AN AXIOMATIC CHARACTERIZATION OF THE GABRIEL-ROITER MEASURE

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Abstract. Given an abelian length category $\mathcal{A}$, the Gabriel-Roiter measure with respect to a length function $\ell$ is characterized as a universal morphism $\text{ind}\mathcal{A} \rightarrow P$ of partially ordered sets. The map is defined on the isomorphism classes of indecomposable objects of $\mathcal{A}$ and is a suitable refinement of the length function $\ell$.

In his proof of the first Brauer-Thrall conjecture [5], Roiter used an induction scheme which Gabriel formalized in his report on abelian length categories [1]. The first Brauer-Thrall conjecture asserts that every finite dimensional algebra of bounded representation type is of finite representation type. Ringel noticed (see the footnote on p. 91 of [1]) that the formalism of Gabriel and Roiter works equally well for studying the representations of algebras having unbounded representation type. We refer to recent work [2, 3, 4] for some beautiful applications.

In this note we present an axiomatic characterization of the Gabriel-Roiter measure which reveals its combinatorial nature. Given a finite dimensional algebra $\Lambda$, the Gabriel-Roiter measure is characterized as a universal morphism $\text{ind}\Lambda \rightarrow P$ of partially ordered sets. The map is defined on the isomorphism classes of finite dimensional indecomposable $\Lambda$-modules and is a suitable refinement of the length function $\text{ind}\Lambda \rightarrow \mathbb{N}$ which sends a module to its composition length.

The first part of this paper is purely combinatorial and might be of independent interest. We study length functions $\lambda: S \rightarrow T$ on a fixed partially ordered set $S$. Such a length function takes its values in another partially ordered set $T$, for example $T = \mathbb{N}$. We denote by $\text{Ch}(T)$ the set of finite chains in $T$, together with the lexicographic ordering. The map $\lambda$ induces a new length function $\lambda^*: S \rightarrow \text{Ch}(T)$, which we call chain length function because each value $\lambda^*(x)$ measures the lengths $\lambda(x_i)$ of the elements $x_i$ occurring in some finite chain $x_1 < x_2 < \ldots < x_n = x$ of $x$ in $S$. We think of $\lambda^*$ as a specific refinement of $\lambda$ and provide an axiomatic characterization. It is interesting to observe that this construction can be iterated. Thus we may consider $(\lambda^*)^*, ((\lambda^*)^*)^*$, and so on.

The second part of the paper discusses the Gabriel-Roiter measure for a fixed abelian length category $\mathcal{A}$, for example the category of finite dimensional $\Lambda$-modules over some algebra $\Lambda$. For each length function $\ell$ on $\mathcal{A}$, we consider its restriction to the partially ordered set $\text{ind}\mathcal{A}$ of isomorphism classes of indecomposable objects of $\mathcal{A}$. Then the Gabriel-Roiter measure with respect to $\ell$ is by definition the corresponding chain length function $\ell^*$. In particular, we obtain an axiomatic characterization of $\ell^*$ and use it to reprove Gabriel’s main property of the Gabriel-Roiter measure. Note that we work with a slight generalization of Gabriel’s original definition. This enables us to characterize the injective objects of $\mathcal{A}$ as those objects where $\ell^*$ takes maximal values for some

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length function \( \ell \). This is a remarkable fact because the Gabriel-Roiter measure is a combinatorial invariant, depending only on the poset of indecomposable objects and some length function, whereas the notion of injectivity involves all morphisms of the category \( \mathcal{A} \).

1. Chains and length functions

**The lexicographic order on finite chains.** Let \((S, \leq)\) be a partially ordered set. A subset \(X \subseteq S\) is a *chain* if \(x_1 \leq x_2\) or \(x_2 \leq x_1\) for each pair \(x_1, x_2 \in X\). For a finite chain \(X\), we denote by \(\min X\) its minimal and by \(\max X\) its maximal element, using the convention

\[
\max \emptyset < x < \min \emptyset \quad \text{for all} \quad x \in S.
\]

We write \(\text{Ch}(S)\) for the set of all finite chains in \(S\) and let

\[
\text{Ch}(S, x) := \{X \in \text{Ch}(S) \mid \max X = x\} \quad \text{for} \quad x \in S.
\]

On \(\text{Ch}(S)\) we consider the *lexicographic order* which is defined by

\[
X \leq Y :\iff \min(Y \setminus X) \leq \min(X \setminus Y) \quad \text{for} \quad X, Y \in \text{Ch}(S).
\]

**Remark 1.1.**

1. \(X \subseteq Y\) implies \(X \leq Y\) for \(X, Y \in \text{Ch}(S)\).
2. Suppose that \(S\) is totally ordered. Then \(\text{Ch}(S)\) is totally ordered. We may think of \(X \in \text{Ch}(S) \subseteq \{0, 1\}^S\) as a string of 0s and 1s which is indexed by the elements in \(S\). The usual lexicographic order on such strings coincides with the lexicographic order on \(\text{Ch}(S)\).

**Example 1.2.** Let \(\mathbb{N} = \{1, 2, 3, \ldots \}\) and \(\mathbb{Q}\) be the set of rational numbers together with the natural ordering. Then the map

\[
\text{Ch}(\mathbb{N}) \longrightarrow \mathbb{Q}, \quad X \mapsto \sum_{x \in X} 2^{-x}
\]

is injective and order preserving, taking values in the interval \([0, 1]\). For instance, the subsets of \(\{1, 2, 3\}\) are ordered as follows:

\[
\emptyset < \{3\} < \{2\} < \{2, 3\} < \{1\} < \{1, 3\} < \{1, 2\} < \{1, 2, 3\}.
\]

We need the following properties of the lexicographic order.

**Lemma 1.3.** Let \(X, Y \in \text{Ch}(S)\) and \(X^* := X \setminus \{\max X\}\).

1. \(X^* = \max\{X' \in \text{Ch}(S) \mid X' < X \text{ and } \max X' < \max X\}\).
2. If \(X^* < Y\) and \(\max X \geq \max Y\), then \(X \leq Y\).

**Proof.**

1. Let \(X' < X\) and \(\max X' < \max X\). We show that \(X' \leq X^*\). This is clear if \(X' \subseteq X^*\). Otherwise, we have

\[
\min(X^* \setminus X') = \min(X \setminus X') < \min(X' \setminus X) = \min(X' \setminus X^*),
\]

and therefore \(X' \leq X^*\).
2. The assumption \(X^* < Y\) implies by definition

\[
\min(Y \setminus X^*) < \min(X^* \setminus Y).
\]

We consider two cases. Suppose first that \(X^* \leq Y\). If \(X \leq Y\), then \(X \leq Y\). Otherwise,

\[
\min(Y \setminus X) < \max X = \min(X \setminus Y)
\]
and therefore $X < Y$. Now suppose that $X^* \not\subseteq Y$. We use again that $\max X \geq \max Y$, exclude the case $Y \subseteq X$, and obtain
\[
\min(Y \setminus X) = \min(Y \setminus X^*) < \min(X^* \setminus Y) = \min(X \setminus Y).
\]
Thus $X \leq Y$ and the proof is complete. \hfill \Box

**Length functions.** Let $(S, \leq)$ be a partially ordered set. A *length function* on $S$ is by definition a map $\lambda: S \rightarrow T$ into a partially ordered set $T$ satisfying for all $x, y \in S$ the following:

- (L1) $x < y$ implies $\lambda(x) < \lambda(y)$.
- (L2) $\lambda(x) \leq \lambda(y)$ or $\lambda(y) \leq \lambda(x)$.
- (L3) $\lambda_0(x) := \text{card}\{\lambda(x') | x' \in S \text{ and } x' \leq x\}$ is finite.

Two length functions $\lambda$ and $\lambda'$ on $S$ are *equivalent* if
\[
\lambda(x) \leq \lambda(y) \iff \lambda'(x) \leq \lambda'(y) \quad \text{for all } x, y \in S.
\]
Observe that (L2) and (L3) are automatically satisfied if $T = \mathbb{N}$. A length function $\lambda: S \rightarrow T$ induces for each $x \in S$ a map
\[
\text{Ch}(S, x) \rightarrow \text{Ch}(T, \lambda(x)), \quad X \mapsto \lambda(X),
\]
and therefore the following *chain length function*
\[
S \rightarrow \text{Ch}(T), \quad x \mapsto \lambda^*(x) := \max\{\lambda(X) | X \in \text{Ch}(S, x)\}.
\]
Note that equivalent length functions induce equivalent chain length functions.

**Example 1.4.** (1) Let $S$ be a poset such that for each $x \in S$ there is a bound $n_x \in \mathbb{N}$ with $\text{card} X \leq n_x$ for all $X \in \text{Ch}(S, x)$. Then the map $S \rightarrow \mathbb{N}$ sending $x$ to $\max\{\text{card} X | X \in \text{Ch}(S, x)\}$ is a length function.

(2) Let $S$ be a poset such that $\{x' \in S | x' \leq x\}$ is a finite chain for each $x \in S$. Then the map $\lambda: S \rightarrow \mathbb{N}$ sending $x$ to $\text{card}\{x' \in S | x' \leq x\}$ is a length function. Moreover, $\lambda^*$ is a length function and equivalent to $\lambda$.

(3) Let $\lambda: S \rightarrow \mathbb{Z}$ be a length function which satisfies in addition the following properties of a *rank function*: $\lambda(x) = \lambda(y)$ for each pair $x, y$ of minimal elements of $S$, and $\lambda(x) = \lambda(y) - 1$ whenever $x$ is an immediate predecessor of $y$ in $S$. Then $\lambda^*$ is a length function and equivalent to $\lambda$.

**Basic properties.** Let $\lambda: S \rightarrow T$ be a length function and $\lambda^*: S \rightarrow \text{Ch}(T)$ the induced chain length function. We collect the basic properties of $\lambda^*$.

**Proposition 1.5.** Let $x, y \in S$.

(C0) $\lambda^*(x) = \max_{x' \leq x} \lambda^*(x') \cup \{\lambda(x)\}$.

(C1) $x \leq y$ implies $\lambda^*(x) \leq \lambda^*(y)$.

(C2) $\lambda^*(x) = \lambda^*(y)$ implies $\lambda(x) = \lambda(y)$.

(C3) $\lambda^*(x') < \lambda^*(y)$ for all $x' < x$ and $\lambda(x) \geq \lambda(y)$ imply $\lambda^*(x) \leq \lambda^*(y)$.

The first property shows that the function $\lambda^*: S \rightarrow \text{Ch}(T)$ can be defined by induction on the length $\lambda_0(x)$ of the elements $x \in S$. The subsequent properties suggest to think of $\lambda^*$ as a refinement of $\lambda$. 
Proof. To prove (C0), let $X = \lambda^*(x)$ and note that $\max X = \lambda(x)$. The assertion follows from Lemma 1.3 because we have

$$X \setminus \{\max X\} = \max\{X' \in \text{Ch}(T) \mid X' < X \text{ and } \max X' < \max X\}.$$ 

Now suppose $x \leq y$ and let $X \in \text{Ch}(S, x)$. Then $Y = X \cup \{y\} \in \text{Ch}(S, y)$ and we have $\lambda(X) \leq \lambda(Y)$ since $\lambda(X) \subseteq \lambda(Y)$. Thus $\lambda^*(x) \leq \lambda^*(y)$. If $\lambda^*(x) = \lambda^*(y)$, then

$$\lambda(x) = \max \lambda^*(x) = \max \lambda^*(y) = \lambda(y).$$

To prove (C3), we use (C0) and apply Lemma 1.3 with $X = \lambda^*(x)$ and $Y = \lambda^*(y)$. In fact, $\lambda^*(x') < \lambda^*(y)$ for all $x' < x$ implies $X < Y$, and $\lambda(x) \geq \lambda(y)$ implies $\max X \geq \max Y$. Thus $X \leq Y$. □

Corollary 1.6. Let $\lambda: S \to T$ be a length function. Then the induced map $\lambda^*$ is a length function.

Proof. (L1) follows from (C1) and (C2). (L2) and (L3) follow from the corresponding conditions on $\lambda$. □

An axiomatic characterization. Let $\lambda: S \to T$ be a length function. We present an axiomatic characterization of the induced chain length function $\lambda^*$. Thus we can replace the original definition in terms of chains by three simple conditions which express the fact that $\lambda^*$ refines $\lambda$.

Theorem 1.7. Let $\lambda: S \to T$ be a length function. Then there exists a map $\mu: S \to U$ into a partially ordered set $U$ satisfying for all $x, y \in S$ the following:

(M1) $x \leq y$ implies $\mu(x) \leq \mu(y)$.
(M2) $\mu(x) = \mu(y)$ implies $\lambda(x) = \lambda(y)$.
(M3) $\mu(x') < \mu(y)$ for all $x' < x$ and $\lambda(x) \geq \lambda(y)$ imply $\mu(x) \leq \mu(y)$.

Moreover, for any map $\mu': S \to U'$ into a partially ordered set $U'$ satisfying the above conditions, we have

$$\mu'(x) \leq \mu'(y) \iff \mu(x) \leq \mu(y) \text{ for all } x, y \in S.$$ 

Proof. We have seen in Proposition 1.5 that $\lambda^*$ satisfies (M1) – (M3). So it remains to show that for any map $\mu: S \to U$ into a partially ordered set $U$, the conditions (M1) – (M3) uniquely determine the relation $\mu(x) \leq \mu(y)$ for any pair $x, y \in S$. We proceed by induction on the length $\lambda(x)$ of the elements $x \in S$ and show in each step the following for $S_n = \{x \in S \mid \lambda_0(x) \leq n\}$.

(i) $\{\mu(x') \mid x' \in S_n \text{ and } x' \leq x\}$ is a finite set for all $x \in S$.
(ii) (M1) – (M3) determine the relation $\mu(x) \leq \mu(y)$ for all $x, y \in S_n$.
(iii) $\mu(x) \leq \mu(y)$ or $\mu(y) \leq \mu(x)$ for all $x, y \in S_n$.

For $n = 1$ the assertion is clear. In fact, $S_1$ is the set of minimal elements in $S$, and $\lambda(x) \geq \lambda(y)$ implies $\mu(x) \leq \mu(y)$ for $x, y \in S_1$, by (M3). Now let $n > 1$ and assume the assertion is true for $S_{n-1}$. To show (i), fix $x \in S$. The map

$$\{\mu(x') \mid x' \in S_n \text{ and } x' \leq x\} \to \{\mu(x') \mid x' \in S_{n-1} \text{ and } x' \leq x\} \times \{\lambda(x') \mid x' \leq x\}$$

sending $\mu(x')$ to the pair $(\max_{y \leq x'} \mu(y), \lambda(x'))$ is well-defined by (i) and (ii); it is injective by (M3). Thus $\{\mu(x') \mid x' \in S_n \text{ and } x' \leq x\}$ is a finite set. In order to verify (ii) and (iii), we choose for each $x \in S_n$ a Gabriel-Roiter filtration, that is, a sequence

$$x_1 < x_2 < \ldots < x_{\gamma(x)-1} < x_{\gamma(x)} = x$$
in \( S \) such that \( x_1 \) is minimal and \( \max_{x' < x} \mu(x') = \mu(x_{i-1}) \) for all \( 1 < i \leq \gamma(x) \). Such a filtration exists because the elements \( \mu(x') \) with \( x' < x \) form a finite chain, by (i) and (iii). Now fix \( x, y \in S_0 \) and let \( I = \{ i \geq 1 \mid \mu(x_i) = \mu(y_i) \} \). We consider \( r = \max I \) and put \( r = 0 \) if \( I = \emptyset \). There are two possible cases. Suppose first that \( r = \gamma(x) \) or \( r = \gamma(y) \). If \( r = \gamma(x) \), then \( \mu(x) = \mu(x_r) = \mu(y_r) \leq \mu(y) \) by (M1). Now suppose \( \gamma(x) \neq r \neq \gamma(y) \). Then we have \( \lambda(x_{r+1}) \neq \lambda(y_{r+1}) \) by (M2) and (M3). If \( \lambda(x_{r+1}) > \lambda(y_{r+1}) \), then we obtain \( \mu(x_{r+1}) < \mu(y_{r+1}) \), again using (M2) and (M3). Iterating this argument, we get \( \mu(x) = \mu(x_{\gamma(x)}) < \mu(y_{r+1}) \). From (M1) we get \( \mu(x) < \mu(y_{r+1}) \leq \mu(y) \). Thus \( \mu(x) \leq \mu(y) \) or \( \mu(x) \geq \mu(y) \) and the proof is complete. \( \square \)

**Corollary 1.8.** Let \( \lambda : S \to T \) be a length function and let \( \mu : S \to U \) be a map into a partially ordered set \( U \) satisfying (M1) – (M3). Then \( \mu \) is a length function. Moreover, we have for all \( x, y \in S \)

\[
\mu(x) = \mu(y) \iff \max_{x' < x} \mu(x') = \max_{y' < y} \mu(y') \text{ and } \lambda(x) = \lambda(y).
\]

**Iterated length functions.** Let \( \lambda \) be a length function. Then \( \lambda^* \) is again a length function by Corollary 1.6. Thus we may define inductively \( \lambda^{(0)} = \lambda \) and \( \lambda^{(n)} = (\lambda^{(n-1)})^* \) for \( n \geq 1 \). In many examples, we have that \( \lambda^{(1)} \) and \( \lambda^{(3)} \) are equivalent. However, this is not a general fact. The author is grateful to Osamu Iyama for suggesting the following example.

**Example 1.9.** The following length functions \( \lambda^{(1)} \) and \( \lambda^{(3)} \) are not equivalent.

\[
\begin{align*}
\lambda^{(0)} : & 4 & 5 & 6 \\
\lambda^{(1)} : & 3 & 6 & 5 \\
\lambda^{(2)} : & 6 & 4 & 2 \\
\lambda^{(3)} : & 3 & 5 & 6 \\
\lambda^{(4)} : & 6 & 4 & 2 \\
\end{align*}
\]

2. **Abelian length categories**

In this section we recall the definition and some basic facts about abelian length categories. We fix an abelian category \( \mathcal{A} \).

**Subobjects.** We say that two monomorphisms \( \phi_1 : X_1 \to X \) and \( \phi_2 : X_2 \to X \) in \( \mathcal{A} \) are **equivalent**, if there exists an isomorphism \( \alpha : X_1 \to X_2 \) such that \( \phi_1 = \phi_2 \circ \alpha \). An equivalence class of monomorphisms into \( X \) is called a **subobject** of \( X \). Given subobjects \( \phi_1 : X_1 \to X \) and \( \phi_2 : X_2 \to X \) of \( X \), we write \( X_1 \subseteq X_2 \) if there is a morphism \( \alpha : X_1 \to X_2 \) such that \( \phi_2 = \phi_1 \circ \alpha \). An object \( X \neq 0 \) is **simple** if \( X' \subseteq X \) implies \( X' = 0 \) or \( X' = X \).

**Length categories.** An object \( X \) of \( \mathcal{A} \) has **finite length** if it has a finite composition series

\[
0 = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_{n-1} \subseteq X_n = X,
\]

that is, each \( X_i/X_{i-1} \) is simple. Note that \( X \) has finite length if and only if \( X \) is both artinian (i.e. it satisfies the descending chain condition on subobjects) and noetherian (i.e. it satisfies the ascending chain condition on subobjects). An abelian category is
called a length category if all objects have finite length and if the isomorphism classes of objects form a set.

Recall that an object $X \neq 0$ is indecomposable if $X = X_1 \oplus X_2$ implies $X_1 = 0$ or $X_2 = 0$. A finite length object admits a finite direct sum decomposition into indecomposable objects having local endomorphism rings. Moreover, such a decomposition is unique up to an isomorphism by the Krull-Remak-Schmidt Theorem.

**Example 2.1.** (1) The finitely generated modules over an artinian ring form a length category.

(2) Let $k$ be a field and $Q$ be any quiver. Then the finite dimensional $k$-linear representations of $Q$ form a length category.

### 3. The Gabriel-Roiter measure

Let $\mathcal{A}$ be an abelian length category. The definition of the Gabriel-Roiter measure of $\mathcal{A}$ is due to Gabriel [1] and was inspired by the work of Roiter [5]. We present a definition which is a slight generalization of Gabriel’s original definition. Then we discuss some specific properties.

**Length functions.** A length function on $\mathcal{A}$ is by definition a map $\ell$ which sends each object $X \in \mathcal{A}$ to some real number $\ell(X) \geq 0$ such that

- $\ell(X) = 0$ if and only if $X = 0$, and
- $\ell(X) = \ell(X') + \ell(X'')$ for every exact sequence $0 \to X' \to X \to X'' \to 0$.

Note that such a length function is determined by the set of values $\ell(S) > 0$, where $S$ runs through the isomorphism classes of simple objects of $\mathcal{A}$. This follows from the Jordan-Hölder Theorem. We write $\ell_1$ for the length function satisfying $\ell_1(S) = 1$ for every simple object $S$. Observe that $\ell_1(X)$ is the usual composition length of an object $X \in \mathcal{A}$.

**The Gabriel-Roiter measure.** We consider the set $\text{ind} \mathcal{A}$ of isomorphism classes of indecomposable objects of $\mathcal{A}$ which is partially ordered via the subobject relation $X \subseteq Y$. Now fix a length function $\ell$ on $\mathcal{A}$. The map $\ell$ induces a length function $\text{ind} \mathcal{A} \to \mathbb{R}$ satisfying (L1) – (L3), and the induced chain length function $\ell^*: \text{ind} \mathcal{A} \to \text{Ch}(\mathbb{R})$ is by definition the Gabriel-Roiter measure of $\mathcal{A}$ with respect to $\ell$. Gabriel’s original definition [1] is based on the length function $\ell_1$. Whenever it is convenient, we substitute $\mu = \ell^*$.

**An axiomatic characterization.** The following axiomatic characterization of the Gabriel-Roiter measure is the main result of this note.

**Theorem 3.1.** Let $\mathcal{A}$ be an abelian length category and $\ell$ a length function on $\mathcal{A}$. Then there exists a map $\mu: \text{ind} \mathcal{A} \to P$ into a partially ordered set $P$ satisfying for all $X, Y \in \text{ind} \mathcal{A}$ the following:

- (GR1) $X \subseteq Y$ implies $\mu(X) \leq \mu(Y)$.
- (GR2) $\mu(X) = \mu(Y)$ implies $\ell(X) = \ell(Y)$.
- (GR3) $\mu(X') < \mu(Y)$ for all $X' \subset X$ and $\ell(X) \geq \ell(Y)$ imply $\mu(X) \leq \mu(Y)$.

Moreover, for any map $\mu': \text{ind} \mathcal{A} \to P'$ into a partially ordered set $P'$ satisfying the above conditions, we have

$$\mu'(X) \leq \mu'(Y) \iff \mu(X) \leq \mu(Y)$$

for all $X, Y \in \text{ind} \mathcal{A}$. 
Gabriel’s main property. Let \( \ell \) be a fixed length function on \( \mathcal{A} \). The following main property of the Gabriel-Roiter measure \( \mu = \ell^\ast \) is crucial; it is the basis for all applications.

**Proposition 3.2** (Gabriel). Let \( X, Y_1, \ldots, Y_r \in \text{ind} \mathcal{A} \). Suppose that \( X \subseteq Y = \bigoplus_{i=1}^r Y_i \). Then \( \mu(X) \leq \max \mu(Y_i) \) and \( X \) is a direct summand of \( Y \) if \( \mu(X) = \max \mu(Y_i) \).

**Proof.** The proof only uses the properties (GR1) – (GR3) of \( \mu \). Fix a monomorphism \( \phi: X \to Y \). We proceed by induction on \( n = \ell_1(X) + \ell_1(Y) \). If \( n = 2 \), then \( \phi \) is an isomorphism and the assertion is clear. Now suppose \( n > 2 \). We can assume that for each \( i \) the \( i \)th component \( \phi_i: X \to Y_i \) of \( \phi \) is an epimorphism. Otherwise choose for each \( i \) a decomposition \( Y_i^\prime = \bigoplus_j Y_{ij} \) of the image of \( \phi_i \) into indecomposables. Then we use (GR1) and have \( \mu(X) \leq \max \mu(Y_{ij}) \leq \max \mu(Y_i) \) because \( \ell_1(X) + \ell_1(Y_i) < n \) and \( Y_{ij} \subseteq Y_i \) for all \( j \). Now suppose that each \( \phi_i \) is an epimorphism. Thus \( \ell(X) \geq \ell(Y_i) \) for all \( i \). Let \( X' \subset X \) be a proper indecomposable subobject. Then \( \mu(X') \leq \max \mu(Y_i) \) because \( \ell_1(X') + \ell_1(Y_i) < n \), and \( X' \) is a direct summand if \( \mu(X') = \max \mu(Y_i) \). We can exclude the case that \( \mu(X') = \max \mu(Y_i) \) because then \( X' \) is a proper direct summand of \( X \), which is impossible. Now we apply (GR3) and obtain \( \mu(X) \leq \max \mu(Y_i) \). Finally, suppose that \( \mu(X) = \max \mu(Y_i) = \mu(Y_k) \) for some \( k \). We claim that we can choose \( k \) such that \( \phi_k \) is an epimorphism. Otherwise, replace all \( Y_i \) with \( \mu(X) = \mu(Y_i) \) by the image \( Y_i^\prime = \bigoplus_j Y_{ij} \) of \( \phi_i \) as before. We obtain \( \mu(X) \leq \max \mu(Y_{ij}) < \mu(Y_k) \) since \( Y_{kj} \subset Y_k \) for all \( j \), using (GR1) and (GR2). This is a contradiction. Thus \( \phi_k \) is an epimorphism and in fact an isomorphism because \( \ell(X) = \ell(Y_k) \) by (GR2). In particular, \( X \) is a direct summand of \( \bigoplus Y_i \). This completes the proof.

**Gabriel-Roiter filtrations.** We keep a length function \( \ell \) on \( \mathcal{A} \) and the corresponding Gabriel-Roiter measure \( \mu = \ell^\ast \). Let \( X, Y \in \text{ind} \mathcal{A} \). We say that \( X \) is a Gabriel-Roiter predecessor of \( Y \) if \( X \subseteq Y \) and \( \mu(X) = \max_{Y' \subseteq Y} \mu(Y') \). Note that each object \( Y \in \text{ind} \mathcal{A} \) which is not simple admits a Gabriel-Roiter predecessor because \( \mu \) is a length function on \( \text{ind} \mathcal{A} \). A Gabriel-Roiter predecessor \( X \) of \( Y \) is usually not unique, but the value \( \mu(X) \) is determined by \( \mu(Y) \).

A sequence

\[
X_1 \subset X_2 \subset \ldots \subset X_{n-1} \subset X_n = X
\]

in \( \text{ind} \mathcal{A} \) is called a Gabriel-Roiter filtration of \( X \) if \( X_1 \) is simple and \( X_{i-1} \) is a Gabriel-Roiter predecessor of \( X_i \) for all \( 1 < i \leq n \). Clearly, each \( X \) admits such a filtration and the values \( \mu(X_i) \) are uniquely determined by \( X \). Note that (C0) implies

\[
(3.1) \quad \mu(X) = \{\ell(X_i) \mid 1 \leq i \leq n\}.
\]

**Injective objects.** In order to illustrate Gabriel’s main property, let us show that the Gabriel-Roiter measure detects injective objects. This is a remarkable fact because the Gabriel-Roiter measure is a combinatorial invariant, depending only on the poset of indecomposable objects and some length function, whereas the notion of injectivity involves all morphisms of the category \( \mathcal{A} \).

**Theorem 3.3.** An indecomposable object \( Q \) of \( \mathcal{A} \) is injective if and only if there is a length function \( \ell \) on \( \mathcal{A} \) such that \( \ell^\ast(X) \leq \ell^\ast(Q) \) for all \( X \in \text{ind} \mathcal{A} \).
We need the following lemma.

**Lemma 3.4.** Let \( \ell \) be a length function on \( \mathcal{A} \) and fix indecomposable objects \( X,Y \in \mathcal{A} \). Suppose that for each pair of simple subobjects \( X' \subseteq X \) and \( Y' \subseteq Y \), we have \( \ell(X') < \ell(Y') \). Then \( \ell^s(X) > \ell^s(Y) \).

**Proof.** We choose Gabriel-Roiter filtrations \( X_1 \subset \ldots \subset X_n = X \) and \( Y_1 \subset \ldots \subset Y_m = Y \). Then \( \ell(X_k) < \ell(Y_1) \) and the formula (3.1) implies

\[
\ell^s(X) = \{ \ell(X_i) \mid 1 \leq i \leq n \} > \{ \ell(Y_i) \mid 1 \leq i \leq m \} = \ell^s(Y).
\]

\( \square \)

**Proof of the theorem.** Suppose first that \( Q \) is injective. Then \( Q \) has a unique simple subobject \( S \) and we define a length function \( \ell = \ell_S \) on \( \mathcal{A} \) by specifying its values on each simple object \( T \in \mathcal{A} \) as follows:

\[
\ell(T) := \begin{cases} 
1 & \text{if } T \cong S, \\
2 & \text{if } T \not\cong S.
\end{cases}
\]

Now let \( X \in \text{ind}\mathcal{A} \). We claim that \( \ell^s(X) \leq \ell^s(Q) \). To see this, let \( X' \subseteq X \) be the maximal subobject of \( X \) having composition factors isomorphic to \( S \). Using induction on the composition length \( n = \ell_1(X') \) of \( X' \), one obtains a monomorphism \( X' \rightarrow Q^n \), and this extends to a map \( \phi: X \rightarrow Q^n \), since \( Q \) is injective. Let \( X/X' = \oplus_i Y_i \) be a decomposition into indecomposables and \( \pi: X \rightarrow X/X' \) be the canonical map. Note that \( \ell^s(Y_i) < \ell^s(Q) \) for all \( i \) by our construction and Lemma 3.4. Then \((\pi, \phi): X \rightarrow (\oplus_i Y_i) \oplus Q^n \) is a monomorphism and therefore \( \ell^s(X) \leq \ell^s(Q) \) by the main property.

Suppose now that \( \ell^s(X) \leq \ell^s(Q) \) for all \( X \in \text{ind}\mathcal{A} \) and some length function \( \ell \) on \( \mathcal{A} \). To show that \( Q \) is injective, suppose that \( Q \subseteq Y \) is the subobject of some \( Y \in \mathcal{A} \). Let \( Y = \oplus Y_i \) be a decomposition into indecomposables. Then the main property implies \( \ell^s(Q) \leq \max \ell^s(Y_i) \leq \ell^s(Q) \) and therefore \( Q \) is a direct summand of \( Y \). Thus \( Q \) is injective and the proof is complete. \( \square \)

Let us mention that there is the following analogous characterization of the simple objects of \( \mathcal{A} \).

**Corollary 3.5.** An indecomposable object \( S \) of \( \mathcal{A} \) is simple if and only if there is a length function \( \ell \) on \( \mathcal{A} \) such that \( \ell^s(S) \leq \ell^s(X) \) for all \( X \in \text{ind}\mathcal{A} \).

**Proof.** Use the property (GR1) of the Gabriel-Roiter measure and apply Lemma 3.4. \( \square \)

**The Kronecker algebra.** Let \( \Lambda = \left[ \begin{array}{cc} k & k^2 \\ 0 & k \end{array} \right] \) be the Kronecker algebra over an algebraically closed field \( k \). We consider the abelian length category which is formed by all finite dimensional \( \Lambda \)-modules. A complete list of indecomposable objects is given by the preprojectives \( P_n \), the regulars \( R_n(\alpha, \beta) \), and the preinjectives \( Q_n \). More precisely,

\[
\text{ind} \Lambda = \{ P_n \mid n \in \mathbb{N} \} \cup \{ R_n(\alpha, \beta) \mid n \in \mathbb{N}, (\alpha, \beta) \in \mathbb{P}_k^1 \} \cup \{ Q_n \mid n \in \mathbb{N} \},
\]
and we obtain the following Hasse diagram.

\[ \begin{array}{cccc}
7 & & & \\
6 & & & \\
5 & \cdots & & \\
4 & \cdots & & \\
3 & \cdots & & \\
2 & \cdots & & \\
1 & & & \ell \ P_n \ R_n(\alpha, \beta) \ Q_n \\
\end{array} \]

The set of indecomposables is ordered as follows via the Gabriel-Roiter measure with respect to $\ell = \ell_1$.

$\ell^*: Q_1 = P_1 < P_2 < P_3 < \ldots \ R_1 < R_2 < R_3 < \ldots \ < Q_4 < Q_3 < Q_2$

$(\ell^*)^*: Q_1 = P_1 < R_1 < Q_2 < P_2 < R_2 < Q_3 < P_3 < R_3 < Q_4 < \ldots$

Moreover, $((\ell^*)^*)^*$ and $\ell^*$ are equivalent length functions.

**Remark 3.6.** While $\ell^*$ has been successfully employed for proving the first Brauer-Thrall conjecture, Hubery points out that $(\ell^*)^*$ might be useful for proving the second. In fact, one needs to find a value $(\ell^*)^*(X)$ such that the set $\{X' \in \text{ind } \Lambda \mid (\ell^*)^*(X') = (\ell^*)^*(X)\}$ is infinite. The example of the Kronecker algebra shows that there exists such a value having only finitely many predecessors $(\ell^*)^*(Y) < (\ell^*)^*(X)$. Note that in all known examples $((\ell^*)^*)^*$ and $\ell^*$ are equivalent.

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