AN AXIOMATIC CHARACTERIZATION OF THE GABRIEL-ROITER MEASURE

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Abstract. Given an abelian length category $\mathcal{A}$, the Gabriel-Roiter measure with respect to a length function ℓ is characterized as a universal morphism $\operatorname{ind}\mathcal{A} \to P$ of partially ordered sets. The map is defined on the isomorphism classes of indecomposable objects of $\mathcal{A}$ and is a suitable refinement of the length function ℓ.

In his proof of the first Brauer-Thrall conjecture [5], Roiter used an induction scheme which Gabriel formalized in his report on abelian length categories [1]. The first Brauer-Thrall conjecture asserts that every finite dimensional algebra of bounded representation type is of finite representation type. Ringel noticed (see the footnote on p. 91 of [1]) that the formalism of Gabriel and Roiter works equally well for studying the representations of algebras having unbounded representation type. We refer to recent work [2, 3, 4] for some beautiful applications.

In this note we present an axiomatic characterization of the Gabriel-Roiter measure which reveals its combinatorial nature. Given a finite dimensional algebra Λ, the Gabriel-Roiter measure is characterized as a universal morphism $\operatorname{ind}\Lambda \to P$ of partially ordered sets. The map is defined on the isomorphism classes of finite dimensional indecomposable Λ-modules and is a suitable refinement of the length function $\operatorname{ind}\Lambda \to \mathbb{N}$ which sends a module to its composition length.

The first part of this paper is purely combinatorial and might be of independent interest. We study length functions $\lambda\colon S \to T$ on a fixed partially ordered set S. Such a length function takes its values in another partially ordered set T, for example $T = \mathbb{N}$. We denote by $\operatorname{Ch}(T)$ the set of finite chains in T, together with the lexicographic ordering. The map λ induces a new length function $\lambda^*\colon S \to \operatorname{Ch}(T)$, which we call chain length function because each value $\lambda^*(x)$ measures the lengths $\lambda(x_i)$ of the elements x_i occuring in some finite chain $x_1 < x_2 < \ldots < x_n = x$ of x in S. We think of λ^* as a specific refinement of λ and provide an axiomatic characterization. It is interesting to observe that this construction can be iterated. Thus we may consider $(\lambda^*)^*$, $((\lambda^*)^*)^*$, and so on.

The second part of the paper discusses the Gabriel-Roiter measure for a fixed abelian length category $\mathcal{A}$, for example the category of finite dimensional Λ-modules over some algebra Λ. For each length function ℓ on $\mathcal{A}$, we consider its restriction to the partially ordered set $\operatorname{ind}\mathcal{A}$ of isomorphism classes of indecomposable objects of $\mathcal{A}$. Then the Gabriel-Roiter measure with respect to ℓ is by definition the corresponding chain length function ℓ^*. In particular, we obtain an axiomatic characterization of ℓ^* and use it to reprove Gabriel's main property of the Gabriel-Roiter measure. Note that we work with a slight generalization of Gabriel's original definition. This enables us to characterize the injective objects of $\mathcal{A}$ as those objects where ℓ^* takes maximal values for some

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length function ℓ. This is a remarkable fact because the Gabriel-Roiter measure is a combinatorial invariant, depending only on the poset of indecomposable objects and some length function, whereas the notion of injectivity involves all morphisms of the category $\mathcal{A}$.

1. Chains and length functions

The lexicographic order on finite chains. Let $(S, \leqslant)$ be a partially ordered set. A subset $X \subseteq S$ is a *chain* if $x_1 \leqslant x_2$ or $x_2 \leqslant x_1$ for each pair $x_1, x_2 \in X$. For a finite chain X, we denote by $\min X$ its minimal and by $\max X$ its maximal element, using the convention

$$\max \emptyset < x < \min \emptyset \quad \text{for all} \quad x \in S.$$

We write $\operatorname{Ch}(S)$ for the set of all finite chains in S and let

$$\operatorname{Ch}(S, x) := \{X \in \operatorname{Ch}(S) \mid \max X = x\} \quad \text{for} \quad x \in S.$$

On $\operatorname{Ch}(S)$ we consider the *lexicographic order* which is defined by

$$X \leqslant Y \quad :\Longleftrightarrow \quad \min(Y \setminus X) \leqslant \min(X \setminus Y) \quad \text{for} \quad X, Y \in \operatorname{Ch}(S).$$

Remark 1.1. (1) $X \subseteq Y$ implies $X \leqslant Y$ for $X, Y \in \operatorname{Ch}(S)$.

(2) Suppose that S is totally ordered. Then $\operatorname{Ch}(S)$ is totally ordered. We may think of $X \in \operatorname{Ch}(S) \subseteq \{0,1\}^S$ as a string of 0s and 1s which is indexed by the elements in S. The usual lexicographic order on such strings coincides with the lexicographic order on $\operatorname{Ch}(S)$.

Example 1.2. Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{Q}$ be the set of rational numbers together with the natural ordering. Then the map

$$\operatorname{Ch}(\mathbb{N}) \longrightarrow \mathbb{Q}, \quad X \mapsto \sum_{x \in X} 2^{-x}$$

is injective and order preserving, taking values in the interval $[0, 1]$. For instance, the subsets of $\{1, 2, 3\}$ are ordered as follows:

$$\{\} < \{3\} < \{2\} < \{2,3\} < \{1\} < \{1,3\} < \{1,2\} < \{1,2,3\}.$$

We need the following properties of the lexicographic order.

Lemma 1.3. *Let $X, Y \in \operatorname{Ch}(S)$ and $X^* := X \setminus \{\max X\}$.*

(1) *$X^* = \max\{X' \in \operatorname{Ch}(S) \mid X' < X$ and $\max X' < \max X\}$.*

(2) *If $X^* < Y$ and $\max X \geqslant \max Y$, then $X \leqslant Y$.*

Proof. (1) Let $X' < X$ and $\max X' < \max X$. We show that $X' \leqslant X^*$. This is clear if $X' \subseteq X^*$. Otherwise, we have

$$\min(X^* \setminus X') = \min(X \setminus X') < \min(X' \setminus X) = \min(X' \setminus X^*),$$

and therefore $X' \leqslant X^*$.

(2) The assumption $X^* < Y$ implies by definition

$$\min(Y \setminus X^*) < \min(X^* \setminus Y).$$

We consider two cases. Suppose first that $X^* \subseteq Y$. If $X \subseteq Y$, then $X \leqslant Y$. Otherwise,

$$\min(Y \setminus X) < \max X = \min(X \setminus Y)$$

and therefore $X < Y$. Now suppose that $X^* \not\subseteq Y$. We use again that $\max X \geqslant \max Y$, exclude the case $Y \subseteq X$, and obtain

$$\min(Y \setminus X) = \min(Y \setminus X^*) < \min(X^* \setminus Y) = \min(X \setminus Y).$$

Thus $X \leqslant Y$ and the proof is complete. □

Length functions. Let $(S, \leqslant)$ be a partially ordered set. A *length function* on S is by definition a map $\lambda\colon S \to T$ into a partially ordered set T satisfying for all $x, y \in S$ the following:

(L1) $x < y$ implies $\lambda(x) < \lambda(y)$.
(L2) $\lambda(x) \leqslant \lambda(y)$ or $\lambda(y) \leqslant \lambda(x)$.
(L3) $\lambda_0(x) := \operatorname{card}\{\lambda(x') \mid x' \in S \text{ and } x' \leqslant x\}$ is finite.

Two length functions λ and λ' on S are *equivalent* if

$$\lambda(x) \leqslant \lambda(y) \quad \Longleftrightarrow \quad \lambda'(x) \leqslant \lambda'(y) \quad \text{for all} \quad x, y \in S.$$

Observe that (L2) and (L3) are automatically satisfied if $T = \mathbb{N}$. A length function $\lambda\colon S \to T$ induces for each $x \in S$ a map

$$\operatorname{Ch}(S, x) \longrightarrow \operatorname{Ch}(T, \lambda(x)), \quad X \mapsto \lambda(X),$$

and therefore the following *chain length function*

$$S \longrightarrow \operatorname{Ch}(T), \quad x \mapsto \lambda^*(x) := \max\{\lambda(X) \mid X \in \operatorname{Ch}(S, x)\}.$$

Note that equivalent length functions induce equivalent chain length functions.

Example 1.4. (1) Let S be a poset such that for each $x \in S$ there is a bound $n_x \in \mathbb{N}$ with $\operatorname{card} X \leqslant n_x$ for all $X \in \operatorname{Ch}(S, x)$. Then the map $S \to \mathbb{N}$ sending x to $\max\{\operatorname{card} X \mid X \in \operatorname{Ch}(S, x)\}$ is a length function.

(2) Let S be a poset such that $\{x' \in S \mid x' \leqslant x\}$ is a finite chain for each $x \in S$. Then the map $\lambda\colon S \to \mathbb{N}$ sending x to $\operatorname{card}\{x' \in S \mid x' \leqslant x\}$ is a length function. Moreover, λ^* is a length function and equivalent to λ.

(3) Let $\lambda\colon S \to \mathbb{Z}$ be a length function which satisfies in addition the following properties of a *rank function*: $\lambda(x) = \lambda(y)$ for each pair x, y of minimal elements of S, and $\lambda(x) = \lambda(y) - 1$ whenever x is an immediate predecessor of y in S. Then λ^* is a length function and equivalent to λ.

Basic properties. Let $\lambda\colon S \to T$ be a length function and $\lambda^*\colon S \to \operatorname{Ch}(T)$ the induced chain length function. We collect the basic properties of λ^*.

Proposition 1.5. *Let $x, y \in S$.*

(C0) $\lambda^*(x) = \max_{x' < x} \lambda^*(x') \cup \{\lambda(x)\}$.
(C1) $x \leqslant y$ *implies* $\lambda^*(x) \leqslant \lambda^*(y)$.
(C2) $\lambda^*(x) = \lambda^*(y)$ *implies* $\lambda(x) = \lambda(y)$.
(C3) $\lambda^*(x') < \lambda^*(y)$ *for all* $x' < x$ *and* $\lambda(x) \geqslant \lambda(y)$ *imply* $\lambda^*(x) \leqslant \lambda^*(y)$.

The first property shows that the function $\lambda^*\colon S \to \operatorname{Ch}(T)$ can be defined by induction on the length $\lambda_0(x)$ of the elements $x \in S$. The subsequent properties suggest to think of λ^* as a refinement of λ.

Proof. To prove (C0), let $X=\lambda^*(x)$ and note that $\max X=\lambda(x)$. The assertion follows from Lemma 1.3 because we have

$$X\setminus\{\max X\}=\max\{X'\in\mathrm{Ch}(T)\mid X'<X \text{ and } \max X'<\max X\}.$$

Now suppose $x\leqslant y$ and let $X\in\mathrm{Ch}(S,x)$. Then $Y=X\cup\{y\}\in\mathrm{Ch}(S,y)$ and we have $\lambda(X)\leqslant\lambda(Y)$ since $\lambda(X)\subseteq\lambda(Y)$. Thus $\lambda^*(x)\leqslant\lambda^*(y)$. If $\lambda^*(x)=\lambda^*(y)$, then

$$\lambda(x)=\max\lambda^*(x)=\max\lambda^*(y)=\lambda(y).$$

To prove (C3), we use (C0) and apply Lemma 1.3 with $X=\lambda^*(x)$ and $Y=\lambda^*(y)$. In fact, $\lambda^*(x')<\lambda^*(y)$ for all $x'<x$ implies $X^*<Y$, and $\lambda(x)\geqslant\lambda(y)$ implies $\max X\geqslant\max Y$. Thus $X\leqslant Y$. ☐

Corollary 1.6. *Let $\lambda\colon S\to T$ be a length function. Then the induced map λ^* is a length function.*

Proof. (L1) follows from (C1) and (C2). (L2) and (L3) follow from the corresponding conditions on λ. ☐

An axiomatic characterization. Let $\lambda\colon S\to T$ be a length function. We present an axiomatic characterization of the induced chain length function λ^*. Thus we can replace the original definition in terms of chains by three simple conditions which express the fact that λ^* refines λ.

Theorem 1.7. *Let $\lambda\colon S\to T$ be a length function. Then there exists a map $\mu\colon S\to U$ into a partially ordered set U satisfying for all $x,y\in S$ the following:*

- (M1) *$x\leqslant y$ implies $\mu(x)\leqslant\mu(y)$.*
- (M2) *$\mu(x)=\mu(y)$ implies $\lambda(x)=\lambda(y)$.*
- (M3) *$\mu(x')<\mu(y)$ for all $x'<x$ and $\lambda(x)\geqslant\lambda(y)$ imply $\mu(x)\leqslant\mu(y)$.*

Moreover, for any map $\mu'\colon S\to U'$ into a partially ordered set U' satisfying the above conditions, we have

$$\mu'(x)\leqslant\mu'(y)\quad\iff\quad\mu(x)\leqslant\mu(y)\quad\textit{for all}\quad x,y\in S.$$

Proof. We have seen in Proposition 1.5 that λ^* satisfies (M1) – (M3). So it remains to show that for any map $\mu\colon S\to U$ into a partially ordered set U, the conditions (M1) – (M3) uniquely determine the relation $\mu(x)\leqslant\mu(y)$ for any pair $x,y\in S$. We proceed by induction on the length $\lambda_0(x)$ of the elements $x\in S$ and show in each step the following for $S_n=\{x\in S\mid\lambda_0(x)\leqslant n\}$.

- (i) $\{\mu(x')\mid x'\in S_n \text{ and } x'\leqslant x\}$ is a finite set for all $x\in S$.
- (ii) (M1) – (M3) determine the relation $\mu(x)\leqslant\mu(y)$ for all $x,y\in S_n$.
- (iii) $\mu(x)\leqslant\mu(y)$ or $\mu(y)\leqslant\mu(x)$ for all $x,y\in S_n$.

For $n=1$ the assertion is clear. In fact, S_1 is the set of minimal elements in S, and $\lambda(x)\geqslant\lambda(y)$ implies $\mu(x)\leqslant\mu(y)$ for $x,y\in S_1$, by (M3). Now let $n>1$ and assume the assertion is true for S_{n-1}. To show (i), fix $x\in S$. The map

$$\{\mu(x')\mid x'\in S_n \text{ and } x'\leqslant x\}\longrightarrow\{\mu(x')\mid x'\in S_{n-1}\text{ and } x'\leqslant x\}\times\{\lambda(x')\mid x'\leqslant x\}$$

sending $\mu(x')$ to the pair $(\max_{y<x'}\mu(y),\lambda(x'))$ is well-defined by (i) and (iii); it is injective by (M3). Thus $\{\mu(x')\mid x'\in S_n\text{ and } x'\leqslant x\}$ is a finite set. In order to verify (ii) and (iii), we choose for each $x\in S_n$ a *Gabriel-Roiter filtration*, that is, a sequence

$$x_1<x_2<\ldots<x_{\gamma(x)-1}<x_{\gamma(x)}=x$$

in S such that x_1 is minimal and $\max_{x'<x_i} \mu(x') = \mu(x_{i-1})$ for all $1 < i \leqslant \gamma(x)$. Such a filtration exists because the elements $\mu(x')$ with $x' < x$ form a finite chain, by (i) and (iii). Now fix $x, y \in S_n$ and let $I = \{i \geqslant 1 \mid \mu(x_i) = \mu(y_i)\}$. We consider $r = \max I$ and put $r = 0$ if $I = \emptyset$. There are two possible cases. Suppose first that $r = \gamma(x)$ or $r = \gamma(y)$. If $r = \gamma(x)$, then $\mu(x) = \mu(x_r) = \mu(y_r) \leqslant \mu(y)$ by (M1). Now suppose $\gamma(x) \neq r \neq \gamma(y)$. Then we have $\lambda(x_{r+1}) \neq \lambda(y_{r+1})$ by (M2) and (M3). If $\lambda(x_{r+1}) > \lambda(y_{r+1})$, then we obtain $\mu(x_{r+1}) < \mu(y_{r+1})$, again using (M2) and (M3). Iterating this argument, we get $\mu(x) = \mu(x_{\gamma(x)}) < \mu(y_{r+1})$. From (M1) we get $\mu(x) < \mu(y_{r+1}) \leqslant \mu(y)$. Thus $\mu(x) \leqslant \mu(y)$ or $\mu(x) \geqslant \mu(y)$ and the proof is complete. □

Corollary 1.8. *Let $\lambda\colon S \to T$ be a length function and let $\mu\colon S \to U$ be a map into a partially ordered set U satisfying* (M1) – (M3). *Then μ is a length function. Moreover, we have for all $x, y \in S$*

$$\mu(x) = \mu(y) \quad \Longleftrightarrow \quad \max_{x'<x} \mu(x') = \max_{y'<y} \mu(y') \text{ and } \lambda(x) = \lambda(y).$$

Iterated length functions. Let λ be a length function. Then λ^* is again a length function by Corollary 1.6. Thus we may define inductively $\lambda^{(0)} = \lambda$ and $\lambda^{(n)} = (\lambda^{(n-1)})^*$ for $n \geqslant 1$. In many examples, we have that $\lambda^{(1)}$ and $\lambda^{(3)}$ are equivalent. However, this is not a general fact. The author is grateful to Osamu Iyama for suggesting the following example.

Example 1.9. The following length functions $\lambda^{(1)}$ and $\lambda^{(3)}$ are not equivalent.

$$\lambda^{(0)}: \begin{array}{ccccc} 4 & & 5 & & 6 \\ | & \diagdown & | & \diagdown & | \\ 3 & & 2 & & 1 \end{array} \qquad \lambda^{(1)}: \begin{array}{ccccc} 3 & & 6 & & 5 \\ | & \diagdown & | & \diagdown & | \\ 1 & & 2 & & 4 \end{array} \qquad \lambda^{(2)}: \begin{array}{ccccc} 6 & & 4 & & 2 \\ | & \diagdown & | & \diagdown & | \\ 5 & & 3 & & 1 \end{array}$$

$$\lambda^{(3)}: \begin{array}{ccccc} 3 & & 5 & & 6 \\ | & \diagdown & | & \diagdown & | \\ 1 & & 2 & & 4 \end{array} \qquad \lambda^{(4)}: \begin{array}{ccccc} 6 & & 4 & & 2 \\ | & \diagdown & | & \diagdown & | \\ 5 & & 3 & & 1 \end{array}$$

2. Abelian length categories

In this section we recall the definition and some basic facts about abelian length categories. We fix an abelian category $\mathcal{A}$.

Subobjects. We say that two monomorphisms $\phi_1\colon X_1 \to X$ and $\phi_2\colon X_2 \to X$ in $\mathcal{A}$ are *equivalent*, if there exists an isomorphism $\alpha\colon X_1 \to X_2$ such that $\phi_1 = \phi_2 \circ \alpha$. An equivalence class of monomorphisms into X is called a *subobject* of X. Given subobjects $\phi_1\colon X_1 \to X$ and $\phi_2\colon X_2 \to X$ of X, we write $X_1 \subseteq X_2$ if there is a morphism $\alpha\colon X_1 \to X_2$ such that $\phi_2 = \phi_1 \circ \alpha$. An object $X \neq 0$ is *simple* if $X' \subseteq X$ implies $X' = 0$ or $X' = X$.

Length categories. An object X of $\mathcal{A}$ has *finite length* if it has a finite composition series

$$0 = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_{n-1} \subseteq X_n = X,$$

that is, each X_i/X_{i-1} is simple. Note that X has finite length if and only if X is both artinian (i.e. it satisfies the descending chain condition on subobjects) and noetherian (i.e. it satisfies the ascending chain condition on subobjects). An abelian category is called a *length category* if all objects have finite length and if the isomorphism classes of objects form a set.

Recall that an object $X \neq 0$ is *indecomposable* if $X = X_1 \oplus X_2$ implies $X_1 = 0$ or $X_2 = 0$. A finite length object admits a finite direct sum decomposition into indecomposable objects having local endomorphism rings. Moreover, such a decomposition is unique up to an isomorphism by the Krull-Remak-Schmidt Theorem.

Example 2.1. (1) The finitely generated modules over an artinian ring form a length category.

(2) Let k be a field and Q be any quiver. Then the finite dimensional k-linear representations of Q form a length category.

3. The Gabriel-Roiter measure

Let $\mathcal{A}$ be an abelian length category. The definition of the Gabriel-Roiter measure of $\mathcal{A}$ is due to Gabriel [1] and was inspired by the work of Roiter [5]. We present a definition which is a slight generalization of Gabriel's original definition. Then we discuss some specific properties.

Length functions. A *length function* on $\mathcal{A}$ is by definition a map ℓ which sends each object $X \in \mathcal{A}$ to some real number $\ell(X) \geqslant 0$ such that

(ℓ1) $\ell(X) = 0$ if and only if $X = 0$, and
(ℓ2) $\ell(X) = \ell(X') + \ell(X'')$ for every exact sequence $0 \to X' \to X \to X'' \to 0$.

Note that such a length function is determined by the set of values $\ell(S) > 0$, where S runs through the isomorphism classes of simple objects of $\mathcal{A}$. This follows from the Jordan-Hölder Theorem. We write ℓ_1 for the length function satisfying $\ell_1(S) = 1$ for every simple object S. Observe that $\ell_1(X)$ is the usual composition length of an object $X \in \mathcal{A}$.

The Gabriel-Roiter measure. We consider the set $\operatorname{ind}\mathcal{A}$ of isomorphism classes of indecomposable objects of $\mathcal{A}$ which is partially ordered via the subobject relation $X \subseteq Y$. Now fix a length function ℓ on $\mathcal{A}$. The map ℓ induces a length function $\operatorname{ind}\mathcal{A} \to \mathbb{R}$ satisfying (L1) – (L3), and the induced chain length function $\ell^*\colon \operatorname{ind}\mathcal{A} \to \operatorname{Ch}(\mathbb{R})$ is by definition the *Gabriel-Roiter measure* of $\mathcal{A}$ with respect to ℓ. Gabriel's original definition [1] is based on the length function ℓ_1. Whenever it is convenient, we substitute $\mu = \ell^*$.

An axiomatic characterization. The following axiomatic characterization of the Gabriel-Roiter measure is the main result of this note.

Theorem 3.1. *Let $\mathcal{A}$ be an abelian length category and ℓ a length function on $\mathcal{A}$. Then there exists a map $\mu\colon \operatorname{ind}\mathcal{A} \to P$ into a partially ordered set P satisfying for all $X, Y \in \operatorname{ind}\mathcal{A}$ the following:*

(GR1) *$X \subseteq Y$ implies $\mu(X) \leqslant \mu(Y)$.*
(GR2) *$\mu(X) = \mu(Y)$ implies $\ell(X) = \ell(Y)$.*
(GR3) *$\mu(X') < \mu(Y)$ for all $X' \subset X$ and $\ell(X) \geqslant \ell(Y)$ imply $\mu(X) \leqslant \mu(Y)$.*

Moreover, for any map $\mu'\colon \operatorname{ind}\mathcal{A} \to P'$ into a partially ordered set P' satisfying the above conditions, we have

$$\mu'(X) \leqslant \mu'(Y) \quad \Longleftrightarrow \quad \mu(X) \leqslant \mu(Y) \quad \text{for all} \quad X, Y \in \operatorname{ind}\mathcal{A}.$$

Proof. Use the axiomatic characterization of the chain length function ℓ^* in Theorem 1.7. □

Gabriel's main property. Let ℓ be a fixed length function on $\mathcal{A}$. The following main property of the Gabriel-Roiter measure $\mu = \ell^*$ is crucial; it is the basis for all applications.

Proposition 3.2 (Gabriel)**.** *Let $X, Y_1, \ldots, Y_r \in \operatorname{ind}\mathcal{A}$. Suppose that $X \subseteq Y = \oplus_{i=1}^r Y_i$. Then $\mu(X) \leqslant \max \mu(Y_i)$ and X is a direct summand of Y if $\mu(X) = \max \mu(Y_i)$.*

Proof. The proof only uses the properties (GR1) – (GR3) of μ. Fix a monomorphism $\phi\colon X \to Y$. We proceed by induction on $n = \ell_1(X) + \ell_1(Y)$. If $n = 2$, then ϕ is an isomorphism and the assertion is clear. Now suppose $n > 2$. We can assume that for each i the ith component $\phi_i\colon X \to Y_i$ of ϕ is an epimorphism. Otherwise choose for each i a decomposition $Y_i' = \oplus_j Y_{ij}$ of the image of ϕ_i into indecomposables. Then we use (GR1) and have $\mu(X) \leqslant \max \mu(Y_{ij}) \leqslant \max \mu(Y_i)$ because $\ell_1(X) + \ell_1(Y') < n$ and $Y_{ij} \subseteq Y_i$ for all j. Now suppose that each ϕ_i is an epimorphism. Thus $\ell(X) \geqslant \ell(Y_i)$ for all i. Let $X' \subset X$ be a proper indecomposable subobject. Then $\mu(X') \leqslant \max \mu(Y_i)$ because $\ell_1(X') + \ell_1(Y) < n$, and X' is a direct summand if $\mu(X') = \max \mu(Y_i)$. We can exclude the case that $\mu(X') = \max \mu(Y_i)$ because then X' is a proper direct summand of X, which is impossible. Now we apply (GR3) and obtain $\mu(X) \leqslant \max \mu(Y_i)$. Finally, suppose that $\mu(X) = \max \mu(Y_i) = \mu(Y_k)$ for some k. We claim that we can choose k such that ϕ_k is an epimorphism. Otherwise, replace all Y_i with $\mu(X) = \mu(Y_i)$ by the image $Y_i' = \oplus_j Y_{ij}$ of ϕ_i as before. We obtain $\mu(X) \leqslant \max \mu(Y_{ij}) < \mu(Y_k)$ since $Y_{kj} \subset Y_k$ for all j, using (GR1) and (GR2). This is a contradiction. Thus ϕ_k is an epimorphism and in fact an isomorphism because $\ell(X) = \ell(Y_k)$ by (GR2). In particular, X is a direct summand of $\oplus_i Y_i$. This completes the proof. □

Gabriel-Roiter filtrations. We keep a length function ℓ on $\mathcal{A}$ and the corresponding Gabriel-Roiter measure $\mu = \ell^*$. Let $X, Y \in \operatorname{ind}\mathcal{A}$. We say that X is a *Gabriel-Roiter predecessor* of Y if $X \subset Y$ and $\mu(X) = \max_{Y' \subset Y} \mu(Y')$. Note that each object $Y \in \operatorname{ind}\mathcal{A}$ which is not simple admits a Gabriel-Roiter predecessor because μ is a length function on $\operatorname{ind}\mathcal{A}$. A Gabriel-Roiter predecessor X of Y is usually not unique, but the value $\mu(X)$ is determined by $\mu(Y)$.

A sequence

$$X_1 \subset X_2 \subset \ldots \subset X_{n-1} \subset X_n = X$$

in $\operatorname{ind}\mathcal{A}$ is called a *Gabriel-Roiter filtration* of X if X_1 is simple and X_{i-1} is a Gabriel-Roiter predecessor of X_i for all $1 < i \leqslant n$. Clearly, each X admits such a filtration and the values $\mu(X_i)$ are uniquely determined by X. Note that (C0) implies

$$\mu(X) = \{\ell(X_i) \mid 1 \leqslant i \leqslant n\}. \tag{3.1}$$

Injective objects. In order to illustrate Gabriel's main property, let us show that the Gabriel-Roiter measure detects injective objects. This is a remarkable fact because the Gabriel-Roiter measure is a combinatorial invariant, depending only on the poset of indecomposable objects and some length function, whereas the notion of injectivity involves all morphisms of the category $\mathcal{A}$.

Theorem 3.3. *An indecomposable object Q of $\mathcal{A}$ is injective if and only if there is a length function ℓ on $\mathcal{A}$ such that $\ell^*(X) \leqslant \ell^*(Q)$ for all $X \in \operatorname{ind}\mathcal{A}$.*

We need the following lemma.

Lemma 3.4. *Let ℓ be a length function on $\mathcal{A}$ and fix indecomposable objects $X, Y \in \mathcal{A}$. Suppose that for each pair of simple subobjects $X' \subseteq X$ and $Y' \subseteq Y$, we have $\ell(X') < \ell(Y')$. Then $\ell^*(X) > \ell^*(Y)$.*

Proof. We choose Gabriel-Roiter filtrations $X_1 \subset \ldots \subset X_n = X$ and $Y_1 \subset \ldots \subset Y_m = Y$. Then $\ell(X_1) < \ell(Y_1)$ and the formula (3.1) implies

$$\ell^*(X) = \{\ell(X_i) \mid 1 \leqslant i \leqslant n\} > \{\ell(Y_i) \mid 1 \leqslant i \leqslant m\} = \ell^*(Y).$$

□

Proof of the theorem. Suppose first that Q is injective. Then Q has a unique simple subobject S and we define a length function $\ell = \ell_S$ on $\mathcal{A}$ by specifying its values on each simple object $T \in \mathcal{A}$ as follows:

$$\ell(T) := \begin{cases} 1 & \text{if } T \cong S, \\ 2 & \text{if } T \ncong S. \end{cases}$$

Now let $X \in \operatorname{ind} \mathcal{A}$. We claim that $\ell^*(X) \leqslant \ell^*(Q)$. To see this, let $X' \subseteq X$ be the maximal subobject of X having composition factors isomorphic to S. Using induction on the composition length $n = \ell_1(X')$ of X', one obtains a monomorphism $X' \to Q^n$, and this extends to a map $\phi \colon X \to Q^n$, since Q is injective. Let $X/X' = \oplus_i Y_i$ be a decomposition into indecomposables and $\pi \colon X \to X/X'$ be the canonical map. Note that $\ell^*(Y_i) < \ell^*(Q)$ for all i by our construction and Lemma 3.4. Then $(\pi, \phi) \colon X \to (\oplus_i Y_i) \oplus Q^n$ is a monomorphism and therefore $\ell^*(X) \leqslant \ell^*(Q)$ by the main property.

Suppose now that $\ell^*(X) \leqslant \ell^*(Q)$ for all $X \in \operatorname{ind} \mathcal{A}$ and some length function ℓ on $\mathcal{A}$. To show that Q is injective, suppose that $Q \subseteq Y$ is the subobject of some $Y \in \mathcal{A}$. Let $Y = \oplus Y_i$ be a decomposition into indecomposables. Then the main property implies $\ell^*(Q) \leqslant \max \ell^*(Y_i) \leqslant \ell^*(Q)$ and therefore Q is a direct summand of Y. Thus Q is injective and the proof is complete. □

Let us mention that there is the following analogous characterization of the simple objects of $\mathcal{A}$.

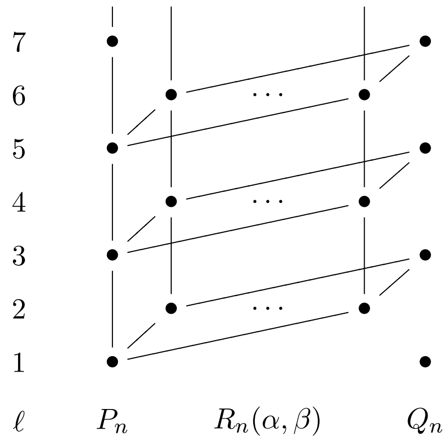
Corollary 3.5. *An indecomposable object S of $\mathcal{A}$ is simple if and only if there is a length function ℓ on $\mathcal{A}$ such that $\ell^*(S) \leqslant \ell^*(X)$ for all $X \in \operatorname{ind} \mathcal{A}$.*

Proof. Use the property (GR1) of the Gabriel-Roiter measure and apply Lemma 3.4. □

The Kronecker algebra. Let $\Lambda = \left[\begin{smallmatrix} k & k^2 \\ 0 & k \end{smallmatrix}\right]$ be the Kronecker algebra over an algebraically closed field k. We consider the abelian length category which is formed by all finite dimensional Λ-modules. A complete list of indecomposable objects is given by the preprojectives P_n, the regulars $R_n(\alpha, \beta)$, and the preinjectives Q_n. More precisely,

$$\operatorname{ind} \Lambda = \{P_n \mid n \in \mathbb{N}\} \cup \{R_n(\alpha, \beta) \mid n \in \mathbb{N},\, (\alpha, \beta) \in \mathbb{P}^1_k\} \cup \{Q_n \mid n \in \mathbb{N}\},$$

and we obtain the following Hasse diagram.



The set of indecomposables is ordered as follows via the Gabriel-Roiter measure with respect to $\ell = \ell_1$.

$$\ell^* : \quad Q_1 = P_1 < P_2 < P_3 < \dots \quad R_1 < R_2 < R_3 < \dots \quad \dots < Q_4 < Q_3 < Q_2$$
$$(\ell^*)^* : \quad Q_1 = P_1 < R_1 < Q_2 < P_2 < R_2 < Q_3 < P_3 < R_3 < Q_4 < \dots$$

Moreover, $((\ell^*)^*)^*$ and ℓ^* are equivalent length functions.

Remark 3.6. While ℓ^* has been successfully employed for proving the first Brauer-Thrall conjecture, Hubery points out that $(\ell^*)^*$ might be useful for proving the second. In fact, one needs to find a value $(\ell^*)^*(X)$ such that the set $\{X' \in \operatorname{ind}\Lambda \mid (\ell^*)^*(X') = (\ell^*)^*(X)\}$ is infinite. The example of the Kronecker algebra shows that there exists such a value having only finitely many predecessors $(\ell^*)^*(Y) < (\ell^*)^*(X)$. Note that in all known examples $((\ell^*)^*)^*$ and ℓ^* are equivalent.

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