An introduction to the Ziegler spectrum of an algebra

Henning Krause

Universität Paderborn

Infinite dimensional analysis and representation theory
Universität Bielefeld
December 10 – 14, 2007

Motivation

The Ziegler spectrum of an algebra ...

- is a topological space
- was introduced by M. Ziegler [Model theory of modules, 1984]
- captures the structure of all representations in a single topological space
- helps to study the interplay between finite and infinite dimensional representations

Further contributions by:
Crawley-Boevey, Herzog, K., Prest, Ringel . . .

The setup

A = an associative k-algebra over a field k

$\operatorname{Mod} A$ = the category of (right) A-modules

$M \mapsto M^* = \operatorname{Hom}_k(M, k)$ *duality* between right and left A-modules

The Ziegler spectrum

Definition

The *Ziegler spectrum* $\operatorname{Zsp} A$ of an algebra A is a topological space.

- The *points* are the isomorphism classes of indecomposable pure-injective A-modules.
- The *topology* is defined in terms of pp-formulas (or finite matrices) over A.

M is *indecomposable* if $M = M_1 \oplus M_2$ implies $M_1 = 0$ or $M_2 = 0$.

Pure injective modules

Definition and characterization

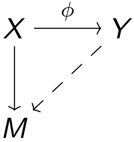
- A monomorphism $M \to N$ is *pure* if it induces a mono $M \otimes_A X \to N \otimes_A X$ for every left A-module X.
- A module M is *pure injective* if every pure mono splits.
- The evaluation map $M \to M^{**}$ is a pure mono.
- M is pure injective iff the evaluation map $M \to M^{**}$ splits.

Examples

- Every injective module is pure injective.
- Every finite dimensional module is pure injective.
- For $A = k[t]$ the free module $M = k[t]$ is not pure injective.
- Products of pure injective modules are pure injective.

Definable classes

Let $\phi\colon X \to Y$ be a map between finitely presented modules. A module M is *ϕ-injective* if every map $X \to M$ factors through ϕ:



Recall: finitely presented = cokernel of a map $A^n \to A^m$.

For a set Φ of maps between finitely presented modules let

$$\operatorname{Inj}\Phi := \{M \in \operatorname{Mod} A \mid M \text{ is } \phi\text{-injective for all } \phi \in \Phi\}.$$

A class $\mathcal{C} \subseteq \operatorname{Mod} A$ is *definable* if $\mathcal{C} = \operatorname{Inj}\Phi$ for some Φ.

Definable classes

Proposition (Crawley-Boevey)

A class $\mathcal{C} \subseteq \operatorname{Mod} A$ *is definable if and only if* $\mathcal{C}$ *is closed under*

- *pure submodules,*
- *direct limits,*
- *products.*

Examples

- Given a finite dimensional indecomposable module M, all direct sums of copies of M form a definable class.
- The injective A-modules form a definable class iff the algebra A is right noetherian.

The Ziegler topology

For a map $\phi\colon X \to Y$ between finitely presented modules define

$$V_\phi := \{M \in \operatorname{Zsp} A \mid M \text{ is } \phi\text{-injective}\}.$$

Lemma

The sets V_ϕ form a basis of closed sets for a topology on $\operatorname{Zsp} A$. The space $\operatorname{Zsp} A$ is quasi-compact.

The fundamental correspondence

Theorem (Ziegler)

The assignments

$$\begin{aligned} \mathcal{C} &\mapsto \mathcal{C} \cap \operatorname{Zsp} A \\ V &\mapsto \{\textit{pure submodules of products of modules in } V\} \end{aligned}$$

induce bijections between

- *definable classes of* Mod A, *and*
- *cosed subsets of* Zsp A.

The Ziegler spectrum of a Dedekind domain

Take as example the ring of integers $\mathbb{Z}$.

$\operatorname{Zsp}\mathbb{Z} = \{\mathbb{Z}/p^n\mathbb{Z} \mid p \text{ prime}, n \geq 1\} \cup \{\mathbb{Z}_{p^\infty}, \mathbb{Z}_p \mid p \text{ prime}\} \cup \{\mathbb{Q}\}$

$\mathbb{Z}_{p^\infty} = \bigcup_{n\geq 1} \mathbb{Z}/p^n\mathbb{Z}$ Prüfer group

$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ p-adic integers

$\{\mathbb{Z}/p^n\mathbb{Z}\}$ is open and closed

$\overline{\{\mathbb{Z}/p^n\mathbb{Z} \mid n \geq 1\}} = \{\mathbb{Z}/p^n\mathbb{Z} \mid n \geq 1\} \cup \{\mathbb{Z}_{p^\infty}, \mathbb{Z}_p, \mathbb{Q}\}$

$\overline{\{\mathbb{Z}_{p^\infty}\}} = \{\mathbb{Z}_{p^\infty}, \mathbb{Q}\}$ and $\overline{\{\mathbb{Z}_p\}} = \{\mathbb{Z}_p, \mathbb{Q}\}$

$\overline{\{\mathbb{Q}\}} = \{\mathbb{Q}\}$

Cantor-Bendixson analysis

Definition

For a topological space X the *Cantor-Bendixson filtration*

$$X = X_{-1} \supseteq X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$$

is defined by

$$X_n = \{x \in X_{n-1} \mid \{x\} \text{ is not open in } X_{n-1}\} \text{ for } n \geq 0.$$

The *Cantor-Bendixson rank* is

$$\operatorname{rank} X = \min\{n \geq 0 \mid X_n = \emptyset\}.$$

Example

$\operatorname{rank} \operatorname{Zsp} \mathbb{Z} = 2.$

Ziegler versus Zariski topology

$A =$ a commutative Noetherian ring

$\operatorname{Spec} A =$ the set of prime ideals with the Zariski topology

The map

$$\operatorname{Spec} A \longrightarrow \operatorname{Zsp} A, \quad \mathfrak{p} \mapsto E(A/\mathfrak{p}) = \text{ inj. envelope of } A/\mathfrak{p},$$

yields identifications:

$$\operatorname{Spec} A \longleftrightarrow \{M \in \operatorname{Zsp} A \mid M \text{ injective}\}$$

$$\text{quasi-compact Zariski open} \;\longleftrightarrow\; \text{basic Ziegler closed}$$

The Ziegler spectrum of a finite dimensional algebra

Some facts

- $\{M\}$ is open iff M is finite dimensional (follows from Auslander-Reiten theory).
- The finite dimensional points form a dense subset.
- $\operatorname{rank}\operatorname{Zsp} A = 0$ iff A is of finite representation type.
- $\operatorname{rank}\operatorname{Zsp} A \neq 1$ for all A.
- $\operatorname{rank}\operatorname{Zsp} A = 2$ for $k[x,y]/(x^2,y^2)$ and $\left[\begin{smallmatrix} k & k^2 \\ 0 & k \end{smallmatrix}\right]$.

Endofinite modules

Definition

A module M is *endofinite* if it is of finite length over $\operatorname{End}_A(M)$.
Let $\ell_{\mathrm{end}}(M) :=$ length of M over $\operatorname{End}_A(M)$.

Some facts

- $\ell_{\mathrm{end}}(M) \leq \dim_k(M)$.
- Every endofinite module is pure injective and decomposes uniquely into indecomposable modules.
- $\{M\}$ is closed for every endofinite $M \in \operatorname{Zsp} A$.

Generic modules

Theorem (Crawley-Boevey)

For a finite dimensional algebra over an infinite field are equivalent:

(1) *There exists an infinite dimensional and endofinite* $G \in \operatorname{Zsp} A$.

(2) $\{M \in \operatorname{Zsp} A \mid \dim_k(M) \leq n\}$ *is infinite for some* $n \in \mathbb{N}$.

Proof.

(2) $\Rightarrow$ (1): The Ziegler closure of $\{M \in \operatorname{Zsp} A \mid \dim_k(M) \leq n\}$ consists of endofinite modules of endolength $\leq n$ and is quasi-compact. Thus there exists an endofinite G which is not isolated and therefore not finite dimensional. □

Brauer-Thrall II

Conjecture

Every algebra of infinite representation type has an infinite dimensional and endofinite indecomposable module.

Oscar Wilde on Purity

A quote from 'A woman of no importance'

LADY S: And what have you been writing about this morning, Mr. K?
K: On the usual subject, Lady S. On Purity.
LADY S: That must be such a very, very interesting thing to write about.
K: It is the one subject of really national importance, nowadays, Lady S.