NOTES ON THE GABRIEL-ROITER MEASURE

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In his proof of the first Brauer-Thrall conjecture [4], Roiter used an induction scheme which Gabriel formalized in his report on abelian length categories [1]. The first Brauer-Thrall conjecture asserts that every finite dimensional algebra of bounded representation type is of finite representation type. Ringel noticed[1] that the formalism of Gabriel and Roiter is useful as well for studying the representations of algebras having unbounded representation type. We report on Ringel's work [2, 3], presenting for instance his refinement of the first Brauer-Thrall conjecture. In order to keep this exposition as elementary as possible, we have chosen an axiomatic approach which seems to be new. Given a finite dimensional algebra Λ, the Gabriel-Roiter measure is treated as a universal morphism $\mu\colon \operatorname{ind}\Lambda \to P$ of partially ordered sets which is defined on isomorphism classes of indecomposable Λ-modules. Thus μ is a suitable refinement of the length function $\operatorname{ind}\Lambda \to \mathbb{N}$ which sends a module to its composition length.

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1. Chains and length functions

We view the Gabriel-Roiter measure as a morphism of partially ordered sets which refines a length function. In this section, the underlying combinatorial concepts are explained.

1.1. **The lexicographic order on finite chains.** Let $(S, \leqslant)$ be a partially ordered set. A subset $X \subseteq S$ is a *chain* if $x_1 \leqslant x_2$ or $x_2 \leqslant x_1$ for each pair $x_1, x_2 \in X$. For a finite chain X, we denote by $\min X$ its minimal and by $\max X$ its maximal element, using the convention

$$\max\emptyset < x < \min\emptyset \quad \text{for all} \quad x \in S.$$

We write $\operatorname{Ch}(S)$ for the set of all finite chains in S and let

$$\operatorname{Ch}(S, x) := \{X \in \operatorname{Ch}(S) \mid \max X = x\} \quad \text{for} \quad x \in S.$$

On $\operatorname{Ch}(S)$ we consider the *lexicographic order* which is defined by

$$X \leqslant Y \quad :\Longleftrightarrow \quad \min(Y \setminus X) \leqslant \min(X \setminus Y) \quad \text{for} \quad X, Y \in \operatorname{Ch}(S).$$

Remark. (1) $X \subseteq Y$ implies $X \leqslant Y$.

(2) Suppose that S is totally ordered. Then $\operatorname{Ch}(S)$ is totally ordered. We may think of $X \in \operatorname{Ch}(S) \subseteq \{0,1\}^S$ as a string of 0s and 1s which is indexed by the elements in S. The usual lexicographic order on such strings coincides with the lexicographic order on $\operatorname{Ch}(\mathbb{N})$.

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[1] Cf. the footnote on p. 91 of [1].

Example. Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{Q}$ be the set of rational numbers together with the natural ordering. Then the map

$$\operatorname{Ch}(\mathbb{N}) \longrightarrow \mathbb{Q}, \quad X \mapsto \sum_{x \in X} 2^{-x}$$

is injective and order preserving, taking values in the interval $[0, 1]$. For instance, the subsets of $\{1, 2, 3\}$ are ordered as follows:

$$\{\} < \{3\} < \{2\} < \{2, 3\} < \{1\} < \{1, 3\} < \{1, 2\} < \{1, 2, 3\}.$$

We need the following properties of the lexicographic order.

Lemma. *Let $X, Y \in \operatorname{Ch}(S)$ and $X^* := X \setminus \{\max X\}$.*

(1) $X^* = \max\{X' \in \operatorname{Ch}(S) \mid X' < X$ *and* $\max X' < \max X\}$.
(2) *If $X^* < Y$ and $\max X \geqslant \max Y$, then $X \leqslant Y$.*

Proof. (1) Let $X' < X$ and $\max X' < \max X$. We show that $X' \leqslant X^*$. This is clear if $X' \subseteq X^*$. Otherwise, we have

$$\min(X^* \setminus X') = \min(X \setminus X') < \min(X' \setminus X) = \min(X' \setminus X^*),$$

and therefore $X' \leqslant X^*$.

(2) The assumption $X^* < Y$ implies by definition

$$\min(Y \setminus X^*) < \min(X^* \setminus Y).$$

We consider two cases. Suppose first that $X^* \subseteq Y$. If $X \subseteq Y$, then $X \leqslant Y$. Otherwise,

$$\min(Y \setminus X) < \max X = \min(X \setminus Y)$$

and therefore $X < Y$. Now suppose that $X^* \not\subseteq Y$. We use again that $\max X \geqslant \max Y$, exclude the case $Y \subseteq X$, and obtain

$$\min(Y \setminus X) = \min(Y \setminus X^*) < \min(X^* \setminus Y) = \min(X \setminus Y).$$

Thus $X \leqslant Y$ and the proof is complete. □

1.2. **Length functions.** Let $(S, \leqslant)$ be a partially ordered set. A *length function* on S is by definition a map $\lambda \colon S \to \mathbb{N}$ such that $x < y$ in S implies $\lambda(x) < \lambda(y)$. A length function $\lambda \colon S \to \mathbb{N}$ induces for each $x \in S$ a map

$$\operatorname{Ch}(S, x) \longrightarrow \operatorname{Ch}(\mathbb{N}, \lambda(x)), \quad X \mapsto \lambda(X),$$

and therefore the following *chain length function*

$$S \longrightarrow \operatorname{Ch}(\mathbb{N}), \quad x \mapsto \lambda^*(x) := \max\{\lambda(X) \mid X \in \operatorname{Ch}(S, x)\}.$$

1.3. **Basic properties.** Let $\lambda \colon S \to \mathbb{N}$ be a length function and $\lambda^* \colon S \to \operatorname{Ch}(\mathbb{N})$ the induced chain length function. We collect a list of properties (C0) – (C5) of λ^*.

Proposition. *Let $x \in S$.*

(C0) $\lambda^*(x) = \max_{x' < x} \lambda^*(x') \cup \{\lambda(x)\}$.

Proof. Let $X = \lambda^*(x)$ and note that $\max X = \lambda(x)$. The assertion follows from Lemma 1.1 because we have

$$X \setminus \{\max X\} = \max\{X' \in \operatorname{Ch}(\mathbb{N}) \mid X' < X \text{ and } \max X' < \max X\}.$$

□

This shows that the function $\lambda^*\colon S \to \mathrm{Ch}(\mathbb{N})$ can be defined by induction on the length of the elements in S. Next we state some basic properties which suggest to think of λ^* as a refinement of λ.

Proposition. *Let $x, y \in S$.*

(C1) *$x \leqslant y$ implies $\lambda^*(x) \leqslant \lambda^*(y)$.*
(C2) *$\lambda^*(x) = \lambda^*(y)$ implies $\lambda(x) = \lambda(y)$.*
(C3) *$\lambda^*(x') < \lambda^*(y)$ for all $x' < x$ and $\lambda(x) \geqslant \lambda(y)$ imply $\lambda^*(x) \leqslant \lambda^*(y)$.*

Proof. Suppose $x \leqslant y$ and let $X \in \mathrm{Ch}(S, x)$. Then $Y = X \cup \{y\} \in \mathrm{Ch}(S, y)$ and we have $\lambda(X) \leqslant \lambda(Y)$ since $\lambda(X) \subseteq \lambda(Y)$. Thus $\lambda^*(x) \leqslant \lambda^*(y)$. If $\lambda^*(x) = \lambda^*(y)$, then

$$\lambda(x) = \max \lambda^*(x) = \max \lambda^*(y) = \lambda(y).$$

To prove (C3), we use (C0) and apply Lemma 1.1 with $X = \lambda^*(x)$ and $Y = \lambda^*(y)$. In fact, $\lambda^*(x') < \lambda^*(y)$ for all $x' < x$ implies $X^* < Y$, and $\lambda(x) \geqslant \lambda(y)$ implies $\max X \geqslant \max Y$. Thus $X \leqslant Y$. □

We state some further elementary properties of the map λ^*.

Proposition. *Let $x, y \in S$.*

(C4) *$\lambda^*(x) \leqslant \lambda^*(y)$ or $\lambda^*(x) \geqslant \lambda^*(y)$.*
(C5) *$\{\lambda^*(x) \mid x \in S$ and $\lambda(x) \leqslant n\}$ is finite for all $n \in \mathbb{N}$.*

Proof. (C4) is clear since $\mathrm{Ch}(\mathbb{N})$ is totally ordered. (C5) follows from the fact that $\{X \in \mathrm{Ch}(\mathbb{N}) \mid \max X \leqslant n\}$ is finite for all $n \in \mathbb{N}$. □

1.4. **A recursive definition.** Let $\lambda\colon S \to \mathbb{N}$ be a length function. The function $\lambda^*\colon S \to \mathrm{Ch}(\mathbb{N})$ can be defined by induction on the length of the elements in S because of the formula (C0). This observation suggests the following recursive definition which avoids any reference to $\mathrm{Ch}(\mathbb{N})$. Note that this definition is not needed for the rest of this exposition. However, it helps to illustrate the partial order on the values of the chain length function λ^*.

We define a surjective map $\mu\colon S \to S/\lambda^*$ and a partial order on S/λ^*. More precisely, we provide an equivalence relation on S such that S/λ^* denotes the set of equivalence classes and $\mu(x)$ denotes the equivalence class of each $x \in S$. The definition of μ is done by induction, that is, in step $n \geqslant 1$ we define $\mu(x)$ and the relation $\mu(x) \leqslant \mu(y)$ for all $x, y \in S$ of length at most n as follows:

(1) If x or y is minimal, then

$$\mu(x) \leqslant \mu(y) \quad :\Longleftrightarrow \quad \min_{x' \leqslant x} \lambda(x') \geqslant \min_{y' \leqslant y} \lambda(y').$$

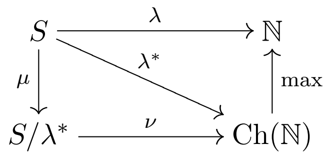
(2) If x and y both are not minimal, then

$$\mu(x) = \mu(y) \quad :\Longleftrightarrow \quad \max_{x' < x} \mu(x') = \max_{y' < y} \mu(y') \text{ and } \lambda(x) = \lambda(y);$$

$$\mu(x) < \mu(y) \quad :\Longleftrightarrow \quad \begin{cases} \max_{x' < x} \mu(x') < \max_{y' < y} \mu(y'), \text{ or} \\ \max_{x' < x} \mu(x') = \max_{y' < y} \mu(y') \text{ and } \lambda(x) > \lambda(y). \end{cases}$$

Note that $\max_{x' < x} \mu(x')$ exists for each $x \in S$ because the set $\{\mu(y) \mid y \in S, \lambda(y) \leqslant n\}$ is finite for all n.

Proposition. *Let $\lambda\colon S \to \mathbb{N}$ be a length function. Then there exists an injective map $\nu\colon S/\lambda^* \to \mathrm{Ch}(\mathbb{N})$ making the following diagram commutative.*



Moreover, $\nu(x) \leqslant \nu(y)$ if and only if $x \leqslant y$ for all $x, y \in S/\lambda^*$.

Proof. First observe, using the equation (C0), that λ^* satisfies the defining relations of μ. Now define $\nu(\mu(x)) = \lambda^*(x)$ for $\mu(x) \in S/\lambda^*$. The map ν is well-defined and injective because λ^* and μ satisfy the same relations. ☐

1.5. **An axiomatic definition.** Let $\lambda\colon S \to \mathbb{N}$ be a length function. We present an axiomatic definition of the induced chain length function λ^*. Thus we can replace the original definition in terms of chains by three simple conditions which express the fact that λ^* refines λ.

Proposition. *Let $\lambda\colon S \to \mathbb{N}$ be a length function. The induced chain length function $\lambda^*\colon S \to \mathrm{Ch}(\mathbb{N})$ is the universal map $\mu\colon S \to P$ into a partially ordered set P satisfying for all $x, y \in S$ the following:*

(P1) *$x \leqslant y$ implies $\mu(x) \leqslant \mu(y)$.*
(P2) *$\mu(x) = \mu(y)$ implies $\lambda(x) = \lambda(y)$.*
(P3) *$\mu(x') < \mu(y)$ for all $x' < x$ and $\lambda(x) \geqslant \lambda(y)$ imply $\mu(x) \leqslant \mu(y)$.*

More precisely, for any such map μ we have

$$\mu(x) \leqslant \mu(y) \quad \Longleftrightarrow \quad \lambda^*(x) \leqslant \lambda^*(y) \quad \textit{for all} \quad x, y \in S.$$

Proof. We have seen in (1.3) that λ^* satisfies (P1) – (P3). So it remains to show that for any map $\mu\colon S \to P$ into a partially ordered set P, the conditions (P1) – (P3) uniquely determine the relation $\mu(x) \leqslant \mu(y)$ for any pair $x, y \in S$. In fact, we claim that (P1) – (P3) imply $\mu(x) \leqslant \mu(y)$ or $\mu(x) \geqslant \mu(y)$. We proceed by induction on the length of the elements in S. For elements of length $n = 1$, the assertion is clear. In fact, $\lambda(x) = 1 = \lambda(y)$ implies $\mu(x) = \mu(y)$ by (P3). Now let $n > 1$ and assume the assertion is true for all elements $x \in S$ of length $\lambda(x) < n$. We choose for each $x \in S$ of length $\lambda(x) \leqslant n$ a *Gabriel-Roiter filtration*, that is, a sequence

$$x_1 < x_2 < \ldots < x_{\gamma(x)-1} < x_{\gamma(x)} = x$$

in S such that x_1 is minimal and $\max_{x' < x_i} \mu(x') = \mu(x_{i-1})$ for all $1 < i \leqslant \gamma(x)$. Such a filtration exists because the elements $\mu(x')$ with $x' < x$ are totally ordered. Now fix $x, y \in S$ of length at most n and let $I = \{i \geqslant 1 \mid \mu(x_i) = \mu(y_i)\}$. We consider $r = \max I$ and put $r = 0$ if $I = \emptyset$. There are two possible cases. Suppose first that $r = \gamma(x)$ or $r = \gamma(y)$. If $r = \gamma(x)$, then $\mu(x) = \mu(x_r) = \mu(y_r) \leqslant \mu(y)$ by (P1). Now suppose $\gamma(x) \neq r \neq \gamma(y)$. Then we have $\lambda(x_{r+1}) \neq \lambda(y_{r+1})$ by (P2) and (P3). If $\lambda(x_{r+1}) > \lambda(y_{r+1})$, then we obtain $\mu(x_{r+1}) < \mu(y_{r+1})$, again using (P2) and (P3). Iterating this argument, we get $\mu(x) = \mu(x_{\gamma(x)}) < \mu(y_{r+1})$. From (P1) we get $\mu(x) < \mu(y_{r+1}) \leqslant \mu(y)$. Thus $\mu(x) \leqslant \mu(y)$ or $\mu(x) \geqslant \mu(y)$ and the proof is complete. ☐

2. Abelian length categories

2.1. **Additive categories.** A category $\mathcal{A}$ is *additive* if every finite family $X_1, X_2, \dots, X_n$ of objects has a coproduct

$$X_1 \oplus X_2 \oplus \dots \oplus X_n,$$

each set $\operatorname{Hom}_{\mathcal{A}}(A, B)$ is an abelian group, and the composition maps

$$\operatorname{Hom}_{\mathcal{A}}(B, C) \times \operatorname{Hom}_{\mathcal{A}}(A, B) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(A, C)$$

are bilinear.

2.2. **Abelian categories.** An additive category $\mathcal{A}$ is *abelian*, if every map $\phi\colon A \to B$ has a kernel and a cokernel, and if the canonical factorization

$$\begin{array}{ccccccc} \operatorname{Ker}\phi & \xrightarrow{\phi'} & A & \xrightarrow{\phi} & B & \xrightarrow{\phi''} & \operatorname{Coker}\phi \\ & & \downarrow & & \uparrow & & \\ & & \operatorname{Coker}\phi' & \xrightarrow{\bar{\phi}} & \operatorname{Ker}\phi'' & & \end{array}$$

of ϕ induces an isomorphism $\bar{\phi}$.

Example. The category of modules over any associative ring is an abelian category.

2.3. **Subobjects.** Let $\mathcal{A}$ be an abelian category. We say that two monomorphisms $X_1 \to X$ and $X_2 \to X$ are *equivalent*, if there exists an isomorphism $X_1 \to X_2$ making the following diagram commutative.

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

An equivalence class of monomorphisms into X is called a *subobject* of X. Given subobjects $X_1 \to X$ and $X_2 \to X$, we write $X_1 \subseteq X_2$ if there is a morphism $X_1 \to X_2$ making the above diagram commutative. An object $X \neq 0$ is *simple* if $X' \subseteq X$ implies $X' = 0$ or $X' = X$.

2.4. **Length categories.** Let $\mathcal{A}$ be an abelian category. An object X has *finite length* if it has a finite composition series

$$0 = X_0 \subseteq X_1 \subseteq \dots \subseteq X_{n-1} \subseteq X_n = X$$

(i.e. each X_i/X_{i-1} is simple). In this case the length of a composition series is an invariant of X by the Jordan-Hölder Theorem; it is called the *length* of X and is denoted by $\ell(X)$. For instance, X is simple if and only if $\ell(X) = 1$. Note that X has finite length if and only if X is both artinian (i.e. satisfies the descending chain condition on subobjects) and noetherian (i.e. satisfies the ascending chain condition on subobjects).

Definition. An abelian category is called a *length category* if all objects have finite length and if the isomorphism classes of objects form a set.

An object $X \neq 0$ is called *indecomposable* if $X = X_1 \oplus X_2$ implies $X_1 = 0$ or $X_2 = 0$. A finite length object admits a finite direct sum decomposition into indecomposable objects having local endomorphism rings. Moreover, such a decomposition is unique up to an isomorphism by the Krull-Remak-Schmidt Theorem.

We denote by $\operatorname{ind}\mathcal{A}$ the set of isomorphism classes of indecomposable objects of $\mathcal{A}$.

Example. (1) Let Λ be a artinian ring. Then the category of finitely generated Λ-modules form a length category which we denote by $\operatorname{mod}\Lambda$.

(2) Let k be a field and Q be any quiver. Then the finite dimensional k-linear representations of Q form a length category.

3. The Gabriel-Roiter measure

Let $\mathcal{A}$ be an abelian length category. We give the definition of the Gabriel-Roiter measure of $\mathcal{A}$ which is due to Gabriel [1] and was inspired by the work of Roiter [4]. Then we discuss some specific properties, including Ringel's results about Gabriel-Roiter inclusions [2].

3.1. **The definition.** Let $\mathcal{A}$ be an abelian length category. The isomorphism classes of objects of $\mathcal{A}$ are partially ordered as follows:

$$X \leqslant Y \quad :\Longleftrightarrow \quad \text{there exists a monomorphism } X \to Y.$$

We consider the length function $\ell\colon \operatorname{ind}\mathcal{A} \to \mathbb{N}$. Then the induced map $\ell^*\colon \operatorname{ind}\mathcal{A} \to \operatorname{Ch}(\mathbb{N})$ is by definition the *Gabriel-Roiter measure* of $\mathcal{A}$. We will not work with this definition but take instead the properties (C1) – (C5) of ℓ^*. Thus we think of the Gabriel-Roiter measure as a map $\mu\colon \operatorname{ind}\mathcal{A} \to P$ into a partially ordered set P, specifying only the relation $\mu(X) \leqslant \mu(Y)$ for pairs $X, Y \in \operatorname{ind}\mathcal{A}$. This approach is justified by Proposition 1.5.

Next we establish further properties (C6) – (C8) of the Gabriel-Roiter measure which depend on the fact that $\mathcal{A}$ is a length category.

3.2. **Gabriel-Roiter filtrations.** Let $X, Y \in \operatorname{ind}\mathcal{A}$. We say that X is a *Gabriel-Roiter predecessor* of Y if $X < Y$ and $\mu(X) = \max_{Y'<Y} \mu(Y')$. Note that each object $Y \in \operatorname{ind}\mathcal{A}$ which is not simple admits a Gabriel-Roiter predecessor, by (C4) and (C5). A Gabriel-Roiter predecessor X of Y is usually not unique, but the value $\mu(X)$ is determined by $\mu(Y)$.

A sequence

$$X_1 < X_2 < \ldots < X_{n-1} < X_n = X$$

in $\operatorname{ind}\mathcal{A}$ is called a *Gabriel-Roiter filtration* of X if X_1 is simple and X_{i-1} is a Gabriel-Roiter predecessor of X_i for all $1 < i \leqslant n$. Clearly, each X admits such a filtration and the values $\mu(X_i)$ are uniquely determined by X.

Proposition. *Let $X, Y \in \operatorname{ind}\mathcal{A}$.*

- (C6) *$X \in \operatorname{ind}\mathcal{A}$ is simple if and only if $\mu(X) \leqslant \mu(Y)$ for all $Y \in \operatorname{ind}\mathcal{A}$.*
- (C7) *Suppose that $\mu(X) < \mu(Y)$. Then there are $Y' < Y'' \leqslant Y$ in $\operatorname{ind}\mathcal{A}$ such that Y' is a Gabriel-Roiter predecessor of Y'' with $\mu(Y') \leqslant \mu(X) < \mu(Y'')$ and $\ell(Y') \leqslant \ell(X)$.*

Proof. For (C6), one uses that each indecomposable object has a simple subobject. To prove (C7), fix a Gabriel-Roiter filtration $Y_1 < Y_2 < \ldots < Y_n = Y$ of Y. We have $\mu(Y_1) \leqslant \mu(X)$ because Y_1 is simple. Using (C4), there exists some i such that $\mu(Y_i) \leqslant \mu(X) < \mu(Y_{i+1})$. Now put $Y' = Y_i$ and $Y'' = Y_{i+1}$. Comparing the filtration of Y with a Gabriel-Roiter filtration of X (as in the proof of Proposition 1.5), we find that $\ell(Y') \leqslant \ell(X)$. $\square$

Example. Let $X \in \mathcal{A}$ be *uniserial*, that is, X has a unique composition series. Then the composition series is a Gabriel-Roiter filtration of X.

3.3. **The main property.** The following main property of the Gabriel-Roiter measure is crucial for the whole theory.

Proposition (Gabriel)**.** *Let $X, Y_1, \dots, Y_r \in \operatorname{ind} \mathcal{A}$.*

(C8) *Suppose that $X \subseteq Y = \oplus_{i=1}^r Y_i$. Then $\mu(X) \leqslant \max \mu(Y_i)$ and X is a direct summand of Y if $\mu(X) = \max \mu(Y_i)$.*

Proof. The proof only uses the properties (C1) – (C3) of μ. Fix a monomorphism $\phi\colon X \to Y$. We proceed by induction on $n = \ell(X) + \ell(Y)$. If $n = 2$, then ϕ is an isomorphism and the assertion is clear. Now suppose $n > 2$. We can assume that for each i the ith component $\phi_i\colon X \to Y_i$ of ϕ is an epimorphism. Otherwise choose for each i a decomposition $Y_i' = \oplus_j Y_{ij}$ of the image of ϕ_i into indecomposables. Then we use (C1) and have $\mu(X) \leqslant \max \mu(Y_{ij}) \leqslant \max \mu(Y_i)$ because $\ell(X) + \ell(Y') < n$ and $Y_{ij} \leqslant Y_i$ for all j. Now suppose that each ϕ_i is an epimorphism. Thus $\ell(X) \geqslant \ell(Y_i)$ for all i. Let $X' \subset X$ be a proper indecomposable subobject. Then $\mu(X') \leqslant \max \mu(Y_i)$ because $\ell(X') + \ell(Y) < n$, and X' is a direct summand if $\mu(X') = \max \mu(Y_i)$. We can exclude the case that $\mu(X') = \max \mu(Y_i)$ because then X' is a proper direct summand of X, which is impossible. Now we apply (C3) and obtain $\mu(X) \leqslant \max \mu(Y_i)$. Finally, suppose that $\mu(X) = \max \mu(Y_i) = \mu(Y_k)$ for some k. We claim that we can choose k such that ϕ_k is an epimorphism. Otherwise, replace all Y_i with $\mu(X) = \mu(Y_i)$ by the image $Y_i' = \oplus_j Y_{ij}$ of ϕ_i as before. We obtain $\mu(X) \leqslant \max \mu(Y_{ij}) < \mu(Y_k)$ since $Y_{kj} < Y_k$ for all j, using (C1) and (C2). This is a contradiction. Thus ϕ_k is an epimorphism and in fact an isomorphism because $\ell(X) = \ell(Y_k)$ by (C2). In particular, X is a direct summand of $\oplus_i Y_i$. This completes the proof. □

Corollary. *Let $X, Y \in \operatorname{ind} \mathcal{A}$ and suppose that $X \subset Y$ with $\mu(X) = \max_{Y' < Y} \mu(Y')$. If $X \subseteq U \subset Y$ in $\mathcal{A}$, then X is a direct summand of U.*

Proof. Let $U = \oplus_i U_i$ be a decomposition into indecomposables. Now apply (C8). We obtain $\mu(X) \leqslant \max \mu(U_i) < \mu(Y)$ and our assumption on $X \subset Y$ implies that X is a direct summand of U. □

Example. (1) Let $Y \in \operatorname{ind} \mathcal{A}$ and suppose that $\mu(X) \leqslant \mu(Y)$ for all $X \in \operatorname{ind} \mathcal{A}$. Then Y is an injective object, because every monomorphism $Y \to Z$ splits by (C8).

(2) Suppose that $\mathcal{A}$ has a cogenerator Q, that is, each object in $\mathcal{A}$ admits a monomorphism into a direct sum of copies of Q. Let $Q = \oplus_i Q_i$ be a decomposition into indecomposable objects. Then $\mu(X) \leqslant \max \mu(Q_i)$ for all $X \in \operatorname{ind} \mathcal{A}$.

3.4. **Gabriel-Roiter inclusions.** Let $X, Y \in \operatorname{ind} \mathcal{A}$. An inclusion $X \subseteq Y$ is called *Gabriel-Roiter inclusion* if $\mu(X) = \max_{Y' < Y} \mu(Y')$. Thus we have a Gabriel-Roiter inclusion $X \subseteq Y$ if and only if X is a Gabriel-Roiter predecessor of Y.

Proposition (Ringel)**.** *Let $X, Y \in \operatorname{ind} \mathcal{A}$ and suppose that $X \subset Y$ is a Gabriel-Roiter inclusion. Then Y/X is an indecomposable object.*

Proof. Let $Z = Y/X$ and assume that $Z = Z' \oplus Z''$ with $Z'' \neq 0$. We obtain the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \mathrm{inc}} & & \\
0 & \longrightarrow & X & \xrightarrow{\mathrm{inc}} & Y & \longrightarrow & Z & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Z'' & = & Z'' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

We have $X \subseteq Y' \subset Y$ and therefore the monomorphism $X \to Y'$ splits by Corollary 3.3. Thus the inclusion $Z' \to Z$ factors through $Y \to Z$ via a split monomorphism $Z' \to Y$. We conclude that $Z' = 0$ since Y is indecomposable. $\square$

Remark. The argument is borrowed from Auslander and Reiten. They show that the cokernel of an irreducible monomorphism between indecomposable objects is indecomposable.

Corollary. *Let Y be an indecomposable object in $\mathcal{A}$ which is not simple. Then there exists a short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ such that X and Z are indecomposable.*

Proof. Take $X \subset Y$ with $\mu(X) = \max_{Y' < Y} \mu(Y')$. $\square$

4. Finiteness results

In this section, Ringel's refinement of the first Brauer-Thrall conjecture is presented [2]. More precisely, we prove a structural result about the partial order of the values of the Gabriel-Roiter measure.

4.1. **Covariant finiteness.** A subcategory $\mathcal{C}$ of $\mathcal{A}$ is called *covariantly finite* if every object $X \in \mathcal{A}$ admits a *left $\mathcal{C}$-approximation*, that is, a map $X \to Y$ with $Y \in \mathcal{C}$ such that the induced map $\mathrm{Hom}_{\mathcal{A}}(Y, C) \to \mathrm{Hom}_{\mathcal{A}}(X, C)$ is surjective for all $C \in \mathcal{C}$. We have also the dual notion: a subcategory $\mathcal{C}$ is *contravariantly finite* if every object in $\mathcal{A}$ admits a *right $\mathcal{C}$-approximation.*

Lemma. *Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$ which is closed under taking direct sums and subobjects. Then $\mathcal{C}$ is a covariantly finite subcategory of $\mathcal{A}$.*

Proof. Fix $X \in \mathcal{A}$. Let $X' \subseteq X$ be minimal among the kernels of all maps $X \to Y$ with $Y \in \mathcal{C}$. Then the canonical map $X \to X/X'$ is a left $\mathcal{C}$-approximation. $\square$

Remark. The proof shows that the inclusion functor $\mathcal{C} \to \mathcal{A}$ admits a left adjoint $F \colon \mathcal{A} \to \mathcal{C}$ which takes $X \in \mathcal{A}$ to X/X'. Note that the adjunction map $X \to FX$ is a left $\mathcal{C}$-approximation.

Let M be any set of values $\mu(X)$. Then we define the subcategory

$$\mathcal{A}(M) := \{\oplus_i X_i \in \mathcal{A} \mid \mu(X_i) \in M \text{ for all } i\}.$$

Proposition (Ringel)**.** *Let M be a set of values $\mu(X)$ which is closed under predecessors, that is, $\mu(Y) \leqslant \mu(X) \in M$ implies $\mu(Y) \in M$. Then $\mathcal{A}(M)$ is a covariantly finite subcategory of $\mathcal{A}$.*

Proof. The subcategory $\mathcal{A}(M)$ is closed under taking subobjects by (C8). □

4.2. **Almost split morphisms.** A map $\phi\colon X \to Y$ in $\mathcal{A}$ is called *left almost split* if ϕ is not a split monomorphism and every map $X \to Y'$ in $\mathcal{A}$ which is not a split monomorphism factors through ϕ. Dually, a map $\psi\colon Y \to Z$ is called *right almost split* if ψ is not a split epimorphism and every map $Y' \to Z$ which is not a split epimorphism factors through ψ. For example, if $\mathcal{A} = \operatorname{mod}\Lambda$ for some artin algebra Λ, then every indecomposable object $X \in \mathcal{A}$ admits a left almost split map starting at X and a right almost split map ending at X.

4.3. **Immediate successors.** Let $X \in \operatorname{ind}\mathcal{A}$. An *immediate successor* of $\mu(X)$ is by definition a minimal element in

$$\{\mu(Y) \mid Y \in \operatorname{ind}\mathcal{A} \text{ and } \mu(X) < \mu(Y)\}.$$

Lemma. *Let $X, Y \in \operatorname{ind}\mathcal{A}$ and suppose that X is a Gabriel-Roiter predecessor of Y. If $X \to \bar{X}$ is a left almost split map in $\mathcal{A}$, then Y is a factor object of $\bar{X}$.*

Proof. The monomorphism $X \to Y$ factors through $X \to \bar{X}$ via a map $\phi\colon \bar{X} \to Y$. Let U be the image of ϕ. Applying Corollary 3.3, we find that $U = Y$. □

Proposition. *Let $X \in \mathcal{A}$ and suppose there exists $n_X \in \mathbb{N}$ such that each $V \in \operatorname{ind}\mathcal{A}$ with $\mu(V) \leqslant \mu(X)$ and $\ell(V) \leqslant \ell(X)$ admits a left almost split map $V \to \bar{V}$ with $\ell(\bar{V}) \leqslant n_X$. Then there exists an immediate successor of $\mu(X)$ provided that $\mu(X)$ is not maximal.*

Proof. Let $\mu(X) < \mu(Y)$. We apply (C7) and find $Y' < Y'' \leqslant Y$ in $\operatorname{ind}\mathcal{A}$ such that Y' is a Gabriel-Roiter predecessor of Y'' with $\mu(Y') \leqslant \mu(X) < \mu(Y'') \leqslant \mu(Y)$ and $\ell(Y') \leqslant \ell(X)$. The preceding lemma implies $\ell(Y'') \leqslant n_X$, and (C5) implies that the number of values $\mu(Y'')$ is finite. Thus there exists a minimal element among those $\mu(Y'')$. □

Corollary (Ringel)**.** *Let Λ be an artin algebra and $X \in \operatorname{ind}\Lambda$. Then there exists an immediate successor of $\mu(X)$ provided that $\mu(X)$ is not maximal.*

Proof. Use that there exists $n_\Lambda \in \mathbb{N}$ having the following property: for each indecomposable $V \in \operatorname{mod}\Lambda$, there exists a left almost split map $V \to \bar{V}$ satisfying $\ell(\bar{V}) \leqslant n_\Lambda \ell(V)$. In fact, one takes $n_\Lambda = pq$, where p denotes the maximal length of an indecomposable projective Λ-module and q denotes the maximal length of an indecomposable injective Λ-module. □

4.4. **A finiteness criterion.** We present a criterion for a subcategory $\mathcal{C}$ of $\mathcal{A}$ such that the number of indecomposable objects in $\mathcal{C}$ is finite. This is based on the following classical lemma.

Lemma (Harada-Sai)**.** *Let $n \in \mathbb{N}$. A composition $X_1 \to X_2 \to \ldots \to X_{2^n}$ of non-invertible maps between indecomposable objects of length at most n is zero.*

Proposition. *Let $\mathcal{A}$ be a length category with left almost split maps and only finitely many isomorphism classes of simple objects. Suppose that $\mathcal{C}$ is a subcategory such that*

1. *$\mathcal{C}$ is covariantly finite, and*
2. *there exists $n \in \mathbb{N}$ such that $\ell(X) \leqslant n$ for all indecomposable $X \in \mathcal{C}$.*

Then there are only finitely many isomorphism classes of indecomposable objects in $\mathcal{C}$.

Proof. We claim that we can construct all indecomposable objects $X \in \mathcal{C}$ in at most 2^n steps from the finitely many simple objects in $\mathcal{A}$ as follows. Choose a non-zero map $S \to X$ from a simple object S and factor this map through the left $\mathcal{C}$-approximation $S \to S'$. Take an indecomposable direct summand X_0 of S' such that the component $S \to X_0 \to X$ of the composition $S \to S' \to X$ is non-zero. Stop if $X_0 \to X$ is an isomorphism. Otherwise take a left almost split map $X_0 \to Y_0$ and a left $\mathcal{C}$-approximation $Y_0 \to Z_0$. The map $X_0 \to X$ factors through the composition $X_0 \to Y_0 \to Z_0$ and we choose an indecomposable direct summand X_1 of Z_0 such that the component $X_0 \to Y_0 \to X_1 \to X$ is non-zero. Again, we stop if $X_1 \to X$ is an isomorphism. Otherwise, we continue as before and obtain in step r a sequence of non-invertible maps

$$X_0 \to X_1 \to X_2 \to \ldots \to X_r$$

such that the compsition is non-zero. The Harada-Sai lemma implies that $r < 2^n$ because $\ell(X_i) \leqslant n$ for all i by our assumption. Thus X is isomorphic to X_i for some $i < 2^n$, and we obtain X in at most 2^n steps, having in each step only finitely many choices by taking an indecomposable direct summand. We conclude that $\mathcal{C}$ has only a finite number of indecomposable objects. □

Remark. This classical argument provides a quick proof of the first Brauer-Thrall conjecture; it is due to Auslander and Yamagata.

4.5. **The initial segment.**

Theorem (Ringel)**.** *Let $\mathcal{A}$ be a length category such that* $\operatorname{ind}\mathcal{A}$ *is infinite. Suppose also that $\mathcal{A}$ has only finitely many isomorphism classes of simple objects and that every indecomposable object admits a left almost split map. Then there exist infinitely many values $\mu(X_1) < \mu(X_2) < \mu(X_3) < \ldots$ of the Gabriel-Roiter measure of $\mathcal{A}$ having the following properties.*

1. *If $\mu(X) \neq \mu(X_i)$ for all i, then $\mu(X_i) < \mu(X)$ for all i.*
2. *The set $\{X \in \operatorname{ind}\mathcal{A} \mid \mu(X) = \mu(X_i)\}$ is finite for all i.*

Proof. We construct the values $\mu(X_i)$ by induction as follows. Take for X_1 any simple object. Observe that $\mu(X_1)$ is minimal among all $\mu(X)$ by (C6) and that only finitely many $X \in \operatorname{ind}\mathcal{A}$ satisfy $\mu(X) = \mu(X_1)$ because $\mathcal{A}$ has only finitely many simple objects. Now suppose that $\mu(X_1) < \ldots < \mu(X_n)$ have been constructed, satisfying the conditions (1) and (2) for all $1 \leqslant i \leqslant n$. We can apply Proposition 4.3 and find an immediate successor $\mu(X_{n+1})$ of $\mu(X_n)$. It remains to show that the set $\{X \in \operatorname{ind}\mathcal{A} \mid \mu(X) = \mu(X_{n+1})\}$ is finite. To this end consider $M = \{\mu(X_1), \ldots, \mu(X_{n+1})\}$. We know from Proposition 4.1 that $\mathcal{A}(M)$ is a covariantly finite subcategory. Clearly, $\ell(X)$ is bounded by $\max\{\ell(X_i), \ldots, \ell(X_{n+1})\}$ for all indecomposable $X \in \mathcal{A}(M)$ by (C2). We conclude from Proposition 4.4 that the number of indecomposables in $\mathcal{A}(M)$ is finite. Thus $\{X \in \operatorname{ind}\mathcal{A} \mid \mu(X) = \mu(X_{n+1})\}$ is finite and the proof is complete. □

Corollary (Brauer-Thrall I). *Let $\mathcal{A}$ be a length category satisfying the above conditions. Then for every $n \in \mathbb{N}$ there exists an indecomposable object $X \in \mathcal{A}$ with $\ell(X) > n$.*

Proof. Use that for fixed $n \in \mathbb{N}$, there are only finitely many values $\mu(X)$ with $\ell(X) \leqslant n$, by (C5). □

4.6. **The terminal segment.**

Theorem (Ringel). *Let $\mathcal{A}$ be a length category such that $\operatorname{ind}\mathcal{A}$ is infinite. Suppose also that $\mathcal{A}$ has a cogenerator (i.e. an object Q such that each object in $\mathcal{A}$ admits a monomorphism into a direct sum of copies of Q) and that every indecomposable object admits a right almost split map. Then there exist infinitely many values $\mu(X^1) > \mu(X^2) > \mu(X^3) > \dots$ of the Gabriel-Roiter measure of $\mathcal{A}$ having the following properties.*

1. *If $\mu(X) \neq \mu(X^i)$ for all i, then $\mu(X^i) > \mu(X)$ for all i.*
2. *The set $\{X \in \operatorname{ind}\mathcal{A} \mid \mu(X) = \mu(X^i)\}$ is finite for all i.*

The proof is based on the following lemma.

Lemma (Auslander-Smalø). *Let $\mathcal{A}$ be a length category and let $X \in \mathcal{A}$. Denote by $\mathcal{A}_X$ the subcategory formed by all objects in $\mathcal{A}$ having no indecomposable direct summand which is isomorphic to a direct summand of X. If every indecomposable direct summand of X admits a right almost split map, then $\mathcal{A}_X$ is contravariantly finite.*

Proof. Let $X = \oplus_{i_0=1}^{r} X_{i_0}$ be a decomposition into indecomposables. It is sufficient to construct a right $\mathcal{A}_X$-approximation for each indecomposable object $Z \in \mathcal{A}$. We take the identity map if $Z \in \mathcal{A}_X$. Otherwise, Z is isomorphic to X_{i_0} for some i_0 and we proceed as follows. Let $\phi_{i_0} \colon \bar{X}_{i_0} \to X_{i_0}$ be a right almost split map and choose a decomposition

$$\bar{X}_{i_0} = Y_{i_0} \oplus (\oplus_{i_1} X_{i_0 i_1})$$

such that $Y_{i_0} \in \mathcal{A}_X$ and $i_0 i_1 \in \{1, \dots, r\}$ for all i_1. Note that each map $V \to X_{i_0}$ with $V \in \mathcal{A}_X$ factors through ϕ_{i_0}. Also, each component $X_{i_0 i_1} \to X_{i_0}$ of ϕ_{i_0} is non-invertible. Now compose ϕ_{i_0} with $\operatorname{id}_{Y_{i_0}} \oplus (\oplus_{i_1} \phi_{i_0 i_1})$ to obtain a map

$$Y_{i_0} \oplus (\oplus_{i_1} (Y_{i_0 i_1} \oplus (\oplus_{i_2} X_{i_0 i_1 i_2}))) \to Y_{i_0} \oplus (\oplus_{i_1} X_{i_0 i_1}) \to X_{i_0}.$$

Again, each map $V \to X_{i_0}$ with $V \in \mathcal{A}_X$ factors through this new map, and each component $X_{i_0 i_1 i_2} \to X_{i_0 i_1}$ is non-invertible. We continue this procedure, compose this map with

$$\operatorname{id}_{Y_{i_0}} \oplus (\oplus_{i_1} (\operatorname{id}_{Y_{i_0 i_1}} \oplus (\oplus_{i_2} \phi_{i_0 i_1 i_2}))),$$

and so on. Now let $n = 2^m$ where $m = \max\{\ell(X_1), \dots, \ell(X_r)\}$. Then the Harada-Sai lemma implies that any composition

$$X_{i_0 i_1 \dots i_n} \to X_{i_0 i_1 \dots i_{n-1}} \to \dots \to X_{i_0 i_1} \to X_{i_0}$$

is zero. Thus the induced map

$$\oplus_{j=0}^{n} (\oplus_{i_1, i_2, \dots, i_j} Y_{i_0 i_1 \dots i_j}) \longrightarrow X_{i_0}$$

is a right $\mathcal{A}_X$-approximation of X_{i_0}. □

Proof of the theorem. We construct the values $\mu(X^i)$ by induction as follows. Let $n \geqslant 0$ and suppose that $\mu(X^1) > \dots > \mu(X^n)$ have been constructed, satisfying the conditions (1) and (2) for all $1 \leqslant i \leqslant n$. Denote by P the direct sum of all $X \in \operatorname{ind}\mathcal{A}$ with $\mu(X) \geqslant \mu(X^n)$, and let $P = 0$ if $n = 0$. Choose a right $\mathcal{A}_P$-approximation $P' \to Q$ and take for X^{n+1} any indecomposable direct summand X of P' such that $\mu(X)$ is maximal. Observe that every indecomposable object $X \in \mathcal{A}_P$ is cogenerated by Q and therefore by P'. Thus (C8) implies that $\mu(X)$ is bounded by $\mu(X^{n+1})$. Moreover, if $\mu(X) = \mu(X^{n+1})$, then X is isomorphic to a direct summand of P'. Thus $\{X \in \operatorname{ind}\mathcal{A} \mid \mu(X) = \mu(X^{n+1})\}$ is finite and the proof is complete. □

Let Λ be an artin algebra of infinite representation type. Then $\mathcal{A} = \operatorname{mod}\Lambda$ satisfies the assumptions of Theorems 4.5 and 4.6. Let us summarize the structure of the partial order on the values of the Gabriel-Roiter measure as follows. We have

$$\operatorname{ind}\mathcal{A}/\mu := \{\mu(X) \mid X \in \operatorname{ind}\mathcal{A}\} = S_{\mathrm{init}} \sqcup S_{\mathrm{cent}} \sqcup S_{\mathrm{term}} \cong \mathbb{N} \sqcup S_{\mathrm{cent}} \sqcup \mathbb{N}^{\mathrm{op}},$$

where the notation $S = S_1 \sqcup S_2$ for a poset S means $S = S_1 \cup S_2$ and $x_1 < x_2$ for all $x_1 \in S_1$, $x_2 \in S_2$.

Example. Let $\Lambda = \left[\begin{smallmatrix} k & k^2 \\ 0 & k \end{smallmatrix}\right]$ be the Kronecker algebra over an algebraically closed field k. Denote by $\dim X = (x_1, x_2) \in \mathbb{N}_0 \times \mathbb{N}_0$ the dimension vector of $X \in \operatorname{ind}\Lambda$. Then $\dim X = \dim Y$ implies $\mu(X) = \mu(Y)$, and we obtain the following ordering on the set of indecomposables via the Gabriel-Roiter measure:

$$(1,0) = (0,1) < (1,2) < (2,3) < \dots \quad (1,1) < (2,2) < \dots \quad \dots < (3,2) < (2,1).$$

5. The Gabriel-Roiter measure for derived categories

Let $\mathcal{A}$ be an abelian length category. We propose a definition for the Gabriel-Roiter measure of the bounded derived category $\mathbf{D}^b(\mathcal{A})$. The derived Gabriel-Roiter measure extends the Gabriel-Roiter measure of the underlying abelian category $\mathcal{A}$.

5.1. **The definition.** The bounded derived category $\mathbf{D}^b(\mathcal{A})$ of $\mathcal{A}$ is by definition the full subcategory of the derived category $\mathbf{D}(\mathcal{A})$ which is formed by all complexes X such that $H^n X = 0$ for $|n| \gg 0$. Note that each object of $\mathbf{D}^b(\mathcal{A})$ admits a finite direct sum decomposition into indecomposable objects having local endomorphism rings. Moreover, such a decomposition is unique up to an isomorphism. We denote by $\operatorname{ind}\mathbf{D}^b(\mathcal{A})$ the set of isomorphism classes of indecomposable objects of $\mathbf{D}^b(\mathcal{A})$.

We consider the functor

$$\mathbf{D}^b(\mathcal{A}) \longrightarrow \mathcal{A}, \quad X \mapsto H^*X = \oplus_{n\in\mathbb{Z}} H^n X,$$

and the isomorphism classes of objects of $\mathbf{D}^b(\mathcal{A})$ are partially ordered via

$$X \leqslant Y \quad :\Longleftrightarrow \quad \begin{cases} \text{there exists a map } X \to Y \text{ inducing} \\ \text{a monomorphism } H^*X \to H^*Y. \end{cases}$$

We have the length function

$$\ell_{H^*} \colon \operatorname{ind}\mathbf{D}^b(\mathcal{A}) \longrightarrow \mathbb{N}, \quad X \mapsto \ell(H^*X)$$

and the induced chain length function $\ell^*_{H^*} \colon \operatorname{ind}\mathbf{D}(\mathcal{A}) \to \operatorname{Ch}(\mathbb{N})$ is by definition the *Gabriel-Roiter measure* of $\mathbf{D}^b(\mathcal{A})$.

5.2. **Derived versus abelian Gabriel-Roiter measure.**

Proposition. *The Gabriel-Roiter measure of* $\mathbf{D}^b(\mathcal{A})$ *extends the Gabriel-Roiter measure of* $\mathcal{A}$*. More precisely, the canonical functor* $\mathcal{A} \to \mathbf{D}^b(\mathcal{A})$ *sending an object of* $\mathcal{A}$ *to the corresponding complex concentrated in degree zero induces an inclusion* $\operatorname{ind}\mathcal{A} \to \operatorname{ind}\mathbf{D}^b(\mathcal{A})$ *of partially ordered sets, which makes the following diagram commutative.*

$$\begin{array}{ccc} \operatorname{ind}\mathcal{A} & \xrightarrow{\text{inc}} & \operatorname{ind}\mathbf{D}^b(\mathcal{A}) \\ & {\scriptstyle \ell^*}\searrow \quad \swarrow{\scriptstyle \ell^*_{H^*}} & \\ & \operatorname{Ch}(\mathbb{N}) & \end{array}$$

Proof. Use the fact that the diagram

$$\begin{array}{ccc} \operatorname{ind}\mathcal{A} & \xrightarrow{\text{inc}} & \operatorname{ind}\mathbf{D}^b(\mathcal{A}) \\ & {\scriptstyle \ell}\searrow \quad \swarrow{\scriptstyle \ell_{H^*}} & \\ & \mathbb{N} & \end{array}$$

is commutative and that $\operatorname{ind}\mathcal{A}$ is closed under predeccessors in $\operatorname{ind}\mathbf{D}^b(\mathcal{A})$. □

5.3. **An alternative definition.** For an alternative definition of the Gabriel-Roiter measure of $\mathbf{D}^b(\mathcal{A})$, consider the lexicographic order on

$$\coprod_{\mathbb{Z}} \mathbb{N}_0 := \{(x_n) \in \prod_{\mathbb{Z}} \mathbb{N}_0 \mid x_n = 0 \text{ for } |n| \gg 0\}, \text{ with}$$

$$(x_n) \leqslant (y_n) \quad :\Longleftrightarrow \quad \begin{cases} x_i = y_i \text{ for all } i \in \mathbb{Z}, \text{ or} \\ x_i \leqslant y_i \text{ for } i = \min\{n \in \mathbb{Z} \mid x_n \neq y_n\}. \end{cases}$$

Take instead of ℓ_{H^*} the length function

$$\lambda\colon \operatorname{ind}\mathbf{D}^b(\mathcal{A}) \longrightarrow \coprod_{\mathbb{Z}} \mathbb{N}_0, \quad X \mapsto (\ell(H^n X)),$$

and instead of $\ell^*_{H^*}$ the induced chain length function

$$\lambda^*\colon \operatorname{ind}\mathbf{D}^b(\mathcal{A}) \longrightarrow \operatorname{Ch}(\coprod_{\mathbb{Z}} \mathbb{N}_0).$$

We illustrate the difference between both definitions by taking a hereditary length category $\mathcal{A}$, that is $\operatorname{Ext}^2_{\mathcal{A}}(-,-) = 0$. Then

$$\operatorname{ind}\mathbf{D}^b(\mathcal{A})/\ell^*_{H^*} = \operatorname{ind}\mathcal{A}/\ell^*,$$

whereas

$$\operatorname{ind}\mathbf{D}^b(\mathcal{A})/\lambda^* = \quad \ldots \sqcup \operatorname{ind}\mathcal{A}/\ell^* \sqcup \operatorname{ind}\mathcal{A}/\ell^* \sqcup \operatorname{ind}\mathcal{A}/\ell^* \sqcup \ldots .$$

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