_2 in S, there is also a map $(\phi_1, \phi_2, \phi_3')$ with ϕ_3' in S.

Lemma. *Let $H \colon \mathcal{T} \to \mathcal{A}$ be a cohomological functor. Then the class S of maps σ in $\mathcal{T}$ such that $H(\Sigma^n \sigma)$ is an isomorphism for all $n \in \mathbb{Z}$ form a multiplicative system, compatible with the triangulation.*

Proof. (MS1) and (MS4) are immediate consequences of the definition. (MS5) follows from the 5-lemma. To show (MS2), let $\sigma \colon X \to Y$ be in S and $Y' \to Y$ be an arbitrary map. Complete $Y' \to Y$ to an exact triangle. Applying (TR2) and

(TR3), we obtain the following morphism between exact triangles

$$\begin{array}{ccccccc} X' & \longrightarrow & X & \longrightarrow & Y'' & \longrightarrow & \Sigma X' \\ \downarrow{\scriptstyle \sigma'} & & \downarrow{\scriptstyle \sigma} & & \| & & \downarrow{\scriptstyle \Sigma\sigma'} \\ Y' & \longrightarrow & Y & \longrightarrow & Y'' & \longrightarrow & \Sigma Y' \end{array}$$

and the 5-lemma shows that σ' belongs to S. It remains to check (MS3). Let $\alpha, \beta\colon X \to Y$ be maps in $\mathcal{T}$ and $\sigma\colon X' \to X$ in S such that $\alpha \circ \sigma = \beta \circ \sigma$. Complete σ to an exact triangle $X' \xrightarrow{\sigma} X \xrightarrow{\phi} X'' \to \Sigma X'$. Then $\alpha - \beta$ factors through ϕ via some map $\psi\colon X'' \to Y$. Now complete ψ to an exact triangle $X'' \xrightarrow{\psi} Y \xrightarrow{\tau} Y' \to \Sigma X''$. Then τ belongs to S and $\tau \circ \alpha = \tau \circ \beta$. □

3.2. $\mathbf{D}(\mathcal{A})$ is triangulated. An *exact* functor $\mathcal{T} \to \mathcal{U}$ between triangulated categories is a pair (F, η) consisting of a functor $F\colon \mathcal{T} \to \mathcal{U}$ and a natural isomorphism $\eta\colon F \circ \Sigma_{\mathcal{T}} \to \Sigma_{\mathcal{U}} \circ F$ such that for every exact triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ in $\mathcal{T}$ the triangle

$$FX \xrightarrow{F\alpha} FY \xrightarrow{F\beta} FZ \xrightarrow{\eta_X \circ F\gamma} \Sigma(FX)$$

is exact in $\mathcal{U}$.

Lemma. *Let $\mathcal{T}$ be a triangulated category and S be a multiplicative system of maps which is compatible with the triangulation. Then the localization $\mathcal{T}[S^{-1}]$ carries a unique triangulated structure such that the canonical functor $\mathcal{T} \to \mathcal{T}[S^{-1}]$ is exact.*

Proof. The equivalence $\Sigma\colon \mathcal{T} \to \mathcal{T}$ induces a unique equivalence $\mathcal{T}[S^{-1}] \to \mathcal{T}[S^{-1}]$ which commutes with the canonical functor $Q\colon \mathcal{T} \to \mathcal{T}[S^{-1}]$. This follows from (MS4). Now take as exact triangles in $\mathcal{T}[S^{-1}]$ all those isomorphic to images of exact triangles in $\mathcal{T}$. It is straightforward to verify the axioms (TR1) – (TR4); see [**29**, II.2.2.6]. The functor Q is exact by construction. □

Let $\mathcal{A}$ be an abelian category. The quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$ form a multiplicative system which is compatible with the triangulation, by lemma (3.1). In fact, $H^0\colon \mathbf{K}(\mathcal{A}) \to \mathcal{A}$ is a cohomological functor by lemma (2.5), and a map σ in $\mathbf{K}(\mathcal{A})$ is a quasi-isomorphism if and only if $H^0(\Sigma^n \sigma) = H^n \sigma$ is an isomorphism for all $n \in \mathbb{Z}$. Thus $\mathbf{D}(\mathcal{A})$ is triangulated and the canonical functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ is exact.

The canonical functor $\mathcal{A} \to \mathbf{D}(\mathcal{A})$ sends every exact sequence

$$E\colon 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

in $\mathcal{A}$ to an exact triangle

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A$$

in $\mathbf{D}(\mathcal{A})$, where γ denotes the map corresponding to E under the canonical isomorphism

$$\operatorname{Ext}^1_{\mathcal{A}}(C, A) \cong \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(C, \Sigma A).$$

In fact, the mapping cone of α produces the following exact triangle

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & B & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle -\mathrm{id}} & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

and the map β induces a quasi-isomorphism between the mapping cone of α and the complex corresponding to C.

EXAMPLE. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between additive categories. Then F induces an exact functor $\mathbf{K}(F)\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$. Now suppose that $\mathcal{A}$ and $\mathcal{B}$ are abelian and F is exact. Then $\mathbf{K}(F)$ sends quasi-isomorphisms to quasi-isomorphisms and induces therefore an exact functor $\mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$.

3.3. Triangulated and thick subcategories. Let $\mathcal{T}$ be a triangulated category. A non-empty full subcategory $\mathcal{S}$ is a *triangulated subcategory* if the following conditions hold.

(TS1) $\Sigma^n X \in \mathcal{S}$ for all $X \in \mathcal{S}$ and $n \in \mathbb{Z}$.

(TS2) Let $X \to Y \to Z \to \Sigma X$ be an exact triangle in $\mathcal{T}$. If two objects from $\{X, Y, Z\}$ belong to $\mathcal{S}$, then also the third.

A triangulated subcategory $\mathcal{S}$ is *thick* if in addition the following condition holds.

(TS3) Every direct factor of an object in $\mathcal{S}$ belongs to $\mathcal{S}$, that is, a decomposition $X = X' \amalg X''$ for $X \in \mathcal{S}$ implies $X' \in \mathcal{S}$.

Note that a triangulated subcategory $\mathcal{S}$ inherits a canonical triangulated structure from $\mathcal{T}$.

Given a class $\mathcal{S}_0$ of objects in $\mathcal{T}$, one can construct inductively the triangulated subcategory *generated by* $\mathcal{S}_0$ as follows. Denote for two classes $\mathcal{U}$ and $\mathcal{V}$ of objects in $\mathcal{T}$ by $\mathcal{U} * \mathcal{V}$ the class of objects X occuring in an exact triangle $U \to X \to V \to \Sigma U$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Note that the operation $*$ is associative by the octahedral axiom. Now let $\mathcal{S}_1$ be the class of all $\Sigma^n X$ with $X \in \mathcal{S}_0$ and $n \in \mathbb{Z}$. For $r > 0$, let $\mathcal{S}_r = \mathcal{S}_1 * \mathcal{S}_1 * \cdots * \mathcal{S}_1$ be the product with r factors.

LEMMA. *Let $\mathcal{S}_0$ be a class of objects in $\mathcal{T}$.*

(1) *The full subcategory of objects in $\mathcal{S} = \bigcup_{r \geqslant 0} \mathcal{S}_r$ is the smallest full triangulated subcategory of $\mathcal{T}$ which contains $\mathcal{S}_0$.*

(2) *The full subcategory of direct factors of objects in $\mathcal{S}$ is the smallest full thick subcategory of $\mathcal{T}$ which contains $\mathcal{S}_0$.*

PROOF. (1) is clear. To prove (2), we need to show that all direct factors of objects in $\mathcal{S}$ form a triangulated subcategory. To this end fix an exact triangle

$$\Delta'\colon X' \xrightarrow{\alpha'} Y' \xrightarrow{\beta'} Z' \xrightarrow{\gamma'} \Sigma X'$$

such that $X = X' \amalg X''$ and $Y = Y' \amalg Y''$ belong to $\mathcal{S}$. We need to show that Z' is a direct factor of some object in $\mathcal{S}$. Complete $\alpha = \left[\begin{smallmatrix} \alpha' & 0 \\ 0 & 0 \end{smallmatrix}\right]$ to an exact triangle

$$\Delta\colon X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

with Z in $\mathcal{S}$. We have obvious maps $\iota\colon \Delta' \to \Delta$ and $\pi\colon \Delta \to \Delta'$ such that $\pi \circ \iota$ is invertible, by lemma (2.4). Thus Z' is a direct factor of Z. □

Example. (1) A complex X in some abelian category $\mathcal{A}$ is *acyclic* if $H^n X = 0$ for all $n \in \mathbb{Z}$. The acyclic complexes form a thick subcategory in $\mathbf{K}(\mathcal{A})$. A map between complexes is a quasi-isomorphism if and only if its mapping cone is acyclic. The canonical functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ annihilates a map in $\mathbf{K}(\mathcal{A})$ if and only if it factors through an acyclic complex; see (3.4).

(2) The bounded derived category $\mathbf{D}^b(\mathcal{A})$ of an abelian category $\mathcal{A}$ is generated as a triangulated category by the objects in $\mathcal{A}$, viewed as complexes concentrated in degree zero.

(3) Let $\mathcal{A}$ be an abelian category with enough injective objects. Then $\mathbf{D}^b(\mathcal{A})$ is generated as a triangulated category by all injective objects, if and only if every object in $\mathcal{A}$ has finite injective dimension.

3.4. The kernel of a localization. Let $\mathcal{T}$ be a triangulated category and let $F\colon \mathcal{T} \to \mathcal{U}$ be an additive functor. The *kernel* $\operatorname{Ker} F$ of F is by definition the full subcategory of $\mathcal{T}$ which is formed by all objects X such that $FX = 0$. If F is an exact functor into a triangulated category, then $\operatorname{Ker} F$ is a thick subcategory of $\mathcal{T}$. Also, if F is a cohomological functor into an abelian category, then $\bigcap_{n\in\mathbb{Z}} \operatorname{Ker}(F \circ \Sigma^n)$ is a thick subcategory of $\mathcal{T}$.

Next we assume that F is a localization functor and describe the maps α in $\mathcal{T}$ such that $F\alpha = 0$. Note that $FX = 0$ for an object X if and only if $F\mathrm{id}_X = 0$.

Lemma. *Let $\mathcal{T}$ be a triangulated category and S be a multiplicative system of maps in $\mathcal{T}$ which is compatible with the triangulation. The following are equivalent for a map $\alpha\colon X \to Y$ in $\mathcal{T}$.*

(1) *The canonical functor $Q\colon \mathcal{T} \to \mathcal{T}[S^{-1}]$ annihilates α.*
(2) *The exists a map $\sigma\colon Y \to Z$ in S with $\sigma \circ \alpha = 0$.*
(3) *The map α factors through the cone of a map in S.*

Proof. (1) ⇔ (2): The functor Q sends α to the fraction (α, id_Y). Thus Q annihilates α if and only if the fractions (α, id_Y) and $(0, \mathrm{id}_Y)$ are equivalent. Now observe that both fractions are equivalent if and only if $\sigma \circ \alpha = 0$ for some $\sigma \in S$.

(2) ⇔ (1): Use lemma (2.3). □

3.5. Verdier localization. Let $\mathcal{T}$ be a triangulated category. Given a triangulated subcategory $\mathcal{S}$, we denote by $S(\mathcal{S})$ the class of maps in $\mathcal{T}$ such that the cone of α belongs to $\mathcal{S}$.

Lemma. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ be a triangulated subcategory. Then $S(\mathcal{S})$ is a multiplicative system which is compatible with the triangulation of $\mathcal{T}$.*

Proof. See [**29**, II.2.1.8] □

The *Verdier localization* of $\mathcal{T}$ with respect to a triangulated subcategory $\mathcal{S}$ is by definition the localization

$$\mathcal{T}/\mathcal{S} = \mathcal{T}[S(\mathcal{S})^{-1}]$$

together with the canonical functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$.

PROPOSITION. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a full triangulated subcategory. Then the category $\mathcal{T}/\mathcal{S}$ and the canonical functor $Q\colon \mathcal{T} \to \mathcal{T}/\mathcal{S}$ have the following properties.*

(1) *The category $\mathcal{T}/\mathcal{S}$ carries a unique triangulated structure such that Q is exact.*
(2) *The kernel* $\operatorname{Ker} Q$ *is the smallest thick subcategory containing $\mathcal{S}$.*
(3) *Every exact functor $\mathcal{T} \to \mathcal{U}$ annihilating $\mathcal{S}$ factors uniquely through Q via an exact functor $\mathcal{T}/\mathcal{S} \to \mathcal{U}$.*
(4) *Every cohomological functor $\mathcal{T} \to \mathcal{A}$ annihilating $\mathcal{S}$ factors uniquely through Q via a cohomological functor $\mathcal{T}/\mathcal{S} \to \mathcal{A}$.*

PROOF. (1) follows from lemma (3.2) and (2) from lemma (3.4).

(3) An exact functor $F\colon \mathcal{T} \to \mathcal{U}$ annihilating $\mathcal{S}$ inverts every map in $S(\mathcal{S})$. Thus there exists a unique functor $\bar{F}\colon \mathcal{T}/\mathcal{S} \to \mathcal{U}$ such that $F = \bar{F} \circ Q$. The functor $\bar{F}$ is exact because an exact triangle Δ in $\mathcal{T}/\mathcal{S}$ is up to isomorphism of the form $Q\Gamma$ for some exact triangle Γ in $\mathcal{T}$. Thus $\bar{F}\Delta = F\Gamma$ is exact.

(4) Analogous to (3). □

3.6. Notes. Localizations of triangulated categories are discussed in Verdier's thèse [**29**]. In particular, he introduced the localization $\mathcal{T}/\mathcal{S}$ of a triangulated category $\mathcal{T}$ with respect to a triangulated subcategory $\mathcal{S}$.

4. Brown representability

The Brown representability theorem provides a useful characterization of the representable functors $\operatorname{Hom}_{\mathcal{T}}(-, X)$ for a class of triangulated categories $\mathcal{T}$. The proof is based on a universal embedding of $\mathcal{T}$ into an abelian category which is of independent interest. Note that such an embedding reverses the direction of the construction of the derived category which provides an embedding of an abelian category into a triangulated category.

4.1. Coherent functors. Let $\mathcal{A}$ be an additive category. We consider functors $F\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$ into the category of abelian groups and call a sequence $F' \to F \to F''$ of functors *exact* if the induced sequence $F'X \to FX \to F''X$ of abelian groups is exact for all X in $\mathcal{A}$. A functor F is said to be *coherent* if there exists an exact sequence (called *presentation*)

$$\mathcal{A}(-, X) \longrightarrow \mathcal{A}(-, Y) \longrightarrow F \longrightarrow 0.$$

Here, we simplify our notation and write $\mathcal{A}(X, Y)$ for the set of maps $X \to Y$. The natural transformations between two coherent functors form a set by Yoneda's lemma, and the coherent functors $\mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$ form an additive category with cokernels. We denote this category by $\widehat{\mathcal{A}}$. A basic tool is the fully faithful *Yoneda functor*

$$\mathcal{A} \longrightarrow \widehat{\mathcal{A}}, \quad X \mapsto \mathcal{A}(-, X).$$

Recall that a map $X \to Y$ is a *weak kernel* for $Y \to Z$ if the induced sequence

$$\mathcal{A}(-,X) \longrightarrow \mathcal{A}(-,Y) \longrightarrow \mathcal{A}(-,Z)$$

is exact.

LEMMA. (1) *If $\mathcal{A}$ has weak kernels, then $\widehat{\mathcal{A}}$ is an abelian category.*
(2) *If $\mathcal{A}$ has arbitrary coproducts, then $\widehat{\mathcal{A}}$ has arbitrary coproducts and the Yoneda functor preserves all coproducts.*

PROOF. (1) The category $\widehat{\mathcal{A}}$ has cokernels, and it is therefore sufficient to show that $\widehat{\mathcal{A}}$ has kernels. To this end fix a map $F_1 \to F_2$ with the following presentation.

$$\begin{array}{ccccccc}
\mathcal{A}(-,X_1) & \longrightarrow & \mathcal{A}(-,Y_1) & \longrightarrow & F_1 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{A}(-,X_2) & \longrightarrow & \mathcal{A}(-,Y_2) & \longrightarrow & F_2 & \longrightarrow & 0
\end{array}$$

We construct the kernel $F_0 \to F_1$ by specifying the following presentation.

$$\begin{array}{ccccccc}
\mathcal{A}(-,X_0) & \longrightarrow & \mathcal{A}(-,Y_0) & \longrightarrow & F_0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{A}(-,X_1) & \longrightarrow & \mathcal{A}(-,Y_1) & \longrightarrow & F_1 & \longrightarrow & 0
\end{array}$$

First the map $Y_0 \to Y_1$ is obtained from the weak kernel sequence

$$Y_0 \longrightarrow X_2 \amalg Y_1 \longrightarrow Y_2.$$

Then the maps $X_0 \to X_1$ and $X_0 \to Y_0$ are obtained from the weak kernel sequence

$$X_0 \longrightarrow X_1 \amalg Y_0 \longrightarrow Y_1.$$

(2) For every family of functors F_i having a presentation

$$\mathcal{A}(-,X_i) \stackrel{(-,\phi_i)}{\longrightarrow} \mathcal{A}(-,Y_i) \longrightarrow F_i \longrightarrow 0,$$

the coproduct $F = \coprod_i F_i$ has a presentation

$$\mathcal{A}(-,\coprod_i X_i) \stackrel{(-,\amalg\phi_i)}{\longrightarrow} \mathcal{A}(-,\coprod_i Y_i) \longrightarrow F \longrightarrow 0.$$

Thus coproducts in $\widehat{\mathcal{A}}$ are not computed pointwise. □

EXAMPLE. (1) Let $\mathcal{A}$ be an abelian category and suppose $\mathcal{A}$ has enough projective objects. Let $\mathcal{P}$ denote the full subcategory of projective objects in $\mathcal{A}$. Then the functor

$$\mathcal{A} \longrightarrow \widehat{\mathcal{P}}, \quad X \mapsto \mathrm{Hom}_{\mathcal{A}}(-,X)|_{\mathcal{P}},$$

is an equivalence.

(2) Let $X \to Y \to Z \to \Sigma X$ be an exact triangle in some triangulated category. Then $X \to Y$ is a weak kernel of $Y \to Z$; see lemma (2.3).

4.2. The abelianization of a triangulated category. Let $\mathcal{T}$ be a triangulated category. The Yoneda functor $\mathcal{T} \to \widehat{\mathcal{T}}$ is the universal cohomological functor for $\mathcal{T}$.

LEMMA. (1) *The category $\widehat{\mathcal{T}}$ is abelian and the Yoneda functor $H_{\mathcal{T}} \colon \mathcal{T} \to \widehat{\mathcal{T}}$ is cohomological.*
(2) *Let $\mathcal{A}$ be an abelian category and $H \colon \mathcal{T} \to \mathcal{A}$ be a cohomological functor. Then there is (up to a unique isomorphism) a unique exact functor $\bar{H} \colon \widehat{\mathcal{T}} \to \mathcal{A}$ such that $H = \bar{H} \circ H_{\mathcal{T}}$.*

PROOF. The category $\mathcal{T}$ has weak kernels and therefore $\widehat{\mathcal{T}}$ is abelian. Note that the weak kernel of a map $Y \to Z$ is obtained by completing the map to an exact triangle $X \to Y \to Z \to \Sigma X$.

Now let $H \colon \mathcal{T} \to \mathcal{A}$ be a cohomological functor. Extend H to $\bar{H}$ by sending F in $\widehat{\mathcal{T}}$ with presentation

$$\mathcal{T}(-, X) \xrightarrow{(-,\phi)} \mathcal{T}(-, Y) \longrightarrow F \longrightarrow 0$$

to the cokernel of $H\phi$. The functor $\bar{H}$ is automatically right exact, and it is exact because H is cohomological. □

The category $\widehat{\mathcal{T}}$ is called the *abelianization* of $\mathcal{T}$. It has some special homological properties. Recall that an abelian category is a *Frobenius category* if there are enough projectives and enough injectives, and both coincide.

LEMMA. *The abelianization of a triangulated category is an abelian Frobenius category.*

PROOF. The representable functors are projective objects in $\widehat{\mathcal{T}}$ by Yoneda's lemma. Thus $\widehat{\mathcal{T}}$ has enough projectives. Using the fact that the Yoneda functors $\mathcal{T} \to \widehat{\mathcal{T}}$ and $\mathcal{T}^{\mathrm{op}} \to \widehat{\mathcal{T}^{\mathrm{op}}}$ are universal cohomological functors, we obtain an equivalence $\widehat{\mathcal{T}}^{\mathrm{op}} \to \widehat{\mathcal{T}^{\mathrm{op}}}$ which sends $\mathcal{T}(-, X)$ to $\mathcal{T}(X, -)$ for all X in $\mathcal{T}$. Thus the representable functors are injective objects, and $\widehat{\mathcal{T}}$ has enough injectives. □

4.3. The idempotent completion of a triangulated category. Let $\mathcal{T}$ be a triangulated category. We identify $\mathcal{T}$ via the Yoneda functor with a full subcategory of projective objects in the abelianization $\widehat{\mathcal{T}}$. Recall that an additive category has *split idempotents* if every idempotent map $\phi^2 = \phi \colon X \to X$ has a kernel. Note that in this case $X = \operatorname{Ker} \phi \amalg \operatorname{Ker}(\mathrm{id}_X - \phi)$.

LEMMA. *The full subcategory $\bar{\mathcal{T}}$ of all projective objects in $\widehat{\mathcal{T}}$ is a triangulated category with respect to the class of triangles which are direct factors of exact triangles in $\mathcal{T}$. The category $\bar{\mathcal{T}}$ has split idempotents, and every exact functor $\mathcal{T} \to \mathcal{U}$ into a triangulated category with split idempotents extends (up to a unique isomorphism) uniquely to an exact functor $\bar{\mathcal{T}} \to \mathcal{U}$.*

PROOF. The proof is straightforward, except for the verification of the octahedral axiom which requires some work; see [**3**]. □

EXAMPLE. Let $\mathcal{A}$ be an abelian category. Then $\mathbf{D}^b(\mathcal{A})$ has split idempotents; see [**3**].

4.4. Homotopy colimits. Let $\mathcal{T}$ be a triangulated category and suppose countable coproducts exist in $\mathcal{T}$. Let

$$X_0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} \cdots$$

be a sequence of maps in $\mathcal{T}$. A *homotopy colimit* of this sequence is by definition an object X which occurs in an exact triangle

$$\coprod_{i\geqslant 0} X_i \xrightarrow{(\mathrm{id}-\phi_i)} \coprod_{i\geqslant 0} X_i \longrightarrow X \longrightarrow \Sigma(\coprod_{i\geqslant 0} X_i).$$

Here, the ith component of the map $(\mathrm{id}-\phi_i)$ is the composite

$$X_i \xrightarrow{\left[\begin{smallmatrix}\mathrm{id}\\ -\phi_i\end{smallmatrix}\right]} X_i \amalg X_{i+1} \xrightarrow{\mathrm{inc}} \coprod_{i\geqslant 0} X_i.$$

Note that a homotopy colimit is unique up to a (non-unique) isomorphism.

Example. (1) Let $\phi\colon X \to X$ be an idempotent map in $\mathcal{T}$, and denote by X' a homotopy colimit of the sequence

$$X \xrightarrow{\phi} X \xrightarrow{\phi} X \xrightarrow{\phi} \cdots .$$

The canonical map $X \to X'$ gives rise to a split exact triangle $X'' \to X \to X' \to \Sigma X''$, where $X'' = \operatorname{Ker}\phi$. Thus $0 \to X'' \to X \to X' \to 0$ is a split exact sequence and $X \cong X' \amalg X''$.

(2) Let $\mathcal{A}$ be an additive category and suppose countable coproducts exist in $\mathcal{A}$. For a complex X in $\mathcal{A}$ and $n \in \mathbb{Z}$, denote by $\tau_n X$ the truncation of X such that $(\tau_n X)^p = 0$ for $p < n$ and $(\tau_n X)^p = X^p$ for $p \geqslant n$. For each $n \in \mathbb{Z}$, there is a sequence of canonical maps

$$\tau_n X \longrightarrow \tau_{n-1} X \longrightarrow \tau_{n-2} X \longrightarrow \cdots$$

which are compatible with the canonical maps $\phi_p\colon \tau_p X \to X$. If X' denotes a homotopy colimit, then the family $(\phi_{n-i})_{i\geqslant 0}$ induces an isomorphism $X' \to X$ in $\mathbf{K}(\mathcal{A})$.

4.5. Brown representability. Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts. An object S in $\mathcal{T}$ satisfying the following conditions is called a *perfect generator*.

(PG1) There is no proper full triangulated subcategory of $\mathcal{T}$ which contains S and is closed under taking coproducts.

(PG2) Given a countable family of maps $X_i \to Y_i$ in $\mathcal{T}$ such that the map $\mathrm{Hom}_{\mathcal{T}}(S, X_i) \to \mathrm{Hom}_{\mathcal{T}}(S, Y_i)$ is surjective for all i, the induced map

$$\mathrm{Hom}_{\mathcal{T}}(S, \coprod_i X_i) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(S, \coprod_i Y_i)$$

is surjective.

We have the following *Brown representability theorem* for triangulated categories with a perfect generator.

Theorem (Brown representability). *Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts and a perfect generator.*

(1) *A functor* $F\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ *is cohomological and sends all coproducts in* $\mathcal{T}$ *to products if and only if* $F \cong \mathrm{Hom}_{\mathcal{T}}(-,X)$ *for some object* X *in* $\mathcal{T}$.
(2) *An exact functor* $\mathcal{T} \to \mathcal{U}$ *between triangulated categories preserves all coproducts if and only if it has a right adjoint.*

We give a complete proof. This is based on the following lemma, which explains the condition (PG2) and is independent from the triangulated structure of $\mathcal{T}$.

Lemma. *Let* $\mathcal{T}$ *be an additive category with arbitrary coproducts and weak kernels. Let* S *be an object in* $\mathcal{T}$, *and denote by* $\mathcal{S}$ *the full subcategory of all coproducts of copies of* S.

(1) *The category* $\mathcal{S}$ *has weak kernels and* $\widehat{\mathcal{S}}$ *is an abelian category.*
(2) *The assignment* $F \mapsto F|_{\mathcal{S}}$ *induces an exact functor* $\widehat{\mathcal{T}} \longrightarrow \widehat{\mathcal{S}}$.
(3) *The functor* $\mathcal{T} \to \widehat{\mathcal{S}}$ *sending* X *to* $\mathcal{T}(-,X)|_{\mathcal{S}}$ *preserves countable coproducts if and only if* (PG2) *holds.*

Proof. First observe that for every X in $\mathcal{T}$, there exists an *approximation* $X' \to X$ such that $X' \in \mathcal{S}$ and $\mathcal{T}(T,X') \to \mathcal{T}(T,X)$ is surjective for all $T \in \mathcal{S}$. Take $X' = \coprod_{\alpha\in\mathcal{T}(S,X)} S$ and the canonical map $X' \to X$.

(1) To prove that $\widehat{\mathcal{S}}$ is abelian, it is sufficient to show that every map in $\mathcal{S}$ has a weak kernel; see (4.1). To obtain a weak kernel of a map $Y \to Z$ in $\mathcal{S}$, take the composite of a weak kernel $X \to Y$ in $\mathcal{T}$ and an approximation $X' \to X$.

(2) We need to check that for F in $\widehat{\mathcal{T}}$, the restriction $F|_{\mathcal{S}}$ belongs to $\widehat{\mathcal{S}}$. It is sufficient to prove this for $F = \mathcal{T}(-,Y)$. To obtain a presentation, let $X \to Y'$ be a weak kernel of an approximation $Y' \to Y$. The composite $X' \to Y'$ with an approximation $X' \to X$ gives an exact sequence

$$\mathcal{T}(-,X')|_{\mathcal{S}} \longrightarrow \mathcal{T}(-,Y')|_{\mathcal{S}} \longrightarrow F|_{\mathcal{S}} \longrightarrow 0,$$

which is a presentation

$$\mathcal{S}(-,X') \longrightarrow \mathcal{S}(-,Y') \longrightarrow F|_{\mathcal{S}} \longrightarrow 0$$

of $F|_{\mathcal{S}}$. Clearly, the assignment $F \mapsto F|_{\mathcal{S}}$ is exact.

(3) We denote by $I\colon \mathcal{S} \to \mathcal{T}$ the inclusion and write $I_*\colon \widehat{\mathcal{T}} \to \widehat{\mathcal{S}}$ for the restriction functor. Note that I induces a right exact functor $I^*\colon \widehat{\mathcal{S}} \to \widehat{\mathcal{T}}$ by sending each representable functor $\mathcal{S}(-,X)$ to $\mathcal{T}(-,X)$.

The functor $\mathcal{T} \to \widehat{\mathcal{S}}$ preserves countable coproducts if and only if I_* preserves countable coproducts. We have $I_* \circ I^* \cong \mathrm{Id}_{\widehat{\mathcal{S}}}$ and this implies that I_* preserves countable coproducts if and only if $\mathrm{Ker}\, I_*$ is closed under countable coproducts, since I_* induces an equivalence $\widehat{\mathcal{T}}/\mathrm{Ker}\, I_* \to \widehat{\mathcal{S}}$; see [**9**, III.2]. Now observe that $\mathrm{Ker}\, I_*$ being closed under countable coproducts is a reformulation of the condition (PG2). □

Proof of the Brown representability theorem. (1) Fix a perfect generator S. First observe that we may assume $\Sigma S \cong S$. Otherwise, replace S by $\coprod_{n\in\mathbb{Z}} \Sigma^n S$. This does not affect the condition (PG2). Also, it does not affect the condition (PG1), because a triangulated subcategory closed under countable coproducts is closed under direct factors; see (4.4).

We construct inductively a sequence

$$X_0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} \cdots$$

of maps in $\mathcal{T}$ and elements π_i in FX_i as follows. Let $X_0 = 0$ and $\pi_0 = 0$. Let $X_1 = S^{[FS]}$ be the coproduct of copies of S indexed by the elements in FS, and let π_1 be the element corresponding to id_{FS} in $FX_1 \cong (FS)^{FS}$. Suppose we have already constructed ϕ_{i-1} and π_i for some $i > 0$. Let

$$K_i = \{\alpha \in \mathcal{T}(S, X_i) \mid (F\alpha)\pi_i = 0\}$$

and complete the canonical map $\chi_i \colon S^{[K_i]} \to X_i$ to an exact triangle

$$S^{[K_i]} \xrightarrow{\chi_i} X_i \xrightarrow{\phi_i} X_{i+1} \longrightarrow \Sigma S^{[K_i]}.$$

Now choose an element π_{i+1} in FX_{i+1} such that $(F\phi_i)\pi_{i+1} = \pi_i$. This is possible since $(F\chi_i)\pi_i = 0$ and F is cohomological.

We identify each π_i via Yoneda's lemma with a map $\mathcal{T}(-, X_i) \to F$ and obtain the following commutative diagram with exact rows in $\widehat{\mathcal{S}}$, where $\mathcal{S}$ denotes the full subcategory of all coproducts of copies of S in $\mathcal{T}$ and $\psi_i = \mathcal{T}(-, \phi_i)|_{\mathcal{S}}$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ker} \pi_i|_{\mathcal{S}} & \longrightarrow & \mathcal{T}(-, X_i)|_{\mathcal{S}} & \xrightarrow{\pi_i} & F|_{\mathcal{S}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \psi_i} & & \| & & \\ 0 & \longrightarrow & \operatorname{Ker} \pi_{i+1}|_{\mathcal{S}} & \longrightarrow & \mathcal{T}(-, X_{i+1})|_{\mathcal{S}} & \xrightarrow{\pi_{i+1}} & F|_{\mathcal{S}} & \longrightarrow & 0 \end{array}$$

We compute the colimit of the sequence $(\psi_i)_{i \geqslant 0}$ in $\widehat{\mathcal{S}}$ and obtain an exact sequence

$$0 \longrightarrow \coprod_i \mathcal{T}(-, X_i)|_{\mathcal{S}} \xrightarrow{(\mathrm{id} - \psi_i)} \coprod_i \mathcal{T}(-, X_i)|_{\mathcal{S}} \longrightarrow F|_{\mathcal{S}} \longrightarrow 0 \tag{4.1}$$

because the sequence $(\psi_i)_{i \geqslant 0}$ is a coproduct of a sequence of zero maps and a sequence of identity maps.

Next consider the exact triangle

$$\coprod_i X_i \xrightarrow{(\mathrm{id} - \phi_i)} \coprod_i X_i \longrightarrow X \longrightarrow \Sigma(\coprod_i X_i)$$

and observe that

$$(\pi_i) \in \prod_i FX_i \cong F(\coprod_i X_i)$$

induces a map

$$\pi \colon \mathcal{T}(-, X) \longrightarrow F$$

by Yoneda's lemma. We have an isomorphism

$$\coprod_i \mathcal{T}(-, X_i)|_{\mathcal{S}} \cong \mathcal{T}(-, \coprod_i X_i)|_{\mathcal{S}}$$

because of the reformulation of condition (PG2) in lemma (4.5), and we obtain in $\widehat{\mathcal{S}}$ the following exact sequence.

$$\coprod_i \mathcal{T}(-,X_i)|_{\mathcal{S}} \xrightarrow{(\mathrm{id}-\psi_i)} \coprod_i \mathcal{T}(-,X_i)|_{\mathcal{S}} \longrightarrow \mathcal{T}(-,X)|_{\mathcal{S}} \longrightarrow$$
$$\coprod_i \mathcal{T}(-,\Sigma X_i)|_{\mathcal{S}} \xrightarrow{(\mathrm{id}-\Sigma\psi_i)} \coprod_i \mathcal{T}(-,\Sigma X_i)|_{\mathcal{S}} \longrightarrow \mathcal{T}(-,\Sigma X)|_{\mathcal{S}}$$

A comparison with the exact sequence (4.1) shows that

$$\pi|_{\mathcal{S}} \colon \mathcal{T}(-,X)|_{\mathcal{S}} \longrightarrow F|_{\mathcal{S}}$$

is an isomorphism, because $(\mathrm{id}-\Sigma\psi_i)$ is a monomorphism. Here, we use that $\Sigma S \cong S$.

Next observe that the class of objects Y in $\mathcal{T}$ such that π_Y is an isomorphism forms a triangulated subcategory of $\mathcal{T}$ which is closed under taking coproducts. Using condition (PG1), we conclude that π is an isomorphism.

(2) Let $F\colon \mathcal{T} \to \mathcal{U}$ be an exact functor. If F preserves all coproducts, then one defines the right adjoint G by sending an object X in $\mathcal{U}$ to the object in $\mathcal{T}$ representing $\mathrm{Hom}_{\mathcal{U}}(F-,X)$. Thus

$$\mathrm{Hom}_{\mathcal{U}}(F-,X) \cong \mathrm{Hom}_{\mathcal{T}}(-,GX).$$

Conversely, given a right adjoint of F, it is automatic that F preserves all coproducts. □

Remark. (1) In the presence of (PG2), the condition (PG1) is equivalent to the following condition.

(PG1') Let X be in $\mathcal{T}$ and suppose $\mathrm{Hom}_{\mathcal{T}}(\Sigma^n S, X) = 0$ for all $n \in \mathbb{Z}$. Then $X = 0$.

(2) The Brown representability theorem implies that a triangulated category $\mathcal{T}$ with a perfect generator has arbitrary products. In fact, given a family of objects X_i in $\mathcal{T}$, let $\prod_i X_i$ be the object representing $\prod_i \mathrm{Hom}_{\mathcal{T}}(-,X_i)$.

(3) There is the dual concept of a *perfect cogenerator* for a triangulated category. The dual Brown representability theorem for triangulated categories $\mathcal{T}$ with a perfect cogenerator characterizes the representable functors $\mathrm{Hom}_{\mathcal{T}}(X,-)$ as the cohomological and product preserving functors $\mathcal{T} \to \mathrm{Ab}$.

4.6. Notes. The abelianization of a triangulated category appears in Verdier's thèse [**29**] and in Freyd's work on the stable homotopy category [**8**]. Note that their construction is slightly different from the one given here, which is based on coherent functors in the sense of Auslander [**1**]. The idempotent completion of a triangulated category is studied by Balmer and Schlichting in [**3**]. Homotopy colimits appear in work of Bökstedt and Neeman [**4**]. The Brown representability theorem in homotopy theory is due to Brown [**5**]. Generalizations of the Brown representability theorem for triangulated categories can be found in work of Franke [**7**], Keller [**17**], and Neeman [**23, 24**]. The proof given here follows [**19**].

5. Resolutions

Resolutions are used to replace a complex in some abelian category $\mathcal{A}$ by another one which is quasi-isomorphic but easier to handle. Depending on properties of $\mathcal{A}$, injective and projective resolutions are constructed via Brown representability.

5.1. Injective resolutions. Let $\mathcal{A}$ be an abelian category. Suppose that $\mathcal{A}$ has arbitrary products which are exact, that is, for every family of exact sequences $X_i \to Y_i \to Z_i$ in $\mathcal{A}$, the sequence $\prod_i X_i \to \prod_i Y_i \to \prod_i Z_i$ is exact. Suppose in addition that $\mathcal{A}$ has an injective cogenerator which we denote by U. Thus $\operatorname{Hom}_{\mathcal{A}}(X,U) = 0$ implies $X = 0$ for every object X in $\mathcal{A}$. Observe that

$$\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(-,U) \cong \operatorname{Hom}_{\mathcal{A}}(H^0-,U). \tag{5.1}$$

Denote by $\mathbf{K}_{\mathrm{inj}}(\mathcal{A})$ the smallest full triangulated subcategory of $\mathbf{K}(\mathcal{A})$ which is closed under taking products and contains all injective objects of $\mathcal{A}$ (viewed as complexes concentrated in degree zero).

LEMMA. *The triangulated category $\mathbf{K}_{\mathrm{inj}}(\mathcal{A})$ is perfectly cogenerated by U. Therefore the inclusion $\mathbf{K}_{\mathrm{inj}}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ has a left adjoint $\mathbf{i}\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}_{\mathrm{inj}}(\mathcal{A})$.*

PROOF. Fix a family of maps $X_i \to Y_i$ in $\mathbf{K}(\mathcal{A})$ and suppose that

$$\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(Y_i,U) \longrightarrow \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(X_i,U)$$

is surjective for all i. The isomorphism (5.1) shows that $H^0X_i \to H^0Y_i$ is a monomorphism for all i. Taking products in $\mathcal{A}$ is exact, and therefore $H^0\colon \mathbf{K}(\mathcal{A}) \to \mathcal{A}$ preserves products. Thus $H^0(\prod_i X_i) \to H^0(\prod_i Y_i)$ is a monomorphism and

$$\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(\prod_i Y_i,U) \longrightarrow \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(\prod_i X_i,U)$$

is surjective.

The existence of a left adjoint for the inclusion $\mathbf{K}_{\mathrm{inj}}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ follows from the Brown representability theorem □

The left adjoint $\mathbf{i}\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}_{\mathrm{inj}}(\mathcal{A})$ induces for each complex X in $\mathcal{A}$ a natural map $X \to \mathbf{i}X$, and we may think of this as an injective resolution. Recall that for A in $\mathcal{A}$, a map $A \to I$ of complexes in $\mathcal{A}$ is an *injective resolution* if it is a quasi-isomorphism, each I^n is injective, and $I^n = 0$ for $n < 0$. An injective resolution $A \to I$ induces an isomorphism

$$\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(I,Y) \cong \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(A,Y)$$

for every complex Y in $\mathcal{A}$ with injective components. Moreover, I belongs to $\mathbf{K}_{\mathrm{inj}}(\mathcal{A})$. Therefore $I \cong \mathbf{i}A$ in $\mathbf{K}(\mathcal{A})$.

PROPOSITION. *Let $\mathcal{A}$ be an abelian category. Suppose $\mathcal{A}$ has an injective cogenerator and arbitrary products which are exact. Let X, Y be complexes in $\mathcal{A}$.*

(1) *The natural map $X \to \mathbf{i}X$ is a quasi-isomorphism and we have natural isomorphisms*

$$\operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(X,Y) \cong \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(X,\mathbf{i}Y) \cong \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(X,\mathbf{i}Y). \tag{5.2}$$

(2) *The composite*

$$\mathbf{K}_{\text{inj}}(\mathcal{A}) \xrightarrow{\text{inc}} \mathbf{K}(\mathcal{A}) \xrightarrow{\text{can}} \mathbf{D}(\mathcal{A})$$

is an equivalence of triangulated categories.

Proof. (1) The natural map $X \to \mathbf{i}X$ induces an isomorphism

$$\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(\mathbf{i}X, U) \longrightarrow \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(X, U)$$

and the isomorphism (5.1) shows that $X \to \mathbf{i}X$ is a quasi-isomorphism.

Now consider the class of complexes Y' such that the map $\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(X, Y') \to \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(X, Y')$ is bijective. This class contains U, by lemma (1.5), and the objects form a triangulated subcategory closed under taking products. Thus the map $\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(X, \mathbf{i}Y) \to \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(X, \mathbf{i}Y)$ is bijective for all Y, and

$$\operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(X, Y) \cong \operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(X, \mathbf{i}Y)$$

since $Y \to \mathbf{i}Y$ is an isomorphism in $\mathbf{D}(\mathcal{A})$.

(2) The first part of the proof shows that the functor is fully faithful and, up to isomorphism, surjective on objects. A quasi-inverse $\mathbf{D}(\mathcal{A}) \to \mathbf{K}_{\text{inj}}(\mathcal{A})$ is induced by $\mathbf{i}$. This follows from the universal property of the localization functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$, since $\mathbf{i}$ sends quasi-isomorphisms to isomorphisms. □

Remark. (1) Given an arbitrary abelian category $\mathcal{A}$, the formation of the derived category $\mathbf{D}(\mathcal{A})$, via the localization with respect to all quasi-isomorphisms, leads to a category where maps between two objects not necessarily form a set. Thus appropriate assumptions on $\mathcal{A}$ are needed. An equivalence $\mathbf{K}_{\text{inj}}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ implies that $\operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(X, Y)$ is actually a set for any pair of complexes X, Y in $\mathcal{A}$.

(2) The isomorphism (5.2) shows that the assignment $X \mapsto \mathbf{i}X$ induces a right adjoint for the canonical functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$.

(3) Let $\operatorname{Inj}\mathcal{A}$ denote the full subcategory of injective objects in $\mathcal{A}$. Then

$$\mathbf{K}^{+}(\operatorname{Inj}\mathcal{A}) \subseteq \mathbf{K}_{\text{inj}}(\mathcal{A}) \subseteq \mathbf{K}(\operatorname{Inj}\mathcal{A}).$$

For the first inclusion, see (4.4). Note that $\mathbf{K}_{\text{inj}}(\mathcal{A}) = \mathbf{K}(\operatorname{Inj}\mathcal{A})$ if every object in $\mathcal{A}$ has finite injective dimension.

Example. (1) An abelian category with a projective generator has exact products.

(2) Let $\mathcal{A}$ be the category of modules over a ring Λ. Then Λ is a projective generator and $\operatorname{Hom}_{\mathbb{Z}}(\Lambda^{\text{op}}, \mathbb{Q}/\mathbb{Z})$ is an injective cogenerator for $\mathcal{A}$. Therefore $\mathcal{A}$ has exact products and exact coproducts.

(3) The category of quasi-coherent sheaves on the projective line $\mathbf{P}^1_k$ over a field k does not have exact products.

5.2. Projective Resolutions. The existence of injective resolutions turns into the existence of projective resolutions if one passes from an abelian category to its opposite category. Because of this symmetry, complexes in an abelian category $\mathcal{A}$ admit projective resolutions provided that $\mathcal{A}$ has a projective generator and arbitrary coproducts which are exact. Keep these assumptions on $\mathcal{A}$, and denote by $\mathbf{K}_{\text{proj}}(\mathcal{A})$ the smallest full triangulated subcategory of $\mathbf{K}(\mathcal{A})$ which is closed under taking coproducts and contains all projective objects of $\mathcal{A}$. Then one obtains a right adjoint $\mathbf{p}\colon \mathbf{K}(\mathcal{A}) \to \mathbf{K}_{\text{proj}}(\mathcal{A})$ of the inclusion. For every complex X, the natural map $\mathbf{p}X \to X$ has the dual properties of the injective resolution $X \to \mathbf{i}X$. The precise formulation is left to the reader. Note that any module category has a projective generator and exact coproducts.

5.3. Derived functors. Injective and projective resolutions are used to define derived functors. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories. The *right derived functor* $\mathbf{R}F\colon \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$ of F sends a complex X to $F(\mathbf{i}X)$, and the *left derived functor* $\mathbf{L}F\colon \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$ sends a complex X to $F(\mathbf{p}X)$.

EXAMPLE. Let Λ and Γ be a pair of rings and ${}_\Lambda B_\Gamma$ a bimodule. Then the pair of adjoint functors

$$\operatorname{Mod}\Lambda \underset{T=-\otimes_\Lambda B}{\overset{H=\operatorname{Hom}_\Gamma(B,-)}{\rightleftarrows}} \operatorname{Mod}\Gamma$$

induces a pair of adjoint functors

$$\mathbf{D}(\operatorname{Mod}\Lambda) \underset{\mathbf{L}T=-\otimes^{\mathbf{L}}_\Lambda B}{\overset{\mathbf{R}H=\mathbf{R}\operatorname{Hom}_\Gamma(B,-)}{\rightleftarrows}} \mathbf{D}(\operatorname{Mod}\Gamma)\ .$$

This follows from the fact that $\mathbf{L}T$ and $\mathbf{R}H$ are composed from three pairs

$$\mathbf{D}(\operatorname{Mod}\Lambda) \underset{\mathbf{p}}{\overset{\mathrm{can}}{\rightleftarrows}} \mathbf{K}(\operatorname{Mod}\Lambda) \underset{\mathbf{K}(T)}{\overset{\mathbf{K}(H)}{\rightleftarrows}} \mathbf{K}(\operatorname{Mod}\Gamma) \underset{\mathrm{can}}{\overset{\mathbf{i}}{\rightleftarrows}} \mathbf{D}(\operatorname{Mod}\Gamma)$$

of adjoint functors.

5.4. Notes. Injective and projective resolutions are needed to construct derived functors. The first application of this formalism is Grothendieck's duality theory [**13**]. Resolutions of unbounded complexes are established by Spaltenstein in [**28**], and also by Bökstedt and Neeman in [**4**]. The existence proof given here has the advantage that it generalizes easily to other settings, for instance to differential graded modules.

6. Differential graded algebras and categories

Differential graded algebras arise as complexes with an additional multiplicative structure, and differential graded categories are differential graded algebras with several objects. The concept generalizes that of an ordinary associative algebra, and we study in a similar way categories of modules and derived categories for such differential graded algebras and categories.

6.1. Differential graded algebras and modules. A *differential graded algebra* or *dg algebra* is a $\mathbb{Z}$-graded associative algebra

$$A = \coprod_{n\in\mathbb{Z}} A^n$$

over some commutative ring k, together with a differential $d\colon A \to A$, that is, a homogeneous k-linear map of degree $+1$ satisfying $d^2 = 0$ and the Leibniz rule

$$d(xy) = d(x)y + (-1)^n x d(y) \quad \text{for all} \quad x \in A^n \quad \text{and} \quad y \in A.$$

A *dg A-module* is a $\mathbb{Z}$-graded (right) A-module X, together with a differential $d\colon X \to X$, that is, a homogeneous k-linear map of degree $+1$ satisfying $d^2 = 0$ and the Leibniz rule

$$d(xy) = d(x)y + (-1)^n x d(y) \quad \text{for all} \quad x \in X^n \quad \text{and} \quad y \in A.$$

A morphism of dg A-modules is an A-linear map which is homogeneous of degree 0 and commutes with the differential. We denote by $\mathbf{C}_{\mathrm{dg}}(A)$ the category of dg A-modules.

EXAMPLE. (1) An associative algebra Λ can be viewed as a dg algebra A if one defines $A^0 = \Lambda$ and $A^n = 0$ otherwise. In this case $\mathbf{C}_{\mathrm{dg}}(A) = \mathbf{C}(\operatorname{Mod}\Lambda)$.

(2) Let X, Y be complexes in some additive category $\mathcal{C}$. Define a new complex $\mathcal{H}om_{\mathcal{C}}(X,Y)$ as follows. The nth component is

$$\prod_{p\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{C}}(X^p, Y^{p+n})$$

and the differential is given by

$$d^n(\phi^p) = d_Y \circ \phi^p - (-1)^n \phi^{p+1} \circ d_X.$$

Note that

$$H^n\, \mathcal{H}om_{\mathcal{C}}(X,Y) \cong \operatorname{Hom}_{\mathbf{K}(\mathcal{C})}(X, \Sigma^n Y)$$

because $\operatorname{Ker} d^n$ identifies with $\operatorname{Hom}_{\mathbf{C}(\mathcal{C})}(X, \Sigma^n Y)$ and $\operatorname{Im} d^{n-1}$ with the ideal of null-homotopic maps $X \to \Sigma^n Y$. The composition of graded maps yields a dg algebra structure for

$$\mathcal{E}nd_{\mathcal{C}}(X) = \mathcal{H}om_{\mathcal{C}}(X,X)$$

and $\mathcal{H}om_{\mathcal{C}}(X,Y)$ is a dg module over $\mathcal{E}nd_{\mathcal{C}}(X)$.

A map $\phi\colon X \to Y$ of dg A-modules is *null-homotopic* if there is a map $\rho\colon X \to Y$ of graded A-modules which is homogeneous of degree -1 such that $\phi = d_Y \circ \rho + \rho \circ d_X$. The null-homotopic maps form an ideal and the *homotopy category* $\mathbf{K}_{\mathrm{dg}}(A)$ is the quotient of $\mathbf{C}_{\mathrm{dg}}(A)$ with respect to this ideal. The homotopy category carries a triangulated structure which is defined as before for the homotopy category $\mathbf{K}(\mathcal{A})$ of an additive category $\mathcal{A}$.

A map $X \to Y$ of dg A-modules is a *quasi-isomorphism* if it induces isomorphisms $H^n X \to H^n Y$ in each degree. The *derived category* of the dg algebra A is by definition the localization

$$\mathbf{D}_{\mathrm{dg}}(A) = \mathbf{K}_{\mathrm{dg}}(A)[S^{-1}]$$

of $\mathbf{K}_{\mathrm{dg}}(A)$ with respect to the class S of all quasi-isomorphisms. Note that S is a multiplicative system and compatible with the triangulation. Therefore $\mathbf{D}_{\mathrm{dg}}(A)$ is triangulated and the localization functor $\mathbf{K}_{\mathrm{dg}}(A) \to \mathbf{D}_{\mathrm{dg}}(A)$ is exact.

6.2. Differential graded categories. Let k be a commutative ring. A category is *k-linear* if the maps between any two objects form a k-module and all composition maps are bilinear. A category is *small* if the isomorphism classes of objects form a set. Now fix a small k-linear category $\mathcal{A}$. We think of $\mathcal{A}$ as an *algebra with several objects*, because a k-algebra is nothing but a k-linear category with precisely one object. The *modules* over $\mathcal{A}$ are by definition the k-linear functors $\mathcal{A}^{\mathrm{op}} \to \operatorname{Mod} k$. For example, the *free* $\mathcal{A}$-modules are the representable functors

$$\mathcal{A}(-, A) = \operatorname{Hom}_{\mathcal{A}}(-, A)$$

with A in $\mathcal{A}$. This terminology is justified by the fact that the modules over an algebra Λ can be identified with k-linear functors $\Lambda^{\mathrm{op}} \to \operatorname{Mod} k$, where Λ is viewed as a category with a single object.

A *dg category* $\mathcal{A}$ is by definition a dg algebra with several objects. More precisely, $\mathcal{A}$ is a $\mathbb{Z}$-graded k-linear category, that is,

$$\mathcal{A}(A, B) = \coprod_{n \in \mathbb{Z}} \mathcal{A}(A, B)^n$$

is a $\mathbb{Z}$-graded k-module for all A, B in $\mathcal{A}$, and the composition maps

$$\mathcal{A}(A, B) \times \mathcal{A}(B, C) \longrightarrow \mathcal{A}(A, C)$$

are bilinear and homogeneous of degree 0. In addition, there are differentials $d\colon \mathcal{A}(A, B) \to \mathcal{A}(A, B)$ for all A, B in $\mathcal{A}$, that is, homogeneous k-linear maps of degree $+1$ satisfying $d^2 = 0$ and the Leibniz rule.

The *opposite category* $\mathcal{A}^{\mathrm{op}}$ of a dg category $\mathcal{A}$ has the same objects, the maps are $\mathcal{A}^{\mathrm{op}}(A, B) = \mathcal{A}(B, A)$ for all objects A, B, and the composition maps are

$$\mathcal{A}^{\mathrm{op}}(A, B)^p \times \mathcal{A}^{\mathrm{op}}(B, C)^q \longrightarrow \mathcal{A}^{\mathrm{op}}(A, C)^{p+q}, \quad (\phi, \psi) \mapsto (-1)^{pq} \phi \circ \psi.$$

Keeping the differentials from $\mathcal{A}$, one checks easily that the Leibniz rule holds in $\mathcal{A}^{\mathrm{op}}$.

Let $\mathcal{A}$ be a small dg category. A *dg $\mathcal{A}$-module* is a graded functor $X\colon \mathcal{A}^{\mathrm{op}} \to \operatorname{Gr} k$ into the category of graded k-modules, that is, the maps

$$\mathcal{A}^{\mathrm{op}}(A, B) \longrightarrow \operatorname{Hom}_{\operatorname{Gr} k}(XA, XB)$$

are homogeneous k-linear of degree 0, where

$$\operatorname{Hom}_{\operatorname{Gr} k}(M, N)^p = \{\phi \in \operatorname{Hom}_k(M, N) \mid \phi(M^q) \subseteq N^{q+p} \text{ for all } q \in \mathbb{Z}\}$$

for M, N in $\operatorname{Gr} k$. In addition, there are differentials $d\colon XA \to XA$ for all A in $\mathcal{A}$, that is, homogeneous k-linear maps of degree $+1$ satisfying $d^2 = 0$ and the Leibniz rule

$$d(xy) = d(x)y + (-1)^n x d(y) \quad \text{for all} \quad x \in (XA)^n \quad \text{and} \quad y \in \mathcal{A}(B, A)^p,$$

where $xy = (-1)^{np}(Xy)x$.

Example. (1) A dg algebra can be viewed as a dg category with one object. Conversely, a dg category $\mathcal{A}$ with one object A gives a dg algebra $\mathcal{A}(A, A)$.

(2) A class $\mathcal{A}_0$ of complexes in some additive category $\mathcal{C}$ gives rise to a dg category $\mathcal{A}$. Define $\mathcal{A}(X, Y) = \mathcal{H}om_{\mathcal{C}}(X, Y)$ for X, Y in $\mathcal{A}_0$. The composition in $\mathcal{A}$ is induced by the composition of graded maps in $\mathcal{C}$. If $\mathcal{A}$ is small and Y is any complex in $\mathcal{C}$, the functor sending X in $\mathcal{A}$ to $\mathcal{H}om_{\mathcal{C}}(X, Y)$ is a dg $\mathcal{A}$-module.

Let $\mathcal{A}$ be a small dg category. A map $\phi\colon X \to Y$ between dg $\mathcal{A}$-modules is a natural transformation such that the maps $\phi_A\colon XA \to YA$ are homogeneous of degree zero and commute with the differentials. We denote by $\mathbf{C}_{\mathrm{dg}}(\mathcal{A})$ the category of dg $\mathcal{A}$-modules. The homotopy category $\mathbf{K}_{\mathrm{dg}}(\mathcal{A})$ and the derived category $\mathbf{D}_{\mathrm{dg}}(\mathcal{A})$ are defined as before for dg algebras.

6.3. Duality. Let $\mathcal{A}$ be a small dg category over some commutative ring k. We fix an injective cogenerator E of the category of k-modules. The duality $\operatorname{Hom}_k(-,E)$ between k-modules induces a duality between dg modules over $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ as follows. Let X be a dg $\mathcal{A}$-module. Define a dg $\mathcal{A}^{\mathrm{op}}$-module DX by

$$\big((DX)A\big)^n = \operatorname{Hom}_k\big((XA)^{-n},E\big) \quad \text{and} \quad \big((DX)\phi\big)\alpha = (-1)^{pq}\alpha\circ X\phi$$

for $\phi\in\mathcal{A}(A,B)^p$ and $\alpha\in\big((DX)A\big)^q$. The differential is

$$d^n_{DX} = (-1)^{n+1}\operatorname{Hom}_k(d_X^{-n-1},E).$$

Given k-modules M,N, there is a natural isomorphism

$$\operatorname{Hom}_k\big(M,\operatorname{Hom}_k(N,E)\big)\cong\operatorname{Hom}_k\big(N,\operatorname{Hom}_k(M,E)\big)$$

which sends χ to $\operatorname{Hom}_k(\chi,E)\circ\delta_N$. Here,

$$\delta_N\colon N\longrightarrow\operatorname{Hom}_k\big(\operatorname{Hom}_k(N,E),E\big)$$

is the evaluation map. The isomorphism induces for dg modules X over $\mathcal{A}$ and Y over $\mathcal{A}^{\mathrm{op}}$ a natural isomorphism

$$\operatorname{Hom}_{\mathbf{C}_{\mathrm{dg}}(\mathcal{A})}(X,DY)\cong\operatorname{Hom}_{\mathbf{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}})}(Y,DX).$$

This isomorphism preserves null-homotopic maps and quasi-isomorphisms.

Lemma. *Let X and Y be dg modules over $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ respectively. Then there are natural isomorphisms*

$$\operatorname{Hom}_{\mathbf{K}_{\mathrm{dg}}(\mathcal{A})}(X,DY)\cong\operatorname{Hom}_{\mathbf{K}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}})}(Y,DX),$$
$$\operatorname{Hom}_{\mathbf{D}_{\mathrm{dg}}(\mathcal{A})}(X,DY)\cong\operatorname{Hom}_{\mathbf{D}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}})}(Y,DX).$$

Remark. The duality D is unique up to isomorphism and depends only on k, if we choose for E a minimal injective cogenerator. Note that E is *minimal* if it is an injective envelope of a coproduct of a representative set of all simple k-modules.

6.4. Injective and projective resolutions. Let $\mathcal{A}$ be a small dg category over some commutative ring k. For A,B in $\mathcal{A}$, we define the *free* dg $\mathcal{A}$-module $A^\wedge=\mathcal{A}^{\mathrm{op}}(A,-)$ and the *injective* dg $\mathcal{A}$-module $B^\vee=DB^\wedge$, where $B^\wedge$ denotes the free $\mathcal{A}^{\mathrm{op}}$-module corresponding to B.

Lemma. *Let X be a dg $\mathcal{A}$-module. Then we have for $A\in\mathcal{A}$*

$$\operatorname{Hom}_{\mathbf{K}_{\mathrm{dg}}(\mathcal{A})}(A^\wedge,X)\cong H^0(XA) \quad \textit{and} \quad \operatorname{Hom}_{\mathbf{K}_{\mathrm{dg}}(\mathcal{A})}(X,A^\vee)\cong\operatorname{Hom}_k(H^0(XA),E).$$

Proof. The first isomorphism follows from Yoneda's lemma. In fact, taking morphisms of graded $\mathcal{A}$-modules, we have $\operatorname{Hom}_{\mathcal{A}}(A^\wedge,X)\cong XA$. Restricting to morphisms which commute with the differential, we get $\operatorname{Hom}_{\mathbf{C}_{\mathrm{dg}}(\mathcal{A})}(A^\wedge,X)\cong Z^0(XA)$. Finally, taking morphism up to null-homotopic maps, we get the isomorphism $\operatorname{Hom}_{\mathbf{K}_{\mathrm{dg}}(\mathcal{A})}(A^\wedge,X)\cong H^0(XA)$.

The second isomorphism is an immediate consequence if one uses the isomorphism

$$\operatorname{Hom}_{\mathbf{K}_{\mathrm{dg}}(\mathcal{A})}(X,DA^\wedge)\cong\operatorname{Hom}_{\mathbf{K}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}})}(A^\wedge,DX).$$

□

Let us denote by $\mathbf{K}_{\mathrm{pdg}}(\mathcal{A})$ the smallest full triangulated subcategory of $\mathbf{K}_{\mathrm{dg}}(\mathcal{A})$ which is closed under coproducts and contains all free dg modules $A^{\wedge}$, $A \in \mathcal{A}$. Analogously, $\mathbf{K}_{\mathrm{idg}}(\mathcal{A})$ denotes the smallest full triangulated subcategory of $\mathbf{K}_{\mathrm{dg}}(\mathcal{A})$ which is closed under products and contains all injective dg modules $A^{\vee}$, $A \in \mathcal{A}$.

LEMMA. (1) *The category $\mathbf{K}_{\mathrm{pdg}}(\mathcal{A})$ is perfectly generated by $\coprod_{A\in\mathcal{A}} A^{\wedge}$. Therefore the inclusion has a right adjoint $\mathbf{p}\colon \mathbf{K}_{\mathrm{dg}}(\mathcal{A}) \to \mathbf{K}_{\mathrm{pdg}}(\mathcal{A})$.*

(2) *The category $\mathbf{K}_{\mathrm{idg}}(\mathcal{A})$ is perfectly cogenerated by $\prod_{A\in\mathcal{A}} A^{\vee}$. Therefore the inclusion has a left adjoint $\mathbf{i}\colon \mathbf{K}_{\mathrm{dg}}(\mathcal{A}) \to \mathbf{K}_{\mathrm{idg}}(\mathcal{A})$.*

PROOF. Adapt the proof of lemma (5.1). □

THEOREM. *Let $\mathcal{A}$ be a small dg category and X, Y be dg $\mathcal{A}$-modules.*

(1) *The natural map $\mathbf{p}X \to X$ is a quasi-isomorphism and we have*
$$\mathrm{Hom}_{\mathbf{D}_{\mathrm{dg}}(\mathcal{A})}(X,Y) \cong \mathrm{Hom}_{\mathbf{D}_{\mathrm{dg}}(\mathcal{A})}(\mathbf{p}X,Y) \cong \mathrm{Hom}_{\mathbf{K}_{\mathrm{dg}}(\mathcal{A})}(\mathbf{p}X,Y).$$

(2) *We have for all $A \in \mathcal{A}$ and $n \in \mathbb{Z}$*
$$\mathrm{Hom}_{\mathbf{D}_{\mathrm{dg}}(\mathcal{A})}(A^{\wedge}, \Sigma^n Y) \cong H^n(YA).$$

(3) *The natural map $Y \to \mathbf{i}Y$ is a quasi-isomorphism and we have*
$$\mathrm{Hom}_{\mathbf{D}_{\mathrm{dg}}(\mathcal{A})}(X,Y) \cong \mathrm{Hom}_{\mathbf{D}_{\mathrm{dg}}(\mathcal{A})}(X,\mathbf{i}Y) \cong \mathrm{Hom}_{\mathbf{K}_{\mathrm{dg}}(\mathcal{A})}(X,\mathbf{i}Y).$$

(4) *The functors*
$$\mathbf{K}_{\mathrm{pdg}}(\mathcal{A}) \xrightarrow{\mathrm{inc}} \mathbf{K}_{\mathrm{dg}}(\mathcal{A}) \xrightarrow{\mathrm{can}} \mathbf{D}_{\mathrm{dg}}(\mathcal{A}) \quad \textit{and} \quad \mathbf{K}_{\mathrm{idg}}(\mathcal{A}) \xrightarrow{\mathrm{inc}} \mathbf{K}_{\mathrm{dg}}(\mathcal{A}) \xrightarrow{\mathrm{can}} \mathbf{D}_{\mathrm{dg}}(\mathcal{A})$$
are equivalences of triangulated categories.

PROOF. Adapt the proof of proposition (5.1). □

6.5. Compact objects and perfect complexes. Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts. An object X in $\mathcal{T}$ is called *compact*, if every map $X \to \coprod_{i\in I} Y_i$ factors through $\coprod_{i\in J} Y_i$ for some finite subset of $J \subseteq I$. Note that X is compact if and only if the functor $\mathrm{Hom}_{\mathcal{T}}(X,-)\colon \mathcal{T} \to \mathrm{Ab}$ preserves all coproducts. This characterization implies that the compact objects in $\mathcal{T}$ form a thick subcategory. Let us denote this subcategory by $\mathcal{T}^c$.

LEMMA. *Suppose there is a set $\mathcal{S}$ of compact objects in $\mathcal{T}$ such that $\mathcal{T}$ admits no proper triangulated subcategory which contains $\mathcal{S}$ and is closed under coproducts. Then $\mathcal{T}^c$ coincides with the smallest full and thick subcategory which contains $\mathcal{S}$.*

PROOF. First observe that the coproduct of all objects in $\mathcal{S}$ is a perfect generator for $\mathcal{T}$. Now let X be a compact object in $\mathcal{T}$ and let $F = \mathrm{Hom}_{\mathcal{T}}(-,X)$. Consider the sequence
$$X_0 \xrightarrow{\phi_0} X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} \cdots$$
of maps in the proof of the Brown representability theorem. Thus X is a homotopy colimit of this sequence and fits into the exact triangle
$$\coprod_{i\geqslant 0} X_i \xrightarrow{(\mathrm{id}-\phi_i)} \coprod_{i\geqslant 0} X_i \xrightarrow{\chi} X \xrightarrow{\psi} \Sigma(\coprod_{i\geqslant 0} X_i).$$
The map ψ factors through some finite coproduct $\coprod_i \Sigma X_i$ and this implies $\psi = 0$ because $(\mathrm{id} - \Sigma\phi_i) \circ \psi = 0$. Thus the identity map id_X factors through χ. Therefore

X is a direct factor of some finite coproduct $\coprod_i X_i$. Each X_i is obtained from coproducts of objects in $\mathcal{S}$ by a finite number of extensions. Comparing this with the construction of the thick subcategory generated by $\mathcal{S}$ in (3.3), one can show that X belongs to the thick subcategory generated by $\mathcal{S}$; see [**22**, 2.2] for details. □

We can now describe the compact objects in some derived categories. Let Λ be an associative ring. Then Λ, viewed as a complex concentrated in degree zero, is a compact object in $\mathbf{D}(\operatorname{Mod}\Lambda)$, since

$$\operatorname{Hom}_{\mathbf{D}(\operatorname{Mod}\Lambda)}(\Lambda, X) \cong H^0 X.$$

A complex of Λ-modules is called *perfect* if it is quasi-isomorphic to a bounded complex of finitely generated projective Λ-modules. The perfect complexes are precisely the compact objects in $\mathbf{D}(\operatorname{Mod}\Lambda)$. This description of compact objects extends to the derived category of a dg category.

PROPOSITION. *Let $\mathcal{A}$ be a small dg category. A dg $\mathcal{A}$-module is compact in $\mathbf{D}_{\mathrm{dg}}(\mathcal{A})$ if and only if it belongs to the smallest thick subcategory of $\mathbf{D}_{\mathrm{dg}}(\mathcal{A})$ which contains all free dg modules $A^\wedge$, $A \in \mathcal{A}$.*

PROOF. It follows from theorem (6.4) that each free module is compact, and that $\mathbf{D}_{\mathrm{dg}}(\mathcal{A})$ admits no proper triangulated subcategory which contains all free module and is closed under coproducts. Now apply lemma (6.5). □

6.6. Notes. Differential graded algebras were introduced by Cartan in order to study the cohomology of Eilenberg-MacLane spaces [**6**]. Differential graded categories and their derived categories provide a conceptual framework for tilting theory; they are studied systematically by Keller in [**17**]. In particular, the existence of projective and injective resolutions for dg modules is proved in [**17**]; see also [**2**]. The analysis of compact objects is due to Neeman [**22**].

7. Algebraic triangulated categories

Algebraic triangulated categories are triangulated categories which arise from algebraic constructions. There is a generic construction which assigns to an exact Frobenius category its stable category. On the other hand, every algebraic triangulated category embeds into the derived category of some dg category.

7.1. Exact categories. Let $\mathcal{A}$ be an exact category in the sense of Quillen [**27**]. Thus $\mathcal{A}$ is an additive category, together with a distinguished class of sequences

$$0 \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \longrightarrow 0$$

which are called *exact*. The exact sequences satisfy a number of axioms. In particular, the maps α and β in each exact sequence as above form a *kernel-cokernel pair*, that is α is a kernel of β and β is a cokernel of α. A map in $\mathcal{A}$ which arises as the kernel in some exact sequence is called *admissible mono*; a map arising as a cokernel is called *admissible epi*. A full subcategory $\mathcal{B}$ of $\mathcal{A}$ is *extension-closed* if every exact sequence in $\mathcal{A}$ belongs to $\mathcal{B}$ provided its endterms belongs to $\mathcal{B}$.

Remark. (1) Any abelian category is exact with respect to the class of all short exact sequences.

(2) Any full and extension-closed subcategory $\mathcal{B}$ of an exact category $\mathcal{A}$ is exact with respect to the class of sequences which are exact in $\mathcal{A}$.

(3) Any small exact category arises, up to an exact equivalence, as a full and extension-closed subcategory of a module category; see (7.7).

7.2. Frobenius categories. Let $\mathcal{A}$ be an exact category. An object P is *projective* if the induced map $\operatorname{Hom}_{\mathcal{A}}(P,Y) \to \operatorname{Hom}_{\mathcal{A}}(P,Z)$ is surjective for every admissible epi $Y \to Z$. Dually, an object Q is *injective* if the induced map $\operatorname{Hom}_{\mathcal{A}}(Y,Q) \to \operatorname{Hom}_{\mathcal{A}}(X,Q)$ is surjective for every admissible mono $X \to Y$. The category $\mathcal{A}$ has *enough projectives* if every object Z admits an admissible epi $Y \to Z$ with Y projective. And $\mathcal{A}$ has *enough injectives* if every object X admits an admissible mono $X \to Y$ with Y injective. Finally, $\mathcal{A}$ is called a *Frobenius category*, if $\mathcal{A}$ has enough projectives and enough injectives and if both coincide.

Example. (1) Let $\mathcal{A}$ be an additive category. Then $\mathcal{A}$ is an exact category with respect to the class of all split exact sequences in $\mathcal{A}$. All objects are projective and injective, and $\mathcal{A}$ is a Frobenius category.

(2) Let $\mathcal{A}$ be an additive category. The category $\mathbf{C}(\mathcal{A})$ of complexes is exact with respect to the class of all sequences $0 \to X \to Y \to Z \to 0$ such that $0 \to X^n \to Y^n \to Z^n \to 0$ is split exact for all $n \in \mathbb{Z}$. A typical projective and injective object is a complex of the form

$$I_A\colon \quad \cdots \longrightarrow 0 \longrightarrow A \xrightarrow{\mathrm{id}} A \longrightarrow 0 \longrightarrow \cdots$$

for some A in $\mathcal{A}$. There is an obvious admissible mono $X \to \prod_{n\in\mathbb{Z}} \Sigma^{-n} I_{X^n}$ and also an admissible epi $\coprod_{n\in\mathbb{Z}} \Sigma^{-n-1} I_{X^n} \to X$. Also,

$$\coprod_{n\in\mathbb{Z}} \Sigma^{-n} I_{X^n} \cong \prod_{n\in\mathbb{Z}} \Sigma^{-n} I_{X^n}.$$

Thus $\mathbf{C}(\mathcal{A})$ is a Frobenius category.

7.3. The derived category of an exact category. Let $\mathcal{A}$ be an exact category. A complex X in $\mathcal{A}$ is called *acyclic* if for each $n \in \mathbb{Z}$ there is an exact sequence

$$0 \longrightarrow Z^n \xrightarrow{\alpha^n} X^n \xrightarrow{\beta^n} Z^{n+1} \longrightarrow 0$$

in $\mathcal{A}$ such that $d_X^n = \alpha^{n+1} \circ \beta^n$. A map $Y \to Z$ of complexes in $\mathcal{A}$ is a *quasi-isomorphism*, if it fits into an exact triangle $X \to Y \to Z \to \Sigma X$ in $\mathbf{K}(\mathcal{A})$ where X is isomorphic to an acyclic complex. Note that this definition coincides with our previous one, if $\mathcal{A}$ is abelian. The class S of quasi-isomorphism in $\mathbf{K}(\mathcal{A})$ is a multiplicative system and compatible with the triangulation. The derived category of the exact category $\mathcal{A}$ is by definition the localization

$$\mathbf{D}(A) = \mathbf{K}(\mathcal{A})[S^{-1}]$$

with respect to S.

Example. Let $\mathcal{A}$ be an additive category and view it as an exact category with respect to the class of all split exact sequences. Then $\mathbf{D}(\mathcal{A}) = \mathbf{K}(\mathcal{A})$.

7.4. The stable category of a Frobenius category. Let $\mathcal{A}$ be a Frobenius category. The *stable category* $\mathbf{S}(\mathcal{A})$ is by definition the quotient of $\mathcal{A}$ with respect to the ideal $\mathcal{I}$ of maps which factor through an injective object. Thus

$$\operatorname{Hom}_{\mathbf{S}(\mathcal{A})}(X,Y) = \operatorname{Hom}_{\mathcal{A}}(X,Y)/\mathcal{I}(X,Y)$$

for all X, Y in $\mathcal{A}$. We choose for each X in $\mathcal{A}$ an exact sequence

$$0 \longrightarrow X \longrightarrow E \longrightarrow \Sigma X \longrightarrow 0$$

such that E is injective, and obtain an equivalence $\Sigma\colon \mathbf{S}(\mathcal{A}) \to \mathbf{S}(\mathcal{A})$ by sending X to ΣX. Every exact sequence $0 \to X \to Y \to Z \to 0$ fits into a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & X & \longrightarrow & E & \longrightarrow & \Sigma X & \longrightarrow & 0 \end{array}$$

such that E is injective. A triangle in $\mathbf{S}(\mathcal{A})$ is by definition *exact* if it isomorphic to a sequence of maps

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

as above.

LEMMA. *The stable category of a Frobenius category is triangulated.*

PROOF. It is easy to verify the axioms, once one observes that every map in $\mathbf{S}(\mathcal{A})$ can be represented by an admissible mono in $\mathcal{A}$. Note that a homotopy cartesian square can be represented by a pull-back and push-out square. This gives a proof for (TR4'). □

EXAMPLE. The category of complexes $\mathbf{C}(\mathcal{A})$ of an additive category $\mathcal{A}$ is a Frobenius category with respect to the degree-wise split exact sequences. The maps factoring through an injective object are precisely the null-homotopic maps. Thus the stable category of $\mathbf{C}(\mathcal{A})$ coincides with the homotopy category $\mathbf{K}(\mathcal{A})$. Note that the triangulated structures which have been defined via mapping cones and via exact sequences in $\mathbf{C}(\mathcal{A})$ coincide.

7.5. Algebraic triangulated categories. A triangulated category $\mathcal{T}$ is called *algebraic* if it satisfies the equivalent conditions of the following lemma.

LEMMA. *For a triangulated category $\mathcal{T}$, the following are equivalent.*

(1) *There is an exact equivalence $\mathcal{T} \to \mathbf{S}(\mathcal{A})$ for some Frobenius category $\mathcal{A}$.*
(2) *There is a fully faithful exact functor $\mathcal{T} \to \mathbf{K}(\mathcal{B})$ for some additive category $\mathcal{B}$.*
(3) *There is a fully faithful exact functor $\mathcal{T} \to \mathbf{S}(\mathcal{C})$ for some Frobenius category $\mathcal{C}$.*

PROOF. (1) ⇒ (2): Let $\mathcal{U}$ denote the full subcategory of all objects in $\mathbf{K}(\mathcal{A})$ which are acyclic complexes with injective components. The functor sending a complex X to Z^0X induces an equivalence $\mathcal{U} \to \mathbf{S}(\mathcal{A})$. Composing a quasi-inverse with with the equivalence $\mathcal{T} \to \mathbf{S}(\mathcal{A})$ and the inclusion $\mathcal{U} \to \mathbf{K}(\mathcal{A})$ gives a fully faithful exact functor $\mathcal{T} \to \mathbf{K}(\mathcal{A})$.

(2) $\Rightarrow$ (3): Clear, since $\mathbf{K}(\mathcal{B}) = \mathbf{S}(\mathbf{C}(\mathcal{B}))$.

(3) $\Rightarrow$ (1): We identify $\mathcal{T}$ with a full triangulated subcategory of $\mathbf{S}(\mathcal{C})$. Denote by $F\colon \mathcal{C} \to \mathbf{S}(\mathcal{C})$ the canonical projection. Then the full subcategory $\mathcal{A}$ of all objects X with FX in $\mathcal{T}$ is an extension-closed subcategory of $\mathcal{C}$, containing all projectives and injectives from $\mathcal{C}$. Thus $\mathcal{A}$ is a Frobenius category and $\mathbf{S}(\mathcal{A})$ is equivalent to $\mathcal{T}$. □

Example. (1) Let $\mathcal{A}$ be an abelian category with enough injectives. Then $\mathbf{D}^b(\mathcal{A})$ is algebraic. To see this, observe that the composite

$$\mathbf{K}^+(\operatorname{Inj}\mathcal{A}) \xrightarrow{\text{inc}} \mathbf{K}(\mathcal{A}) \xrightarrow{\text{can}} \mathbf{D}(\mathcal{A})$$

is fully faithful; see lemma (1.5). The image of this functor contains $\mathbf{D}^b(\mathcal{A})$, because we can identify it with the smallest full triangulated subcategory of $\mathbf{D}(\mathcal{A})$ which contains the injective resolutions of all objects in $\mathcal{A}$. Thus $\mathbf{D}^b(\mathcal{A})$ is equivalent to a full triangulated subcategory of $\mathbf{K}(\mathcal{A})$.

(2) The derived category $\mathbf{D}_{\mathrm{dg}}(\mathcal{A})$ of a dg category $\mathcal{A}$ is algebraic. This follows from the fact that the homotopy category $\mathbf{K}_{\mathrm{dg}}(\mathcal{A})$ is algebraic, and that $\mathbf{D}_{\mathrm{dg}}(\mathcal{A})$ is equivalent to the full triangulated subcategory $\mathbf{K}_{\mathrm{pdg}}(\mathcal{A})$.

Theorem. *Let $\mathcal{T}$ be an algebraic triangulated category and $\mathcal{S}$ be a full subcategory which is small. Then there exists a dg category $\mathcal{A}$ and an exact functor $F\colon \mathcal{T} \to \mathbf{D}_{\mathrm{dg}}(\mathcal{A})$ having the following properties.*

1. *F identifies $\mathcal{S}$ with the full subcategory formed by the free dg modules $A^\wedge$, $A \in \mathcal{A}$.*
2. *F identifies (up to direct factors) the smallest full and thick subcategory of $\mathcal{T}$ containing $\mathcal{S}$ with the full subcategory formed by the compact objects in $\mathbf{D}_{\mathrm{dg}}(\mathcal{A})$.*
3. *Suppose $\mathcal{T}$ has arbitrary coproducts and every object in $\mathcal{S}$ is compact. Then F identifies the smallest full triangulated subcategory of $\mathcal{T}$ which contains $\mathcal{S}$ and is closed under all coproducts with $\mathbf{D}_{\mathrm{dg}}(\mathcal{A})$.*

Proof. Let $\mathcal{T} = \mathbf{S}(\mathcal{B})$ for some Frobenius category $\mathcal{B}$. We denote by $\tilde{\mathcal{B}}$ the full subcategory of $\mathbf{C}(\mathcal{B})$ which is formed by all acyclic complexes in $\mathcal{B}$ having injective components. This is a Frobenius category with respect to the degree-wise split exact sequences. The functor

$$\mathbf{S}(\tilde{\mathcal{B}}) \longrightarrow \mathbf{S}(\mathcal{B}), \quad X \mapsto Z^0 X,$$

is an equivalence.

For each X in $\mathcal{T}$, choose a complex $\tilde{X}$ in $\tilde{\mathcal{B}}$ with $Z^0\tilde{X} = X$. Define $\mathcal{A}$ by taking as objects the set $\{\tilde{X} \in \tilde{\mathcal{B}} \mid X \in \mathcal{S}\}$, and let $\mathcal{A}(\tilde{X}, \tilde{Y}) = \mathcal{H}om_{\mathcal{B}}(\tilde{X}, \tilde{Y})$ for all X, Y in $\mathcal{S}$. Then $\mathcal{A}$ is a small dg category. We obtain a functor

$$\tilde{\mathcal{B}} \longrightarrow \mathbf{C}_{\mathrm{dg}}(\mathcal{A}), \quad X \mapsto \mathcal{H}om_{\mathcal{B}}(-, X),$$

and compose it with the canonical functor $\mathbf{C}_{\mathrm{dg}}(\mathcal{A}) \to \mathbf{D}_{\mathrm{dg}}(\mathcal{A})$ to get an exact functor

$$\mathbf{S}(\tilde{\mathcal{B}}) \longrightarrow \mathbf{D}_{\mathrm{dg}}(\mathcal{A})$$

of triangulated categories. We compose this with a quasi-inverse of $\mathbf{S}(\tilde{\mathcal{B}}) \to \mathbf{S}(\mathcal{B})$ and get the functor

$$F\colon \mathcal{T} = \mathbf{S}(\mathcal{B}) \longrightarrow \mathbf{D}_{\mathrm{dg}}(\mathcal{A}), \quad X \mapsto \mathcal{H}om_{\mathcal{B}}(-, \tilde{X}).$$

(1) The functor F sends X in $\mathcal{S}$ to the free $\mathcal{A}$-module $\tilde{X}^\wedge$, and we have for Y in $\mathcal{S}$

$$\mathrm{Hom}_{\mathbf{S}(\mathcal{B})}(X, Y) \cong H^0\, \mathcal{H}om_{\mathcal{B}}(\tilde{X}, \tilde{Y}) \cong H^0 \mathcal{A}(\tilde{X}, \tilde{Y}) \cong \mathrm{Hom}_{\mathbf{D}_{\mathrm{dg}}(\mathcal{A})}(\tilde{X}^\wedge, \tilde{Y}^\wedge).$$

Thus the canonical map

$$\mathrm{Hom}_{\mathbf{S}(\mathcal{B})}(X, Y) \longrightarrow \mathrm{Hom}_{\mathbf{D}_{\mathrm{dg}}(\mathcal{A})}(FX, FY) \tag{7.1}$$

is bijective for all X, Y in $\mathcal{S}$.

(2) The functor F is exact and therefore the bijection (7.1) for objects in $\mathcal{S}$ extends to a bijection for all objects in the thick subcategory $\mathcal{T}'$ generated by $\mathcal{S}$. Now observe that the compact objects in $\mathbf{D}_{\mathrm{dg}}(\mathcal{A})$ are precisely the objects in the thick subcategory generated by the free dg $\mathcal{A}$-modules. Thus F identifies $\mathcal{T}'$, up to direct factors, with the full subcategory formed by the compact objects in $\mathbf{D}_{\mathrm{dg}}(\mathcal{A})$.

(3) The functor F preserves coproducts. To see this, let (X_i) be a family of objects in $\mathcal{T}$ and X in $\mathcal{S}$. The degree n cohomology of the canonical map

$$\coprod_i \mathcal{H}om_{\mathcal{B}}(\tilde{X}, \tilde{X}_i) \longrightarrow \mathcal{H}om_{\mathcal{B}}(\tilde{X}, \coprod_i \tilde{X}_i)$$

identifies with the canonical map

$$\coprod_i \mathrm{Hom}_{\mathbf{S}(\mathcal{B})}(X, \Sigma^n X_i) \longrightarrow \mathrm{Hom}_{\mathbf{S}(\mathcal{B})}(X, \Sigma^n \coprod_i X_i)$$

which is an isomorphism since X is compact. Thus $\coprod_i FX_i \cong F(\coprod_i X_i)$. Using this fact and the exactness of F, one sees that the map (7.1) is a bijection for all X, Y in the triangulated subcategory $\mathcal{T}''$ which contains $\mathcal{S}$ and is closed under coproducts. Now observe that $\mathbf{D}_{\mathrm{dg}}(\mathcal{A})$ is generated by all free dg $\mathcal{A}$-modules, that is, there is no proper triangulated subcategory closed under coproducts and containing all free dg $\mathcal{A}$-modules. Thus F identifies $\mathcal{T}''$ with $\mathbf{D}_{\mathrm{dg}}(\mathcal{A})$. □

Remark. Let $\mathcal{T}$ be a small algebraic triangulated category and fix an embedding $\mathcal{T} \to \mathbf{D}_{\mathrm{dg}}(\mathcal{A})$. Suppose S is a multiplicative system of maps in $\mathcal{T}$ which is compatible with the triangulation. Then the localization functor $\mathcal{T} \to \mathcal{T}[S^{-1}]$ admits a fully faithful and exact "right adjoint" $\mathcal{T}[S^{-1}] \to \mathbf{D}_{\mathrm{dg}}(\mathcal{A})$; see [**22**]. Therefore $\mathcal{T}[S^{-1}]$ is algebraic.

7.6. The stable homotopy category is not algebraic. There are triangulated categories which are not algebraic. For instance, the stable homotopy category of spectra is not algebraic. The following argument has been suggested by Bill Dwyer. Given any endomorphism $\phi\colon X \to X$ in a triangulated category $\mathcal{T}$, denote by X/ϕ its cone. If $\mathcal{T}$ is algebraic, then we can identify X with a complex and ϕ induces an endomorphism of the mapping cone X/ϕ which is null-homotopic. Thus $2 \cdot \mathrm{id}_{X/\phi} = 0$ in $\mathcal{T}$ for $\phi = 2 \cdot \mathrm{id}_X$. On the other hand, let S denote the sphere spectrum. Then it is well-known (and can be shown using Steenrod operations) that the identity map of the mod 2 Moore spectrum $M(2) = S/(2 \cdot \mathrm{id}_S)$ has order 4.

PROPOSITION. *There is no faithful exact functor from the stable homotopy category of spectra into an algebraic triangulated category.*

PROOF. Let $F\colon \mathcal{S} \to \mathcal{T}$ be an exact functor from the stable homotopy category of spectra to an algebraic triangulated category and let $X = F(S)$. Then we have $F(M(2)) = X/(2 \cdot \mathrm{id}_X)$ and therefore $F(2 \cdot \mathrm{id}_{M(2)}) = 2 \cdot \mathrm{id}_{X/(2\cdot\mathrm{id}_X)} = 0$. On the other hand, $2 \cdot \mathrm{id}_{M(2)} \neq 0$. Thus F is not faithful. $\square$

7.7. The differential graded category of an exact category. Let $\mathcal{A}$ be a small exact category. We denote by $\operatorname{Lex}\mathcal{A}$ the category of additive functors $F\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$ to the category of abelian groups which are *left exact*, that is, each exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ induces an exact sequence

$$0 \longrightarrow FZ \longrightarrow FY \longrightarrow FX$$

of abelian groups.

LEMMA. *The category* $\operatorname{Lex}\mathcal{A}$ *is an abelian Grothendieck category. The Yoneda functor*

$$\mathcal{A} \longrightarrow \operatorname{Lex}\mathcal{A}, \quad X \mapsto \operatorname{Hom}_{\mathcal{A}}(-, X),$$

is exact and identifies $\mathcal{A}$ *with a full extension-closed subcategory of* $\operatorname{Lex}\mathcal{A}$*. It induces a fully faithful exact functor*

$$\mathbf{D}^b(\mathcal{A}) \longrightarrow \mathbf{D}^b(\operatorname{Lex}\mathcal{A}).$$

PROOF. For the first part, see [**16**, A.2]. We identify $\mathcal{A}$ with its image in $\operatorname{Lex}\mathcal{A}$ under the Yoneda functor. The proof of the first part shows that for each exact sequence $0 \to X \to Y \to Z \to 0$ in $\operatorname{Lex}\mathcal{A}$ with Z in $\mathcal{A}$, there exists an exact sequence $0 \to X' \to Y' \to Z \to 0$ in $\mathcal{A}$ such that the map $Y' \to Z$ factors through the map $Y \to Z$. This implies that the category $\mathbf{K}^-(\mathcal{A})$ is *left cofinal* in $\mathbf{K}^-(\operatorname{Lex}\mathcal{A})$ with respect to the class of quasi-isomorphisms; see [**29**, III.2.4.1] or [**18**, 12.1]. Thus for every quasi-isomorphism $\sigma\colon X \to X'$ in $\mathbf{K}^-(\operatorname{Lex}\mathcal{A})$ with X' in $\mathbf{K}^-(\mathcal{A})$, there exists a map $\phi\colon X'' \to X$ with X'' in $\mathbf{K}^-(\mathcal{A})$ such that $\sigma \circ \phi$ is a quasi-isomorphism. Therefore the canonical functor $\mathbf{D}^-(\mathcal{A}) \to \mathbf{D}^-(\operatorname{Lex}\mathcal{A})$ is fully faithful. $\square$

An abelian Grothendieck category has enough injective objects. Thus $\mathbf{D}^b(\operatorname{Lex}\mathcal{A})$ is algebraic, and therefore $\mathbf{D}^b(\mathcal{A})$ is algebraic. This has the following consequence.

PROPOSITION. *Let* $\mathcal{A}$ *be a small exact category. Then there exists a dg category* $\bar{\mathcal{A}}$ *and a fully faithful exact functor* $F\colon \mathbf{D}^b(\mathcal{A}) \to \mathbf{D}_{\mathrm{dg}}(\bar{\mathcal{A}})$ *having the following properties.*

1. *F identifies $\mathcal{A}$ with the full subcategory formed by the free dg modules $A^\wedge$, $A \in \bar{\mathcal{A}}$.*
2. *F identifies $\mathbf{D}^b(\mathcal{A})$ (up to direct factors) with the full subcategory formed by the compact objects in $\mathbf{D}_{\mathrm{dg}}(\bar{\mathcal{A}})$.*

EXAMPLE. Let Λ be a right noetherian ring and denote by $\mathcal{A} = \operatorname{mod}\Lambda$ the category of finitely generated Λ-modules. Then the functor

$$\operatorname{Mod}\Lambda \longrightarrow \operatorname{Lex}\mathcal{A}, \quad X \mapsto \operatorname{Hom}_\Lambda(-, X)|_{\mathcal{A}},$$

is an equivalence; see [**9**, II.4]. Thus the exact embedding $\operatorname{mod}\Lambda \to \operatorname{Mod}\Lambda$ induces a fully faithful exact functor

$$\mathbf{D}^b(\operatorname{mod}\Lambda) \longrightarrow \mathbf{D}^b(\operatorname{Mod}\Lambda),$$

which identifies $\mathbf{D}^b(\operatorname{mod}\Lambda)$ with the full subcategory of complexes X in $\mathbf{D}^b(\operatorname{Mod}\Lambda)$ such that H^nX is finitely generated for all n and $H^nX = 0$ for almost all $n \in \mathbb{Z}$.

7.8. Notes. Frobenius categories and their stable categories appear in the work of Heller [**14**], and later in the work of Happel on derived categories of finite dimensional algebras [**12**]. The derived category of an exact category is introduced by Neeman in [**21**]. The characterization of algebraic triangulated categories via dg categories is due to Keller [**17**].

Appendix A. The octahedral axiom

Let $\mathcal{T}$ be a pre-trianglated category. In this appendix, it is shown that the octahedral axiom (TR4) is equivalent to the axiom (TR4'). It is convenient to introduce a further axiom.

(TR4") Every diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & & & \\ X' & \longrightarrow & Y' & & & & \end{array}$$

consisting of a homotopy cartesian square with differential $\delta\colon Y' \to \Sigma X$ and an exact triangle can be completed to a morphism

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \| & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & \Sigma X' \end{array}$$

between exact triangles such that the composite $Y' \to Z \to \Sigma X$ equals δ.

PROPOSITION. *Given a pre-triangulated category, the axioms (TR4), (TR4'), and (TR4") are all equivalent.*

PROOF. (TR4') $\Rightarrow$ (TR4"): Suppose there is a diagram

$$\begin{array}{ccccccc} U & \xrightarrow{\alpha} & V & \xrightarrow{\beta} & W & \xrightarrow{\gamma} & \Sigma U \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} & & & & \\ X & \xrightarrow{\kappa} & Y & & & & \end{array}$$

consisting of a homotopy cartesian square with differential $\delta\colon Y \to \Sigma U$ and an exact triangle. Applying (TR4'), we obtain a morphism

$$\begin{array}{ccccccc} U & \xrightarrow{\alpha} & V & \xrightarrow{\beta'} & W' & \xrightarrow{\gamma'} & \Sigma U \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi'} & & \| & & \downarrow{\scriptstyle\Sigma\phi} \\ X & \xrightarrow{\kappa'} & Y' & \xrightarrow{\lambda'} & W' & \xrightarrow{\mu'} & \Sigma X \end{array}$$

between exact triangles such that the left hand square is homotopy cartesian with differential $\gamma' \circ \lambda'$. We apply (TR3) and lemma (2.4) to obtain an isomorphism $(\mathrm{id}_U, \mathrm{id}_V, \sigma)$ between (α, β, γ) and $(\alpha, \beta', \gamma')$. This yields the following morphism

$$\begin{array}{ccccccc} U & \xrightarrow{\alpha} & V & \xrightarrow{\beta} & W & \xrightarrow{\gamma} & \Sigma U \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi'} & & \| & & \downarrow{\scriptstyle \Sigma\phi} \\ X & \xrightarrow{\kappa'} & Y' & \xrightarrow{\sigma^{-1}\circ\lambda'} & W & \xrightarrow{\mu'\circ\sigma} & \Sigma X \end{array}$$

between exact triangles. Next we apply (TR3) and lemma (2.4) to obtain an isomorphism $(\mathrm{id}_U, \mathrm{id}_{V \amalg X}, \tau)$ between the triangles $(\left[\begin{smallmatrix}\alpha \\ \phi\end{smallmatrix}\right], [\,\psi\ \kappa\,], \delta)$ and $(\left[\begin{smallmatrix}\alpha \\ \phi\end{smallmatrix}\right], [\,\psi'\ \kappa'\,], \gamma' \circ \lambda')$. Let $\lambda = \sigma^{-1} \circ \lambda' \circ \tau$. This yields the following morphism

$$\begin{array}{ccccccc} U & \xrightarrow{\alpha} & V & \xrightarrow{\beta} & W & \xrightarrow{\gamma} & \Sigma U \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \| & & \downarrow{\scriptstyle \Sigma\phi} \\ X & \xrightarrow{\kappa} & Y & \xrightarrow{\lambda} & W & \xrightarrow{\mu'\circ\sigma} & \Sigma X \end{array}$$

between exact triangles. Note that $\gamma \circ \lambda$ is a differential for the left hand square. Thus (TR4") holds.

(TR4") $\Rightarrow$ (TR4): Suppose there are exact triangles $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$, and $(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_1 = \beta_1 \circ \alpha_1$. Use (TR1) and complete the map $Y \to U \amalg Z$ to an exact triangle

$$Y \xrightarrow{\left[\begin{smallmatrix}\alpha_2 \\ \beta_1\end{smallmatrix}\right]} U \amalg Z \xrightarrow{[\,\delta_1\ -\gamma_2\,]} V \xrightarrow{\varepsilon} \Sigma Y.$$

Then we apply (TR4") twice to obtain a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\alpha_1} & Y & \xrightarrow{\alpha_2} & U & \xrightarrow{\alpha_3} & \Sigma X \\ \| & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \delta_1} & & \| \\ X & \xrightarrow{\gamma_1} & Z & \xrightarrow{\gamma_2} & V & \xrightarrow{\gamma_3} & \Sigma X \\ & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \delta_2} & & \\ & & W & = & W & & \\ & & \downarrow{\scriptstyle \beta_3} & & \downarrow{\scriptstyle \delta_3} & & \\ & & \Sigma Y & \xrightarrow{\Sigma\alpha_2} & \Sigma U & & \end{array}$$

such that the first two rows and the two central columns are exact triangles. Moreover, the top central square is homotopy cartesian and the differential satisfies

$$-\Sigma\alpha_1 \circ \gamma_3 = \varepsilon = -\beta_3 \circ \delta_2.$$

Thus (TR4) holds.

(TR4) ⇒ (TR4"): Suppose there is given a diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & \Sigma X \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & & & \\ X' & \xrightarrow{\alpha'} & Y' & & & & \end{array}$$

consisting of a homotopy cartesian square and an exact triangle. Thus we have an exact triangle

$$X \xrightarrow{\left[\begin{smallmatrix}\alpha \\ \phi_1\end{smallmatrix}\right]} Y \amalg X' \xrightarrow{[\,\phi_2\ -\alpha'\,]} Y' \xrightarrow{\delta} \Sigma X.$$

We apply (TR4) and obtain the following commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\left[\begin{smallmatrix}\alpha \\ \phi_1\end{smallmatrix}\right]} & Y \amalg X' & \xrightarrow{[\,\phi_2\ -\alpha'\,]} & Y' & \xrightarrow{\delta} & \Sigma X \\ \| & & \downarrow{\scriptstyle [\,\mathrm{id}\ 0\,]} & & \downarrow{\scriptstyle \beta'} & & \| \\ X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & \Sigma X \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \gamma'} & & \downarrow{\scriptstyle \Sigma\left[\begin{smallmatrix}\alpha \\ \phi_1\end{smallmatrix}\right]} \\ & & \Sigma X' & = & \Sigma X' & \xrightarrow{\left[\begin{smallmatrix}0 \\ \mathrm{id}\end{smallmatrix}\right]} & \Sigma(Y \amalg X') \\ & & \downarrow{\scriptstyle \left[\begin{smallmatrix}0 \\ \mathrm{id}\end{smallmatrix}\right]} & & \downarrow{\scriptstyle -\Sigma\alpha'} & & \\ & & \Sigma(Y \amalg X') & \xrightarrow{\Sigma[\,\phi_2\ -\alpha'\,]} & \Sigma Y' & & \end{array}$$

such that $(\beta', \gamma', -\Sigma\alpha')$ is an exact triangle. This gives the following morphism

$$\begin{array}{ccccccc} X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & \Sigma X \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \| & & \downarrow{\scriptstyle \Sigma\phi_1} \\ X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z & \xrightarrow{\gamma'} & \Sigma X' \end{array}$$

of triangles where $\delta = \gamma \circ \beta'$ is the differential of the homotopy cartesian square. Thus (TR4") holds. In particular, (TR4') holds and therefore the proof is complete. □

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