Exercises on derived categories, resolutions, and Brown representability

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The numbering of the following exercises refers to the article "Derived categories, resolutions, and Brown representability" in this volume.

(1.2.1) Let $\mathcal{A}$ be an abelian category. Show that $\mathbf{K}(\mathcal{A})$ and $\mathbf{D}(\mathcal{A})$ are additive categories and that the canonical functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ is additive.

(1.4.1) Let $\mathcal{A}$ be an abelian category and denote by T the class of all quasi-isomorphisms in $\mathbf{C}(\mathcal{A})$. Show that two maps $\phi, \psi \colon X \to Y$ in $\mathbf{C}(\mathcal{A})$ are identified by the canonical functor $\mathbf{C}(\mathcal{A}) \to \mathbf{C}(\mathcal{A})[T^{-1}]$ if $\phi - \psi$ is null-homotopic.

(1.5.1) Let $\mathcal{A}$ be the module category of a ring Λ. Show that $\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(\Lambda, X) \cong H^0 X$ for every complex X of Λ-modules.

(1.5.2) Let $\mathcal{A}$ be an abelian category. Show that the canonical functor $\mathcal{A} \to \mathbf{D}(\mathcal{A})$ identifies $\mathcal{A}$ with the full subcategory of complexes X in $\mathbf{D}(\mathcal{A})$ such that $H^n X = 0$ for all $n \neq 0$.

(1.6.1) Let $\mathcal{A}$ be the category of vector spaces over a field k. Describe all objects and morphisms in $\mathbf{D}(\mathcal{A})$.

(1.6.2) Let $\mathcal{A}$ be the category of finitely generated abelian groups and $\mathcal{P}$ be the category of finitely generated free abelian groups. Describe all objects and morphisms in $\mathbf{D}^b(\mathcal{A})$. Show that the canonical functor $\mathbf{K}^b(\mathcal{P}) \to \mathbf{D}^b(\mathcal{A})$ is an equivalence.

(1.6.3) Let k be a field and consider the following finite dimensional algebras.

$$\Lambda_1 = \begin{bmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{bmatrix} \quad \Lambda_2 = \begin{bmatrix} k & k & 0 \\ 0 & k & 0 \\ 0 & k & k \end{bmatrix} \quad \Lambda_3 = \Lambda_1/I, \quad I = \begin{bmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Describe in each case the category $\mathcal{A}_i$ of finite dimensional Λ_i-modules and its derived category $\mathbf{D}^b(\mathcal{A}_i)$. Here are some hints.

(1) $\mathcal{A}_1$ and $\mathcal{A}_2$ are hereditary categories, but $\mathcal{A}_3$ is not.
(2) Each object in $\mathcal{A}_i$ or $\mathbf{D}^b(\mathcal{A}_i)$ decomposes essentialy uniquely into a finite number of indecomposable objects.
(3) The indecomposable projective Λ_i-modules are $E_{jj}\Lambda_i$, $j = 1, 2, 3$.
(4) Λ_1 and Λ_2 have each 6 pairwise non-isomorphic indecomposable modules, and Λ_3 has 5.

(5) $\operatorname{Ext}^n_{\Lambda_i}(X,Y)$ has k-dimension at most 1 for all indecomposable Λ_i-modules X, Y and $n \geq 0$.

The *Auslander-Reiten quiver* provides a convenient method to display the categories $\mathcal{A}_i$ and $\mathbf{D}^b(\mathcal{A}_i)$, because the morphism spaces between indecomposable objects are at most one-dimensional. This quiver (=oriented graph) is defined as follows. The vertices correspond to the indecomposable objects. Put an arrow $X \to Y$ between two indecomposable objects if there is an irreducible map $\phi\colon X \to Y$ (where ϕ is *irreducible* if ϕ is not invertible and any factorization $\phi = \phi'' \circ \phi'$ implies that ϕ' is a split monomorphism or ϕ'' is a split epimorphism).

(1.7.1) Let $\mathcal{A}$ be an abelian category. Show that the canonical functor $\mathbf{D}^b(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ is fully faithful.

(1.7.2) Let $\mathcal{A}$ be an abelian category and denote by $\mathcal{I}$ the full subcategory of injective objects. Suppose that $\mathcal{A}$ has enough injective objects. Then the canonical functor $\mathbf{K}^+(\mathcal{I}) \to \mathbf{D}^+(\mathcal{A})$ is an equivalence.

(1.7.3) Let $\mathcal{A}$ be the category of finite dimensional modules over $\Lambda = k[T]/(T^2)$, where k is a field. Describe the derived category $\mathbf{D}^b(\mathcal{A})$. (Hint: Fix an injective resulution I of the unique simple module $k[T]/(T)$ (with $I^n = \Lambda$ or $I^n = 0$ for all n) and build every object in $\mathbf{D}^b(\mathcal{A})$ from I.)

(2.1.1) Let $\mathcal{T}$ be a triangulated category. Show that the coproduct of two exact triangles is an exact triangle. Generalize this as follows. Let $X_i \to Y_i \to Z_i \to \Sigma X_i$ be a family of exact triangles such that the coproducts $\coprod_i X_i$, $\coprod_i Y_i$, and $\coprod_i X_i$ exist in $\mathcal{T}$. Show that

$$\coprod_i X_i \longrightarrow \coprod_i Y_i \longrightarrow \coprod_i Z_i \longrightarrow \Sigma\big(\coprod_i X_i\big)$$

is an exact triangle in $\mathcal{T}$.

(2.1.2) Let $\mathcal{T}$ be a triangulated category. Show that the opposite category $\mathcal{T}^{\mathrm{op}}$ is also triangulated.

(2.3.1) Show that every monomorphism $\phi\colon X \to Y$ in a triangulated category has a left inverse ϕ' such that $\phi' \circ \phi = \mathrm{id}_X$.

(2.4.1) Give an example of an exact triangle Δ and two endomorphisms $(\phi_1, \phi_2, \phi_3')$ and $(\phi_1, \phi_2, \phi_3'')$ of Δ such that $\phi_3' \neq \phi_3''$.

(2.5.1) Let $\mathcal{A}$ be an additive category. Check the axioms (TR1) – (TR4) for $\mathbf{K}(\mathcal{A})$.

(3.1.1) Let $\mathcal{A}$ be an abelian category. Show that a map in $\mathbf{K}(\mathcal{A})$ is a quasi-isomorphism if and only if the canonical functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ sends the map to an isomorphism in $\mathbf{D}(\mathcal{A})$.

(3.2.1) Let $F\colon \mathcal{T} \to \mathcal{U}$ be an exact functor between triangulated categories. Show that a right adjoint of F is an exact functor.

(3.2.2) Let $\mathcal{A}$ be an abelian category. Find a criterion such that $\mathbf{D}(\mathcal{A})$ is an abelian category.

(3.3.1) Let Λ be a noetherian ring and $\mathcal{A}$ be the category of Λ-modules. A complex X in $\mathcal{A}$ has *finite cohomology* if H^nX is finitely generated for all n and vanishes for almost all $n \in \mathbb{Z}$. Show that the complexes with finite cohomology form a thick subcategory of $\mathbf{D}(\mathcal{A})$.

(3.3.2) Let $\mathcal{A}$ be the category of finite dimensional modules over $k[T]/(T^n)$. Describe the thick subcategory of all acyclic complexes in $\mathbf{K}(\mathcal{A})$ which have projective components. Draw the Auslander-Reiten quiver of this category. (Hint: Note that projective and injective modules over $k[T]/(T^n)$ coincide. Each acyclic complex X of injectives is essentially determined by the module Z^0X.)

(3.5.1) Let Λ be a ring and $e = e^2 \in \Lambda$ be an idempotent. Let $\Gamma = e\Lambda e \cong \operatorname{End}_\Lambda(e\Lambda)$. Then $\operatorname{Hom}_\Lambda(e\Lambda, -)$ induces an exact functor $\operatorname{Mod}\Lambda \to \operatorname{Mod}\Gamma$ which extends to an exact functor $F\colon \mathbf{D}(\operatorname{Mod}\Lambda) \to \mathbf{D}(\operatorname{Mod}\Gamma)$. Show that F induces an equivalence

$$\mathbf{D}(\operatorname{Mod}\Lambda)/\operatorname{Ker} F \to \mathbf{D}(\operatorname{Mod}\Gamma).$$

(4.1.1) Let $\mathcal{A}$ be an additive category. Give a presentation of the cokernel of a map between two coherent functors in $\widehat{\mathcal{A}}$.

(4.1.2) Let $\mathcal{A}$ be an additive category. Show that for every family of functors F_i in $\widehat{\mathcal{A}}$ having a presentation

$$\mathcal{A}(-, X_i) \stackrel{(-,\phi_i)}{\longrightarrow} \mathcal{A}(-, Y_i) \longrightarrow F_i \longrightarrow 0,$$

the coproduct $F = \coprod_i F_i$ in $\widehat{\mathcal{A}}$ has a presentation

$$\mathcal{A}(-, \coprod_i X_i) \stackrel{(-,\amalg\phi_i)}{\longrightarrow} \mathcal{A}(-, \coprod_i Y_i) \longrightarrow F \longrightarrow 0.$$

(4.1.3) Let Λ be a ring and $\mathcal{A}$ be the category of free Λ-modules. Show that $\widehat{\mathcal{A}}$ is equivalent to the category of Λ-modules.

(4.2.1) Let $F\colon \mathcal{T} \to \mathcal{U}$ be an exact functor between triangulated categories. Show that the induced functor $\widehat{\mathcal{T}} \to \widehat{\mathcal{U}}$ is exact.

(4.5.1) Let $\mathcal{A}$ be the category of Λ-modules over a ring Λ. Show that Λ is a perfect generator for $\mathbf{D}(\mathcal{A})$.

(4.5.2) Let $\mathcal{T}$ be a triangulated category with arbitrary coproducts. Show that one can replace in the definition of a perfect generator the condition

(PG1) There is no proper full triangulated subcategory of $\mathcal{T}$ which contains S and is closed under taking coproducts.

by the following condition

(PG1') Let X be in $\mathcal{T}$ and suppose $\operatorname{Hom}_{\mathcal{T}}(\Sigma^n S, X) = 0$ for all $n \in \mathbb{Z}$. Then $X = 0$.

(5.1.1) Let $\mathcal{A}$ be an abelian category and I be the injective resolution of an object A. Show that the canonical map $A \to I$ induces an isomorphism

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(I, X) \cong \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(A, X)$$

for every complex X with injective components.

(5.1.2) Let $\mathcal{A}$ be an abelian category and suppose $\mathcal{A}$ has arbitrary products. Then the canonical functor $\mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ preserves products if and only if products in $\mathcal{A}$ are exact.

(5.1.3) Let $\mathcal{A}$ be an abelian category with a projective generator. Show that products in $\mathcal{A}$ are exact.

(5.1.4) Let $\mathcal{A}$ be an abelian category with arbitrary products, and denote by $\mathrm{Inj}\,\mathcal{A}$ the full subcategory of injective objects. Show that

$$\mathbf{K}^+(\mathrm{Inj}\,\mathcal{A}) \subseteq \mathbf{K}_{\mathrm{inj}}(\mathcal{A}) \subseteq \mathbf{K}(\mathrm{Inj}\,\mathcal{A}).$$

(Hint: Write every complex in $\mathbf{K}^+(\mathrm{Inj}\,\mathcal{A})$ as a homotopy limit of truncations from $\mathbf{K}^b(\mathrm{Inj}\,\mathcal{A})$.)

(5.1.5) Let $\mathcal{A}$ be an abelian category with exact products and an injective cogenerator. Denote by $\mathrm{Inj}\,\mathcal{A}$ the full subcategory of injective objects. Suppose every object in $\mathcal{A}$ has finite injective dimension. Show that $\mathbf{K}_{\mathrm{inj}}(\mathcal{A}) = \mathbf{K}(\mathrm{Inj}\,\mathcal{A})$. In particular, $\mathbf{K}(\mathrm{Inj}\,\mathcal{A})$ and $\mathbf{D}(\mathcal{A})$ are equivalent. (Hint: An acyclic complex of injectives is null-homotopic.)

(5.1.6) If a ring Λ has finite global dimension, then $\mathbf{K}(\mathrm{Inj}\,\Lambda)$ and $\mathbf{K}(\mathrm{Proj}\,\Lambda)$ are equivalent.

(5.3.1) Consider the setup from (1.6.3). Define Λ_1-modules

$$B = E_{11}\Lambda_1 \amalg E_{22}\Lambda_1 \amalg (E_{22}\Lambda_1/E_{23}\Lambda_1) \quad \text{and} \quad C = (E_{11}\Lambda_1/E_{12}\Lambda_1) \amalg E_{11}\Lambda_1 \amalg E_{33}\Lambda_1.$$

Show that $\Lambda_2 \cong \mathrm{End}_{\Lambda_1}(B)$ and $\Lambda_3 \cong \mathrm{End}_{\Lambda_1}(C)$. Viewing these isomorphisms as identifications, we have bimodules ${}_{\Lambda_2}B_{\Lambda_1}$ and ${}_{\Lambda_3}C_{\Lambda_1}$ which induce equivalences

$$\mathbf{R}\mathrm{Hom}_{\Lambda_1}(B, -)\colon \mathbf{D}^b(\mathcal{A}_1) \to \mathbf{D}^b(\mathcal{A}_2) \quad \text{and} \quad \mathbf{R}\mathrm{Hom}_{\Lambda_1}(C, -)\colon \mathbf{D}^b(\mathcal{A}_1) \to \mathbf{D}^b(\mathcal{A}_3).$$

(The Λ_1-modules B and C are examples of so-called tilting modules.)

(6.1.1) Let k be a field and consider again the algebra

$$\Lambda = \begin{bmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{bmatrix}.$$

Denote by $S = S_1 \amalg S_2 \amalg S_3$ the coproduct of the three simple Λ-modules. Let $P = \mathbf{p}S$ be a projective resolution of S. Compute $A = \mathcal{E}nd_\Lambda(P)$ and show that $H^n A \cong \mathrm{Ext}^n_\Lambda(S, S)$ for all n. Show that $X \mapsto \mathcal{H}om_\Lambda(P, X)$ induces a functor $\mathbf{K}(\mathrm{Proj}\,\Lambda) \to \mathbf{D}_{\mathrm{dg}}(A)$ which is an equivalence.

(6.2.1) View a k-algebra A as a category $\mathcal{A}$ with a single object $*$ and $\mathcal{A}(*, *) = A$. Establish an equivalence between the category of right A-modules and the category of k-linear functors $\mathcal{A}^{\mathrm{op}} \to \mathrm{Mod}\,k$.

(6.5.1) Let $\mathcal{A}$ be the module category of a noetherian ring, and let A in $\mathcal{A}$ be finitely generated. Show that A is a compact object in $\mathcal{A}$. The object A is compact in $\mathbf{D}(\mathcal{A})$ if and only if A has finite projective dimension.

(6.5.2) Let $\mathcal{A}$ be the module category of a commutative noetherian ring Λ. Show that a complex X in $\mathbf{D}(\mathcal{A})$ has finite cohomology if and only if $\operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(\Sigma^n C, X)$ is finitely generated over Λ for every compact object C and all $n \in \mathbb{Z}$, and if it vanishes for almost all $n \in \mathbb{Z}$.

(7.4.1) Let $\mathcal{A}$ be an additive category. Show that the two triangulated structures on $\mathbf{K}(\mathcal{A})$ (defined via mapping cones sequences and via degree-wise split exact sequences) coincide.

(7.4.2) Let Λ be a ring such that projective and injective Λ-modules coincide. Then Λ is noetherian and the category $\mathcal{A}$ of finitely generated Λ-modules is an abelian Frobenius category. Denote by $\mathbf{D}^b(\operatorname{Proj}\mathcal{A})$ the thick subcategory of $\mathbf{D}^b(\mathcal{A})$ which is generated by all projective modules. Show that the composition

$$\mathcal{A} \longrightarrow \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathbf{D}^b(\mathcal{A})/\mathbf{D}^b(\operatorname{Proj}\mathcal{A})$$

of canonical functors induces an equivalence $\mathbf{S}(\mathcal{A}) \to \mathbf{D}^b(\mathcal{A})/\mathbf{D}^b(\operatorname{Proj}\mathcal{A})$ of triangulated categories.

(7.5.1) Let $\mathcal{A}$ be a Frobenius category and $\tilde{\mathcal{A}}$ the full subcategory of acyclic complexes with injective components in $\mathbf{C}(\mathcal{A})$. Show that $\tilde{\mathcal{A}}$ is a Frobenius category (with respect to the degree-wise split exact sequences) and that the functor $\mathbf{S}(\tilde{\mathcal{A}}) \to \mathbf{S}(\mathcal{A})$ sending X to $Z^0 X$ is an equivalence.