

# THE AUSLANDER-REITEN FORMULA FOR COMPLEXES OF MODULES

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ABSTRACT. An Auslander-Reiten formula for complexes of modules is presented. This formula contains as a special case the classical Auslander Reiten formula. The Auslander-Reiten translate of a complex is described explicitly, and various applications are discussed.

## 1. INTRODUCTION

The classical Auslander-Reiten formula for modules over a noetherian algebra  $\Lambda$  says that

$$D \operatorname{Ext}_{\Lambda}^1(M, N) \cong \overline{\operatorname{Hom}}_{\Lambda}(N, D \operatorname{Tr} M)$$

whenever  $M$  is finitely generated [2]. Here,  $D$  denotes the duality over a fixed commutative ground ring  $k$  (see Section 2) and  $\operatorname{Tr}$  denotes the transpose construction (see Section 4). In this paper, we extend this to a formula for complexes of modules. We take as morphisms chain maps up to homotopy and obtain the formula for  $D \operatorname{Ext}_{\Lambda}^1(M, N)$  as a special case by applying it to injective resolutions  $\mathbf{i}M$  and  $\mathbf{i}N$ .

Various authors noticed the analogy between the Auslander-Reiten formula and Serre duality for categories of sheaves; see for instance [18, 8], and see [6] for a formulation of Serre duality in terms of extension groups. Passing from abelian categories to their derived categories, further analogies have been noticed, in particular in connection with the existence of Auslander-Reiten triangles [13, 17].

There is the following common setting for proving such duality formulas. Let  $\mathcal{T}$  be a  $k$ -linear triangulated category which is compactly generated. Then one can apply Brown's representability theorem and has for any compact object  $X$  a representing object  $\mathbf{t}X$  such that

$$D \operatorname{Hom}_{\mathcal{T}}(X, -) \cong \operatorname{Hom}_{\mathcal{T}}(-, \mathbf{t}X).$$

In this paper, we take for  $\mathcal{T}$  the category  $\mathbf{K}(\operatorname{Inj} \Lambda)$  of complexes of injective  $\Lambda$ -modules up to homotopy. Then we can prove that  $\mathbf{t}X = \mathbf{p}X \otimes_{\Lambda} D\Lambda$ , where  $\mathbf{p}X$  denotes the projective resolution of  $X$ .

There is a good reason to consider the category of complexes  $\mathbf{K}(\operatorname{Inj} \Lambda)$ . The injective resolutions of all finitely generated modules generate the full subcategory of compact objects, which therefore is equivalent to the bounded derived category  $\mathbf{D}^b(\operatorname{mod} \Lambda)$  of the category  $\operatorname{mod} \Lambda$  of finitely generated  $\Lambda$ -modules.

Our identification of the translation  $\mathbf{t}$  has various interesting consequences. For instance, we can describe Auslander-Reiten triangles in  $\mathbf{K}(\operatorname{Inj} \Lambda)$ , and we get a simple recipe for computing almost split sequences in the category  $\operatorname{Mod} \Lambda$  of  $\Lambda$ -modules which seems to be new.

There is another method for computing Auslander-Reiten triangles in  $\mathbf{K}(\text{Inj } \Lambda)$ . This is based on the construction of an adjoint for Happel's functor

$$\mathbf{D}^b(\text{mod } \Lambda) \longrightarrow \underline{\text{mod}} \hat{\Lambda}$$

into the stable module category of the repetitive algebra  $\hat{\Lambda}$  [9]. To be precise, we extend Happel's functor to a functor  $\mathbf{K}(\text{Inj } \Lambda) \rightarrow \underline{\text{Mod}} \hat{\Lambda}$  on unbounded complexes, and this admits a right adjoint which preserves Auslander-Reiten triangles.

The Auslander-Reiten theory for complexes has been initiated by Happel. In [9, 10], he introduced Auslander-Reiten triangles and characterized their existence in the derived category  $\mathbf{D}^b(\text{mod } \Lambda)$ . This pioneering work has been extended by various authors. More recently, Bautista et al. established in [4] the existence of almost split sequences in some categories of complexes of fixed size.

The methods in this paper might be of interest not only for studying module categories. In fact, they can be applied to other more general settings. To illustrate this point, we include an Auslander-Reiten formula for computing  $\text{Ext}_{\mathcal{A}}^1(-, -)$  in any locally noetherian Grothendieck category  $\mathcal{A}$ .

## 2. THE HOMOTOPY CATEGORY OF INJECTIVES

Let  $k$  be a commutative noetherian ring which is complete and local. Throughout this paper, we fix a noetherian  $k$ -algebra  $\Lambda$ , that is, a  $k$ -algebra which is finitely generated as a module over  $k$ .

We consider the category  $\text{Mod } \Lambda$  of (right)  $\Lambda$ -modules and the following full subcategories:

- $\text{mod } \Lambda$  = the finitely presented  $\Lambda$ -modules,
- $\text{Inj } \Lambda$  = the injective  $\Lambda$ -modules,
- $\text{Proj } \Lambda$  = the projective  $\Lambda$ -modules,
- $\text{proj } \Lambda$  = the finitely generated projective  $\Lambda$ -modules.

Note that the assumptions on  $\Lambda$  imply that every finitely generated  $\Lambda$ -module decomposes essentially uniquely into a finite coproduct of indecomposable modules with local endomorphism rings.

In addition, we fix an injective envelope  $E = E(k/\mathfrak{m})$ , where  $\mathfrak{m}$  denotes the unique maximal ideal of  $k$ . We obtain a functor

$$D = \text{Hom}_k(-, E): \text{Mod } k \longrightarrow \text{Mod } k$$

which induces a functor between  $\text{Mod } \Lambda$  and  $\text{Mod } \Lambda^{\text{op}}$ . Note that  $D$  induces a duality between the full subcategories formed by all reflexive modules, which contains all finitely generated modules by our assumption on  $k$ . Recall that a module  $M$  is *reflexive* if the canonical map  $M \rightarrow D^2M$  is an isomorphism.

Given any additive category  $\mathcal{A}$ , we denote by  $\mathbf{C}(\mathcal{A})$  the category of cochain complexes in  $\mathcal{A}$ , and we write  $\mathbf{K}(\mathcal{A})$  for the category of cochain complexes up to homotopy. If  $\mathcal{A}$  is abelian, the derived category is denoted by  $\mathbf{D}(\mathcal{A})$ . We refer to [20] for further notation and terminology concerning categories of complexes.

We denote by

$$\mathbf{p}: \mathbf{D}(\text{Mod } \Lambda) \longrightarrow \mathbf{K}(\text{Proj } \Lambda)$$

the left adjoint of the composite

$$\mathbf{K}(\text{Proj } \Lambda) \xrightarrow{\text{inc}} \mathbf{K}(\text{Mod } \Lambda) \xrightarrow{\text{can}} \mathbf{D}(\text{Mod } \Lambda)$$

which sends a complex  $X$  to its *projective resolution*  $\mathbf{p}X$ . Dually, we denote by

$$\mathbf{i}: \mathbf{D}(\text{Mod } \Lambda) \longrightarrow \mathbf{K}(\text{Inj } \Lambda)$$

the right adjoint of the composite

$$\mathbf{K}(\text{Inj } \Lambda) \xrightarrow{\text{inc}} \mathbf{K}(\text{Mod } \Lambda) \xrightarrow{\text{can}} \mathbf{D}(\text{Mod } \Lambda)$$

which sends a complex  $X$  to its *injective resolution*  $\mathbf{i}X$ . For the existence of  $\mathbf{p}$  and  $\mathbf{i}$ , see [19, 5].

We shall work in the category  $\mathbf{K}(\text{Inj } \Lambda)$ . This is a triangulated category with arbitrary coproducts. We denote by  $\mathbf{K}^c(\text{Inj } \Lambda)$  the full subcategory which is formed by all compact objects. Recall that an object  $X$  is *compact* if every map  $X \rightarrow \coprod_{i \in I} Y_i$  factors through  $\coprod_{i \in J} Y_i$  for some finite  $J \subseteq I$ . Let us collect from [14] the basic properties of  $\mathbf{K}(\text{Inj } \Lambda)$ .

**Lemma 2.1.** *An object in  $\mathbf{K}(\text{Inj } \Lambda)$  is compact if and only if it is isomorphic to a complex  $X$  satisfying*

- (1)  $X^n = 0$  for  $n \ll 0$ ,
- (2)  $H^n X$  is finitely generated over  $\Lambda$  for all  $n$ , and
- (3)  $H^n X = 0$  for  $n \gg 0$ .

**Lemma 2.2.** *The canonical functor  $\mathbf{K}(\text{Inj } \Lambda) \rightarrow \mathbf{D}(\text{Mod } \Lambda)$  induces an equivalence*

$$\mathbf{K}^c(\text{Inj } \Lambda) \longrightarrow \mathbf{D}^b(\text{mod } \Lambda).$$

**Lemma 2.3.** *The triangulated category  $\mathbf{K}(\text{Inj } \Lambda)$  is compactly generated, that is, it coincides with the smallest full triangulated subcategory closed under all coproducts and containing all compact objects.*

### 3. THE AUSLANDER-REITEN FORMULA

In this section, the Auslander-Reiten formula for complexes is proved. We begin with a number of simple lemmas. Given a pair of complexes  $X, Y$  of modules over  $\Lambda$  or  $\Lambda^{\text{op}}$ , we denote by  $\text{Hom}_\Lambda(X, Y)$  and  $X \otimes_\Lambda Y$  the total Hom and the total tensor product respectively, which are complexes of  $k$ -modules.

**Lemma 3.1.** *Let  $X, Y$  be complexes in  $\mathbf{C}(\text{Mod } \Lambda)$ . Then we have in  $\mathbf{C}(\text{Mod } k)$  a natural map*

$$(3.1) \quad Y \otimes_\Lambda \text{Hom}_\Lambda(X, \Lambda) \longrightarrow \text{Hom}_\Lambda(X, Y),$$

*which is an isomorphism if  $X \in \mathbf{C}^-(\text{proj } \Lambda)$  and  $Y \in \mathbf{C}^+(\text{Mod } \Lambda)$ .*

*Proof.* Given  $\Lambda$ -modules  $M$  and  $N$ , we have a map

$$\sigma: N \otimes_\Lambda \text{Hom}_\Lambda(M, \Lambda) \longrightarrow \text{Hom}_\Lambda(M, N)$$

which is defined by

$$\sigma(n \otimes \phi)(m) = n\phi(m).$$

This map is an isomorphism if  $M$  is finitely generated projective and extends to an isomorphism of complexes provided  $X$  and  $Y$  are bounded in the appropriate direction.  $\square$

**Lemma 3.2.** *Let  $M, N$  be  $\Lambda$ -modules and suppose that  $M$  is finitely presented. Then we have an isomorphism*

$$(3.2) \quad M \otimes_\Lambda \text{Hom}_k(N, E) \cong \text{Hom}_k(\text{Hom}_\Lambda(M, N), E).$$

*Proof.* We have the isomorphism for  $M = \Lambda$  and therefore whenever  $M$  has a presentation  $\Lambda^n \rightarrow \Lambda^m \rightarrow M \rightarrow 0$ , since  $-\otimes_{\Lambda} \mathrm{Hom}_k(N, E)$  and  $\mathrm{Hom}_k(\mathrm{Hom}_{\Lambda}(-, N), E)$  are both right exact.  $\square$

**Lemma 3.3.** *Let  $X, Y$  be complexes of  $\Lambda$ -modules. Then we have an isomorphism*

$$(3.3) \quad H^0 \mathrm{Hom}_{\Lambda}(X, Y) \cong \mathrm{Hom}_{\mathbf{K}(\mathrm{Mod} \Lambda)}(X, Y).$$

*Proof.* The cycles of  $\mathrm{Hom}_{\Lambda}(X, Y)$  in degree zero are precisely the chain maps  $X \rightarrow Y$ , and boundaries in degree zero form the subgroup of null-homotopic chain maps. Thus  $H^0 \mathrm{Hom}_{\Lambda}(X, Y)$  equals the set of chain maps  $X \rightarrow Y$  up to homotopy.  $\square$

Let us consider the following commutative diagram

$$\begin{array}{ccccc} & & \mathbf{D}^-(\mathrm{Mod} \Lambda) & \xleftarrow{\sim} & \mathbf{K}^-(\mathrm{Proj} \Lambda) \\ & & \uparrow & & \uparrow \\ \mathbf{K}^c(\mathrm{Inj} \Lambda) & \xrightarrow{\sim} & \mathbf{D}^b(\mathrm{mod} \Lambda) & \xleftarrow{\sim} & \mathbf{K}^{-,b}(\mathrm{proj} \Lambda) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{K}^+(\mathrm{Inj} \Lambda) & \xrightarrow{\sim} & \mathbf{D}^+(\mathrm{Mod} \Lambda) & & \end{array}$$

in which all horizontal functors are obtained by restricting the localization functor  $\mathbf{K}(\mathrm{Mod} \Lambda) \rightarrow \mathbf{D}(\mathrm{Mod} \Lambda)$  to appropriate subcategories. We denote by

$$\pi: \mathbf{K}^c(\mathrm{Inj} \Lambda) \longrightarrow \mathbf{K}^{-,b}(\mathrm{proj} \Lambda)$$

the composite of the equivalence  $\mathbf{K}^c(\mathrm{Inj} \Lambda) \rightarrow \mathbf{D}^b(\mathrm{mod} \Lambda)$  with a quasi-inverse of the equivalence  $\mathbf{K}^{-,b}(\mathrm{proj} \Lambda) \rightarrow \mathbf{D}^b(\mathrm{mod} \Lambda)$ . Note that  $\pi X \cong \mathbf{p}X$ .

**Theorem 3.4.** *Let  $X$  and  $Y$  be complexes of injective  $\Lambda$ -modules. Suppose that  $X^n = 0$  for  $n \ll 0$ , that  $H^n X$  is finitely generated over  $\Lambda$  for all  $n$ , and that  $H^n X = 0$  for  $n \gg 0$ . Then we have an isomorphism*

$$(3.4) \quad D \mathrm{Hom}_{\mathbf{K}(\mathrm{Inj} \Lambda)}(X, Y) \cong \mathrm{Hom}_{\mathbf{K}(\mathrm{Inj} \Lambda)}(Y, (\pi X) \otimes_{\Lambda} D\Lambda)$$

which is natural in  $X$  and  $Y$ .

*Proof.* We use the fact that  $\mathbf{K}(\mathrm{Inj} \Lambda)$  is compactly generated. Therefore it is sufficient to verify the isomorphism for every compact object  $Y$ . This follows from the subsequent Lemma 3.5. Thus we suppose that  $Y$  is a compact object in  $\mathbf{K}(\mathrm{Inj} \Lambda)$ . Note that this implies  $Y^n = 0$  for  $n \ll 0$ , and in particular  $Y \cong \mathbf{i}Y$ . We obtain the following sequence

of isomorphisms, where short arguments are added on the right hand side.

$$\begin{aligned}
D \operatorname{Hom}_{\mathbf{K}(\operatorname{Inj} \Lambda)}(X, Y) &\cong \operatorname{Hom}_k(\operatorname{Hom}_{\mathbf{K}(\operatorname{Inj} \Lambda)}(X, \mathbf{i}Y), E) && Y \text{ compact} \\
&\cong \operatorname{Hom}_k(\operatorname{Hom}_{\mathbf{D}(\operatorname{Mod} \Lambda)}(X, Y), E) && \text{adjunction} \\
&\cong \operatorname{Hom}_k(\operatorname{Hom}_{\mathbf{K}(\operatorname{Mod} \Lambda)}(\pi X, Y), E) && \text{adjunction} \\
&\cong \operatorname{Hom}_k(H^0 \operatorname{Hom}_{\Lambda}(\pi X, Y), E) && \text{from (3.3)} \\
&\cong H^0 \operatorname{Hom}_k(\operatorname{Hom}_{\Lambda}(\pi X, Y), E) && E \text{ injective} \\
&\cong H^0 \operatorname{Hom}_k(Y \otimes_{\Lambda} \operatorname{Hom}_{\Lambda}(\pi X, \Lambda), E) && \text{from (3.1)} \\
&\cong H^0 \operatorname{Hom}_{\Lambda}(Y, \operatorname{Hom}_k(\operatorname{Hom}_{\Lambda}(\pi X, \Lambda), E)) && \text{adjunction} \\
&\cong H^0 \operatorname{Hom}_{\Lambda}(Y, (\pi X) \otimes_{\Lambda} \operatorname{Hom}_k(\Lambda, E)) && \text{from (3.2)} \\
&\cong \operatorname{Hom}_{\mathbf{K}(\operatorname{Inj} \Lambda)}(Y, (\pi X) \otimes_{\Lambda} D\Lambda) && \text{from (3.3)}
\end{aligned}$$

This isomorphism completes the proof.  $\square$

**Lemma 3.5.** *Let  $X, X'$  be objects in a  $k$ -linear compactly generated triangulated category. Suppose that  $X$  is compact. If there is a natural isomorphism*

$$D \operatorname{Hom}_{\mathcal{T}}(X, Y) \cong \operatorname{Hom}_{\mathcal{T}}(Y, X')$$

for all compact  $Y \in \mathcal{T}$ , then  $D \operatorname{Hom}_{\mathcal{T}}(X, -) \cong \operatorname{Hom}_{\mathcal{T}}(-, X')$ .

*Proof.* We shall use Theorem 1.8 in [12], which states the following equivalent conditions for an object  $Y$  in  $\mathcal{T}$ .

- (1) The object  $Y$  is pure-injective.
- (2) The object  $H_Y = \operatorname{Hom}_{\mathcal{T}}(-, Y)|_{\mathcal{T}^c}$  is injective in the category  $\operatorname{Mod} \mathcal{T}^c$  of contravariant additive functors  $\mathcal{T}^c \rightarrow \operatorname{Ab}$ .
- (3) The map  $\operatorname{Hom}_{\mathcal{T}}(Y', Y) \rightarrow \operatorname{Hom}_{\mathcal{T}^c}(H_{Y'}, H_Y)$  sending  $\phi$  to  $H_{\phi}$  is bijective for all  $Y'$  in  $\mathcal{T}$ .

Here,  $\mathcal{T}^c$  denotes the full subcategory of compact objects in  $\mathcal{T}$ .

We apply Brown's representability theorem (see [11, 5.2] or [15, Theorem 3.1]) and obtain an object  $X''$  such that

$$D \operatorname{Hom}_{\mathcal{T}}(X, -) \cong \operatorname{Hom}_{\mathcal{T}}(-, X''),$$

since  $X$  is compact. Condition (2) implies that both objects  $X'$  and  $X''$  are pure-injective, since  $\operatorname{Hom}_{\mathcal{T}^c}(X, -)$  is a projective object in the category of covariant additive functors  $\mathcal{T}^c \rightarrow \operatorname{Ab}$ , by Yoneda's lemma. We have an isomorphism

$$H_{X'} = \operatorname{Hom}_{\mathcal{T}}(-, X')|_{\mathcal{T}^c} \cong D \operatorname{Hom}_{\mathcal{T}}(X, -)|_{\mathcal{T}^c} \cong \operatorname{Hom}_{\mathcal{T}}(-, X'')|_{\mathcal{T}^c} = H_{X''},$$

and (3) implies that this isomorphism is induced by an isomorphism  $X' \rightarrow X''$  in  $\mathcal{T}$ . We conclude that

$$D \operatorname{Hom}_{\mathcal{T}}(X, -) \cong \operatorname{Hom}_{\mathcal{T}}(-, X').$$

$\square$

## 4. THE AUSLANDER-REITEN TRANSLATION

In this section, we investigate the properties of the Auslander-Reiten translation for complexes of  $\Lambda$ -modules. The Auslander-Reiten translation  $D \operatorname{Tr}$  for modules is obtained from the translation for complexes. In particular, we deduce the classical Auslander-Reiten formula.

**Proposition 4.1.** *The functor*

$$\mathbf{t}: \mathbf{K}(\operatorname{Inj} \Lambda) \xrightarrow{\operatorname{can}} \mathbf{D}(\operatorname{Mod} \Lambda) \xrightarrow{\mathbf{P}} \mathbf{K}(\operatorname{Proj} \Lambda) \xrightarrow{-\otimes_{\Lambda} D\Lambda} \mathbf{K}(\operatorname{Inj} \Lambda)$$

has the following properties.

- (1)  $\mathbf{t}$  is exact and preserves all coproducts.
- (2) For compact objects  $X, Y$  in  $\mathbf{K}(\operatorname{Inj} \Lambda)$ , the natural map

$$\operatorname{Hom}_{\mathbf{K}(\operatorname{Inj} \Lambda)}(X, Y) \longrightarrow \operatorname{Hom}_{\mathbf{K}(\operatorname{Inj} \Lambda)}(\mathbf{t}X, \mathbf{t}Y)$$

is bijective.

- (3) For  $X, Y$  in  $\mathbf{K}(\operatorname{Inj} \Lambda)$  with  $X$  compact, there is a natural isomorphism

$$D \operatorname{Hom}_{\mathbf{K}(\operatorname{Inj} \Lambda)}(X, Y) \cong \operatorname{Hom}_{\mathbf{K}(\operatorname{Inj} \Lambda)}(Y, \mathbf{t}X).$$

- (4)  $\mathbf{t}$  admits a right adjoint which is  $\mathbf{i} \operatorname{Hom}_{\Lambda}(D\Lambda, -)$ .

*Proof.* (1) is clear and (3) follows from (3.4). Now observe that for each pair  $X, Y$  of compact objects, the  $k$ -module  $\operatorname{Hom}_{\mathbf{K}(\operatorname{Inj} \Lambda)}(X, Y)$  is finitely generated. Therefore (2) follows from (3), since we have the isomorphism

$$\begin{aligned} \operatorname{Hom}_{\mathbf{K}(\operatorname{Inj} \Lambda)}(X, Y) &\cong D^2 \operatorname{Hom}_{\mathbf{K}(\operatorname{Inj} \Lambda)}(X, Y) \\ &\cong D \operatorname{Hom}_{\mathbf{K}(\operatorname{Inj} \Lambda)}(Y, \mathbf{t}X) \\ &\cong \operatorname{Hom}_{\mathbf{K}(\operatorname{Inj} \Lambda)}(\mathbf{t}X, \mathbf{t}Y). \end{aligned}$$

To prove (4), let  $X, Y$  be objects in  $\mathbf{K}(\operatorname{Inj} \Lambda)$ . Then we have

$$\begin{aligned} \operatorname{Hom}_{\mathbf{K}(\operatorname{Mod} \Lambda)}(\mathbf{p}X \otimes_{\Lambda} D\Lambda, Y) &\cong \operatorname{Hom}_{\mathbf{K}(\operatorname{Mod} \Lambda)}(\mathbf{p}X, \operatorname{Hom}_{\Lambda}(D\Lambda, Y)) \\ &\cong \operatorname{Hom}_{\mathbf{D}(\operatorname{Mod} \Lambda)}(X, \operatorname{Hom}_{\Lambda}(D\Lambda, Y)) \\ &\cong \operatorname{Hom}_{\mathbf{K}(\operatorname{Mod} \Lambda)}(X, \mathbf{i} \operatorname{Hom}_{\Lambda}(D\Lambda, Y)). \end{aligned}$$

Thus  $\mathbf{t}$  and  $\mathbf{i} \operatorname{Hom}_{\Lambda}(D\Lambda, -)$  form an adjoint pair.  $\square$

Let us continue with some definitions. We denote by  $\underline{\operatorname{Mod}} \Lambda$  the *stable module category* modulo projectives which is obtained by forming for each pair of  $\Lambda$ -modules  $M$  and  $N$  the quotient

$$\underline{\operatorname{Hom}}_{\Lambda}(M, N) = \operatorname{Hom}_{\Lambda}(M, N) / \{M \rightarrow P \rightarrow N \mid P \text{ projective}\}.$$

Analogously, the stable module category  $\overline{\operatorname{Mod}} \Lambda$  modulo injectives is defined.

Recall that a  $\Lambda$ -module  $M$  is *finitely presented* if it admits a projective presentation

$$P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

such that  $P_0$  and  $P_1$  are finitely generated. The *transpose*  $\operatorname{Tr} M$  relative to this presentation is the  $\Lambda^{\operatorname{op}}$ -module which is defined by the exactness of the induced sequence

$$\operatorname{Hom}_{\Lambda}(P_0, \Lambda) \rightarrow \operatorname{Hom}_{\Lambda}(P_1, \Lambda) \rightarrow \operatorname{Tr} M \rightarrow 0.$$

Note that the presentation of  $M$  is minimal if and only if the corresponding presentation of  $\text{Tr } M$  is minimal. The construction of the transpose is natural up to maps factoring through a projective and induces a duality  $\underline{\text{mod}} \Lambda \rightarrow \overline{\text{mod}} \Lambda^{\text{op}}$ .

**Proposition 4.2.** *The functor*

$$\mathbf{a}: \text{Mod } \Lambda \xrightarrow{\text{inc}} \mathbf{D}(\text{Mod } \Lambda) \xrightarrow{\mathbf{i}} \mathbf{K}(\text{Inj } \Lambda) \xrightarrow{\mathbf{t}} \mathbf{K}(\text{Inj } \Lambda) \xrightarrow{Z^{-1}} \overline{\text{Mod}} \Lambda$$

has the following properties.

- (1)  $\mathbf{a}M \cong D \text{Tr } M$  for every finitely presented  $\Lambda$ -module  $M$ .
- (2)  $\mathbf{a}$  preserves all coproducts.
- (3)  $\mathbf{a}$  annihilates all projective  $\Lambda$ -modules and induces a functor  $\underline{\text{Mod}} \Lambda \rightarrow \overline{\text{Mod}} \Lambda$ .
- (4) Each exact sequence  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  of  $\Lambda$ -modules induces a sequence

$$0 \rightarrow \mathbf{a}L \rightarrow \mathbf{a}M \rightarrow \mathbf{a}N \rightarrow L \otimes_{\Lambda} D\Lambda \rightarrow M \otimes_{\Lambda} D\Lambda \rightarrow N \otimes_{\Lambda} D\Lambda \rightarrow 0$$

of  $\Lambda$ -modules which is exact.

*Proof.* (1) The functor  $\mathbf{t}$  sends an injective resolution  $\mathbf{i}M$  of  $M$  to  $\mathbf{p}M \otimes_{\Lambda} D\Lambda$ . Using (3.2), we have

$$\mathbf{p}M \otimes_{\Lambda} D\Lambda \cong D \text{Hom}_{\Lambda}(\mathbf{p}M, \Lambda).$$

This implies

$$(4.1) \quad Z^{-1}(\mathbf{p}M \otimes_{\Lambda} D\Lambda) \cong D \text{Tr } M.$$

(2) First observe that  $\coprod_i (\mathbf{i}M_i) \cong \mathbf{i}(\coprod_i M_i)$  for every family of  $\Lambda$ -modules  $M_i$ . Clearly,  $\mathbf{t}$  and  $Z^{-1}$  preserve coproducts. Thus  $\mathbf{a}$  preserves coproducts.

(3) We have  $\mathbf{t}(\mathbf{i}\Lambda) = D\Lambda$  and therefore  $\mathbf{a}\Lambda = 0$  in  $\overline{\text{Mod}} \Lambda$ . Thus  $\mathbf{a}$  annihilates all projectives since it preserves coproducts.

(4) An exact sequence  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  induces an exact triangle  $\mathbf{p}L \rightarrow \mathbf{p}M \rightarrow \mathbf{p}N \rightarrow (\mathbf{p}L)[1]$ . This triangle can be represented by a sequence  $0 \rightarrow \mathbf{p}L \rightarrow \mathbf{p}M \rightarrow \mathbf{p}N \rightarrow 0$  of complexes which is split exact in each degree. Now apply  $- \otimes_{\Lambda} D\Lambda$  and use the snake lemma.  $\square$

We are now in the position to deduce the classical Auslander-Reiten formula for modules [2] from the formula for complexes.

**Corollary 4.3** (Auslander/Reiten). *Let  $M$  and  $N$  be  $\Lambda$ -modules and suppose that  $M$  is finitely presented. Then we have an isomorphism*

$$(4.2) \quad D \text{Ext}_{\Lambda}^1(M, N) \cong \overline{\text{Hom}}_{\Lambda}(N, D \text{Tr } M).$$

*Proof.* Let  $\mathbf{i}M$  and  $\mathbf{i}N$  be injective resolutions of  $M$  and  $N$ , respectively. We apply the Auslander-Reiten formula (3.4) and the formula (4.1) for the Auslander-Reiten translate. Thus we have

$$D \text{Ext}_{\Lambda}^1(M, N) \cong D \text{Hom}_{\mathbf{K}(\text{Inj } \Lambda)}(\mathbf{i}M, (\mathbf{i}N)[1]) \cong \text{Hom}_{\mathbf{K}(\text{Inj } \Lambda)}(\mathbf{i}N, (\mathbf{p}M \otimes_{\Lambda} D\Lambda)[-1]),$$

and the map

$$\text{Hom}_{\mathbf{K}(\text{Inj } \Lambda)}(\mathbf{i}N, (\mathbf{p}M \otimes_{\Lambda} D\Lambda)[-1]) \longrightarrow \overline{\text{Hom}}_{\Lambda}(N, D \text{Tr } M), \quad \phi \mapsto Z^0 \phi,$$

is clearly surjective. The composite is also an injective map, since a map  $N \rightarrow D \text{Tr } M$  factoring through an injective module  $N'$  comes from an element in  $D \text{Ext}_{\Lambda}^1(M, N')$  which vanishes.  $\square$

5. A GENERAL AUSLANDER-REITEN FORMULA FOR  $\text{Ext}_{\mathcal{A}}^1(-, -)$ 

In this section, we extend the classical Auslander-Reiten formula for modules to a formula for a more general class of abelian categories. Thus we fix a locally noetherian Grothendieck category  $\mathcal{A}$ , that is,  $\mathcal{A}$  is an abelian Grothendieck category having a set of generators which are noetherian objects in  $\mathcal{A}$ .

**Theorem 5.1.** *Let  $M$  and  $N$  be objects in  $\mathcal{A}$ . Suppose that  $M$  is noetherian and let  $\Gamma = \text{End}_{\mathcal{A}}(M)$ . Given an injective  $\Gamma$ -module  $I$ , there is an object  $\mathbf{a}_I M$  in  $\mathcal{A}$  and an isomorphism*

$$\text{Hom}_{\Gamma}(\text{Ext}_{\mathcal{A}}^1(M, N), I) \cong \overline{\text{Hom}}_{\mathcal{A}}(N, \mathbf{a}_I M)$$

which is natural in  $I$  and  $N$ .

*Proof.* The category  $\mathbf{K}(\text{Inj } \mathcal{A})$  is compactly generated and an injective resolution  $\mathbf{i}M$  of  $M$  is a compact object; see [14, Proposition 2.3]. The functor

$$\text{Hom}_{\Gamma}(\text{Hom}_{\mathbf{K}(\text{Inj } \mathcal{A})}(\mathbf{i}M, -), I)$$

is cohomological and sends coproducts in  $\mathbf{K}(\text{Inj } \mathcal{A})$  to products of abelian groups. Using Brown representability (see [11, 5.2] or [15, Theorem 3.1]), we have a representing object  $\mathbf{t}_I(\mathbf{i}M)$  in  $\mathbf{K}(\text{Inj } \mathcal{A})$  such that

$$\text{Hom}_{\Gamma}(\text{Hom}_{\mathbf{K}(\text{Inj } \mathcal{A})}(\mathbf{i}M, -), I) \cong \text{Hom}_{\mathbf{K}(\text{Inj } \mathcal{A})}(-, \mathbf{t}_I(\mathbf{i}M)).$$

Now put  $\mathbf{a}_I M = Z^0 \mathbf{t}_I(\mathbf{i}M)$  and adapt the proof of the classical Auslander-Reiten formula (4.2) from the previous section.  $\square$

**Remark 5.2.** If  $\mathcal{A}$  is a  $k$ -linear category, then we can take  $I = D\Gamma$  and obtain

$$D \text{Ext}_{\mathcal{A}}^1(M, N) \cong \overline{\text{Hom}}_{\mathcal{A}}(N, \mathbf{a}_{D\Gamma} M).$$

## 6. AUSLANDER-REITEN TRIANGLES

In [2], Auslander and Reiten used the formula

$$D \text{Ext}_{\Lambda}^1(M, N) \cong \overline{\text{Hom}}_{\Lambda}(N, D \text{Tr } M)$$

to establish the existence of almost split sequences. More precisely, for a finitely presented indecomposable and non-projective  $\Lambda$ -module  $M$ , there exists an almost split sequence

$$0 \rightarrow D \text{Tr } M \rightarrow L \rightarrow M \rightarrow 0$$

in the category of  $\Lambda$ -modules.

In this section, we produce Auslander-Reiten triangles in the category  $\mathbf{K}(\text{Inj } \Lambda)$ , using the Auslander-Reiten formula for complexes. In addition, we show that almost split sequences can be obtained from Auslander-Reiten triangles. This yields a simple recipe for the construction of an almost split sequence.

Let us recall the relevant definitions from Auslander-Reiten theory. A map  $\alpha: X \rightarrow Y$  is called *left almost split*, if  $\alpha$  is not a section and if every map  $X \rightarrow Y'$  which is not a section factors through  $\alpha$ . Dually, a map  $\beta: Y \rightarrow Z$  is *right almost split*, if  $\beta$  is not a retraction and if every map  $Y' \rightarrow Z$  which is not a retraction factors through  $\beta$ .

**Definition 6.1.** (1) An exact sequence  $0 \rightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \rightarrow 0$  in an abelian category is called *almost split sequence*, if  $\alpha$  is left almost split and  $\beta$  is right almost split.

- (2) An exact triangle  $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} X[1]$  in a triangulated category is called *Auslander-Reiten triangle*, if  $\alpha$  is left almost split and  $\beta$  is right almost split.

Happel introduced Auslander-Reiten triangles and studied their existence in  $\mathbf{D}^b(\text{mod } \Lambda)$  [9]. There is a general existence result for Auslander-Reiten triangles in compactly generated triangulated categories; see [13]. This yields the following.

**Proposition 6.2.** *Let  $Z$  be a compact object in  $\mathbf{K}(\text{Inj } \Lambda)$  which is indecomposable. Then there exists an Auslander-Reiten triangle*

$$(6.1) \quad (\mathbf{p}Z \otimes_{\Lambda} D\Lambda)[-1] \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \mathbf{p}Z \otimes_{\Lambda} D\Lambda.$$

*Proof.* First observe that  $\Gamma = \text{End}_{\mathbf{K}(\text{Inj } \Lambda)}(Z)$  is local because it is a noetherian  $k$ -algebra. Let  $I = E(\Gamma/\text{rad } \Gamma)$  and observe that the functor  $\text{Hom}_{\Gamma}(-, I)$  is isomorphic to  $D = \text{Hom}_k(-, E)$ . Now apply Theorem 2.2 from [13] which states the existence of an Auslander-Reiten triangle  $X[-1] \rightarrow Y \rightarrow Z \rightarrow X$  where  $X$  is the object representing the functor  $\text{Hom}_{\Gamma}(\text{Hom}_{\mathbf{K}(\text{Inj } \Lambda)}(Z, -), I)$ . It follows from the formula (3.4) that  $X = \mathbf{p}Z \otimes_{\Lambda} D\Lambda$ .  $\square$

Let us mention that the exact triangle (6.1) is determined by the map

$$\gamma: Z \longrightarrow \mathbf{p}Z \otimes_{\Lambda} D\Lambda$$

which corresponds under the isomorphism

$$D \text{Hom}_{\mathbf{K}(\text{Inj } \Lambda)}(Z, Z) \cong \text{Hom}_{\mathbf{K}(\text{Inj } \Lambda)}(Z, \mathbf{p}Z \otimes_{\Lambda} D\Lambda)$$

to a non-zero map  $\text{End}_{\mathbf{K}(\text{Inj } \Lambda)}(Z) \rightarrow E$  annihilating the radical of  $\text{End}_{\mathbf{K}(\text{Inj } \Lambda)}(Z)$ .

An Auslander-Reiten triangle ending in the injective resolution of a finitely presented indecomposable non-projective module induces an almost split sequence as follows.

**Theorem 6.3.** *Let  $N$  be a finitely presented  $\Lambda$ -module which is indecomposable and non-projective. Then there exists an Auslander-Reiten triangle*

$$(\mathbf{p}N \otimes_{\Lambda} D\Lambda)[-1] \xrightarrow{\alpha} Y \xrightarrow{\beta} \mathbf{i}N \xrightarrow{\gamma} \mathbf{p}N \otimes_{\Lambda} D\Lambda$$

in  $\mathbf{K}(\text{Inj } \Lambda)$  which the functor  $Z^0$  sends to an almost split sequence

$$0 \rightarrow D \text{Tr } N \xrightarrow{Z^0\alpha} Z^0Y \xrightarrow{Z^0\beta} N \rightarrow 0$$

in the category of  $\Lambda$ -modules.

*Proof.* The Auslander-Reiten triangle for  $\mathbf{i}N$  is obtained from the triangle (6.1) by taking  $Z = \mathbf{i}N$ . Let us assume that the projective resolution  $\mathbf{p}N$  is minimal. Note that we have a sequence

$$(6.2) \quad 0 \rightarrow (\mathbf{p}N \otimes_{\Lambda} D\Lambda)[-1] \xrightarrow{\alpha} Y \xrightarrow{\beta} \mathbf{i}N \rightarrow 0$$

of chain maps which is split exact in each degree. This sequence is obtained from the mapping cone construction for  $\gamma: \mathbf{i}N \rightarrow \mathbf{p}N \otimes_{\Lambda} D\Lambda$ .

We know from (4.1) that

$$D \text{Tr } N \cong Z^0(\mathbf{p}N \otimes_{\Lambda} D\Lambda)[-1].$$

This module is indecomposable and has a local endomorphism ring. Here we use that  $N$  is indecomposable and that the resolution  $\mathbf{p}N$  is minimal. The functor  $Z^0$  takes the sequence (6.2) to an exact sequence

$$(6.3) \quad 0 \rightarrow D \operatorname{Tr} N \xrightarrow{Z^0 \alpha} Z^0 Y \xrightarrow{Z^0 \beta} N.$$

Now observe that the map  $Z^0 \beta$  is right almost split. This is clear because  $\beta$  is right almost split and  $Z^0$  induces a bijection  $\operatorname{Hom}_{\mathbf{K}(\operatorname{Inj} \Lambda)}(\mathbf{i}M, \mathbf{i}N) \rightarrow \operatorname{Hom}_{\Lambda}(M, N)$  for all  $M$ . In particular,  $Z^0 \beta$  is an epimorphism since  $N$  is non-projective. We conclude from the following Lemma 6.4 that the sequence (6.3) is almost split.  $\square$

**Lemma 6.4.** *An exact sequence  $0 \rightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \rightarrow 0$  in an abelian category is almost split if and only if  $\beta$  is right almost split and the endomorphism ring of  $X$  is local.*

*Proof.* See Proposition II.4.4 in [1].  $\square$

**Remark 6.5.** There is an analogue of Theorem 6.3 for a projective module  $N$ . Then  $D \operatorname{Tr} N = 0$  and  $Z^0 \beta$  is the right almost split map ending in  $N$ .

From the mapping cone construction for complexes, we get an explicit recipe for the construction of an almost split sequence. Note that the computation of almost split sequences is a classical problem in representation theory; see for instance [7] or [3]. In particular, the middle term is considered to be mysterious.

**Corollary 6.6.** *Let  $N$  be a finitely presented  $\Lambda$ -module which is indecomposable and non-projective. Denote by*

$$P_1 \xrightarrow{\delta_1} P_0 \rightarrow N \rightarrow 0 \quad \text{and} \quad 0 \rightarrow N \rightarrow I^0 \xrightarrow{\delta^0} I^1$$

*a minimal projective presentation and an injective presentation of  $N$  respectively. Choose a non-zero  $k$ -linear map  $\operatorname{End}_{\Lambda}(N) \rightarrow E$  annihilating the radical of  $\operatorname{End}_{\Lambda}(N)$ , and extend it to a  $k$ -linear map  $\phi: \operatorname{Hom}_{\Lambda}(P_0, I^0) \rightarrow E$ . Let  $\bar{\phi}$  denote the image of  $\phi$  under the isomorphism*

$$D \operatorname{Hom}_{\Lambda}(P_0, I^0) \cong \operatorname{Hom}_{\Lambda}(I^0, P_0 \otimes_{\Lambda} D\Lambda).$$

*Then we have a commutative diagram with exact rows and columns*

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & P_1 \otimes_{\Lambda} D\Lambda & \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} & (P_1 \otimes_{\Lambda} D\Lambda) \amalg I^0 & \xrightarrow{\begin{bmatrix} 0 & 1 \end{bmatrix}} & I^0 \longrightarrow 0 \\ & & \downarrow \delta_1 \otimes 1 & & \downarrow \begin{bmatrix} \delta_1 \otimes 1 & \bar{\phi} \\ 0 & \delta^0 \end{bmatrix} & & \downarrow \delta^0 \\ 0 & \longrightarrow & P_0 \otimes_{\Lambda} D\Lambda & \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} & (P_0 \otimes_{\Lambda} D\Lambda) \amalg I^1 & \xrightarrow{\begin{bmatrix} 0 & 1 \end{bmatrix}} & I^1 \longrightarrow 0 \end{array}$$

*such that the upper row is an almost split sequence in the category of  $\Lambda$ -modules.*



and we view  $\Lambda$  as a bimodule  ${}_{\Lambda}\Lambda_{\hat{\Lambda}}$  via  $\phi$ . Let us explain the following diagram.

$$\begin{array}{ccccc}
\text{mod } \Lambda & \xrightarrow{-\otimes_{\Lambda}\Lambda} & \text{mod } \hat{\Lambda} & \xrightarrow{\text{can}} & \underline{\text{mod}} \hat{\Lambda} \\
\downarrow \text{inc} & & \downarrow \text{inc} & & \parallel \\
\mathbf{D}^b(\text{mod } \Lambda) & \xrightarrow{-\otimes_{\Lambda}\Lambda} & \mathbf{D}^b(\text{mod } \hat{\Lambda}) & \longrightarrow & \underline{\text{mod}} \hat{\Lambda} \\
\uparrow \wr & & \uparrow \wr & & \uparrow \wr \\
\mathbf{K}^c(\text{Inj } \Lambda) & \xrightarrow{F^c} & \mathbf{K}^c(\text{Inj } \hat{\Lambda}) & \xrightarrow{G^c} & \mathbf{K}_{\text{ac}}^c(\text{Inj } \hat{\Lambda}) \\
\downarrow \text{inc} & & \downarrow \text{inc} & & \downarrow \text{inc} \\
\mathbf{K}(\text{Inj } \Lambda) & \xrightarrow{F} & \mathbf{K}(\text{Inj } \hat{\Lambda}) & \xrightarrow{G} & \mathbf{K}_{\text{ac}}(\text{Inj } \hat{\Lambda}) \\
\uparrow \mathbf{j}_{\Lambda} & \xrightarrow{\text{Hom}_{\hat{\Lambda}}(\Lambda, -)} & \uparrow \mathbf{j}_{\hat{\Lambda}} & \xrightarrow{\text{inc}} & \downarrow \text{inc} \\
\mathbf{K}(\text{Mod } \Lambda) & \xrightarrow{-\otimes_{\Lambda}\Lambda} & \mathbf{K}(\text{Mod } \hat{\Lambda}) & & \\
& \xleftarrow{\text{Hom}_{\hat{\Lambda}}(\Lambda, -)} & & & 
\end{array}$$

The top squares show the construction of Happel's functor  $\mathbf{D}^b(\text{mod } \Lambda) \rightarrow \underline{\text{mod}} \hat{\Lambda}$  for which we refer to [10, 2.5].

The bimodule  ${}_{\Lambda}\Lambda_{\hat{\Lambda}}$  induces an adjoint pair of functors between  $\mathbf{K}(\text{Mod } \Lambda)$  and  $\mathbf{K}(\text{Mod } \hat{\Lambda})$ . Note that  $\text{Hom}_{\hat{\Lambda}}(\Lambda, -)$  takes injective  $\hat{\Lambda}$ -modules to injective  $\Lambda$ -modules. Thus we get an induced functor  $\mathbf{K}(\text{Inj } \hat{\Lambda}) \rightarrow \mathbf{K}(\text{Inj } \Lambda)$ . This functor preserves products and has therefore a left adjoint  $F$ , by Brown's representability theorem [16, Theorem 8.6.1]. A left adjoint preserves compactness if the right adjoint preserves coproducts; see [15, Theorem 5.1]. Clearly,  $\text{Hom}_{\hat{\Lambda}}(\Lambda, -)$  preserves coproducts since  $\Lambda$  is finitely generated over  $\hat{\Lambda}$ . Thus  $F$  induces a functor  $F^c$ .

The inclusion  $\mathbf{K}(\text{Inj } \Lambda) \rightarrow \mathbf{K}(\text{Mod } \Lambda)$  preserves products and has therefore a left adjoint  $\mathbf{j}_{\Lambda}$ , by Brown's representability theorem [16, Theorem 8.6.1]. Note that  $\mathbf{j}_{\Lambda}M = \mathbf{i}M$  is an injective resolution for every  $\Lambda$ -module  $M$ . We have the same for  $\hat{\Lambda}$ , of course. Thus we have

$$F \circ \mathbf{j}_{\Lambda} = \mathbf{j}_{\hat{\Lambda}} \circ (-\otimes_{\Lambda}\Lambda).$$

It follows that  $F$  takes the injective resolution of a  $\Lambda$ -module  $M$  to the injective resolution of the  $\hat{\Lambda}$ -module  $M \otimes_{\Lambda} \Lambda$ . This shows that  $F^c$  coincides with  $-\otimes_{\Lambda}\Lambda$  when one passes to the derived category  $\mathbf{D}^b(\text{mod } \Lambda)$  via the canonical equivalence  $\mathbf{K}^c(\text{Inj } \Lambda) \rightarrow \mathbf{D}^b(\text{mod } \Lambda)$ .

The inclusion  $\mathbf{K}_{\text{ac}}(\text{Inj } \hat{\Lambda}) \rightarrow \mathbf{K}(\text{Inj } \hat{\Lambda})$  has a left adjoint  $G$ ; see [14, Theorem 4.2]. This left adjoint admits an explicit description. For instance, it takes the injective resolution  $\mathbf{i}M$  of a  $\hat{\Lambda}$ -module  $M$  to the mapping cone of the canonical map  $\mathbf{p}M \rightarrow \mathbf{i}M$ , which is a complete resolution of  $M$ . The functor  $G$  preserves compactness and induces therefore a functor  $G^c$ , because its right adjoint preserves coproducts [15, Theorem 5.1].

The following result summarizes our construction.

**Theorem 7.2.** *The composite*

$$\underline{\text{Mod}} \hat{\Lambda} \xrightarrow{\sim} \mathbf{K}_{\text{ac}}(\text{Inj } \hat{\Lambda}) \xrightarrow{\text{Hom}_{\hat{\Lambda}}(\Lambda, -)} \mathbf{K}(\text{Inj } \Lambda)$$

has a fully faithful left adjoint

$$\mathbf{K}(\text{Inj } \Lambda) \xrightarrow{G \circ F} \mathbf{K}_{\text{ac}}(\text{Inj } \hat{\Lambda}) \xrightarrow{\sim} \underline{\text{Mod}} \hat{\Lambda}$$

which extends Happel's functor

$$\mathbf{D}^b(\text{mod } \Lambda) \xrightarrow{-\otimes_{\Lambda} \hat{\Lambda}} \mathbf{D}^b(\text{mod } \hat{\Lambda}) \rightarrow \underline{\text{mod}} \hat{\Lambda}.$$

## 8. THE COMPUTATION OF AUSLANDER-REITEN TRIANGLES

In this section, we explain a method for computing Auslander-Reiten triangles in  $\mathbf{K}(\text{Inj } \Lambda)$ . It is shown that the adjoint of Happel's functor reduces the computation to the problem of computing almost split sequences in  $\text{mod } \hat{\Lambda}$ . This is based on the following result.

**Proposition 8.1.** *Let  $F: \mathcal{S} \rightarrow \mathcal{T}$  be a fully faithful exact functor between triangulated categories which admits a right adjoint  $G: \mathcal{T} \rightarrow \mathcal{S}$ . Suppose*

$$X_{\mathcal{S}} \xrightarrow{\alpha_{\mathcal{S}}} Y_{\mathcal{S}} \xrightarrow{\beta_{\mathcal{S}}} Z_{\mathcal{S}} \xrightarrow{\gamma_{\mathcal{S}}} X_{\mathcal{S}}[1] \quad \text{and} \quad X_{\mathcal{T}} \xrightarrow{\alpha_{\mathcal{T}}} Y_{\mathcal{T}} \xrightarrow{\beta_{\mathcal{T}}} Z_{\mathcal{T}} \xrightarrow{\gamma_{\mathcal{T}}} X_{\mathcal{T}}[1]$$

are Auslander-Reiten triangles in  $\mathcal{S}$  and  $\mathcal{T}$  respectively, where  $Z_{\mathcal{T}} = FZ_{\mathcal{S}}$ . Then

$$GX_{\mathcal{T}} \xrightarrow{G\alpha_{\mathcal{T}}} GY_{\mathcal{T}} \xrightarrow{G\beta_{\mathcal{T}}} GZ_{\mathcal{T}} \xrightarrow{G\gamma_{\mathcal{T}}} GX_{\mathcal{T}}[1]$$

is the coproduct of  $X_{\mathcal{S}} \xrightarrow{\alpha_{\mathcal{S}}} Y_{\mathcal{S}} \xrightarrow{\beta_{\mathcal{S}}} Z_{\mathcal{S}} \xrightarrow{\gamma_{\mathcal{S}}} X_{\mathcal{S}}[1]$  and a triangle  $W \xrightarrow{\text{id}} W \rightarrow 0 \rightarrow W[1]$ .

*Proof.* We have a natural isomorphism  $\text{Id}_{\mathcal{S}} \cong G \circ F$  which we view as an identification. In particular,  $G$  induces a bijection

$$(8.1) \quad \text{Hom}_{\mathcal{T}}(FX, Y) \rightarrow \text{Hom}_{\mathcal{S}}((G \circ F)X, GY)$$

for all  $X \in \mathcal{S}$  and  $Y \in \mathcal{T}$ . Next we observe that for any exact triangle  $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} X[1]$ , the map  $\beta$  is a retraction if and only if  $\gamma = 0$ .

The map  $F\beta_{\mathcal{S}}$  is not a retraction since  $F\gamma_{\mathcal{S}} \neq 0$ . Thus  $F\beta_{\mathcal{S}}$  factors through  $\beta_{\mathcal{T}}$ , and  $G(F\beta_{\mathcal{S}}) = \beta_{\mathcal{S}}$  factors through  $G\beta_{\mathcal{T}}$ . We obtain the following commutative diagram.

$$\begin{array}{ccccccc} X_{\mathcal{S}} & \xrightarrow{\alpha_{\mathcal{S}}} & Y_{\mathcal{S}} & \xrightarrow{\beta_{\mathcal{S}}} & Z_{\mathcal{S}} & \xrightarrow{\gamma_{\mathcal{S}}} & X_{\mathcal{S}}[1] \\ \downarrow \phi & & \downarrow \psi & & \parallel & & \downarrow \phi[1] \\ GX_{\mathcal{T}} & \xrightarrow{G\alpha_{\mathcal{T}}} & GY_{\mathcal{T}} & \xrightarrow{G\beta_{\mathcal{T}}} & GZ_{\mathcal{T}} & \xrightarrow{G\gamma_{\mathcal{T}}} & GX_{\mathcal{S}}[1] \end{array}$$

On the other hand,  $G\beta_{\mathcal{T}}$  is not a retraction since the bijection (8.1) implies  $G\gamma_{\mathcal{T}} \neq 0$ . Thus  $G\beta_{\mathcal{T}}$  factors through  $\beta_{\mathcal{S}}$ , and we obtain the following commutative diagram.

$$\begin{array}{ccccccc} GX_{\mathcal{T}} & \xrightarrow{G\alpha_{\mathcal{T}}} & GY_{\mathcal{T}} & \xrightarrow{G\beta_{\mathcal{T}}} & GZ_{\mathcal{T}} & \xrightarrow{G\gamma_{\mathcal{T}}} & GX_{\mathcal{T}}[1] \\ \downarrow \phi' & & \downarrow \psi' & & \parallel & & \downarrow \phi'[1] \\ X_{\mathcal{S}} & \xrightarrow{\alpha_{\mathcal{S}}} & Y_{\mathcal{S}} & \xrightarrow{\beta_{\mathcal{S}}} & Z_{\mathcal{S}} & \xrightarrow{\gamma_{\mathcal{S}}} & X_{\mathcal{S}}[1] \end{array}$$

We have  $\beta_{\mathcal{S}} \circ (\psi' \circ \psi) = \beta_{\mathcal{S}}$ , and this implies that  $\psi' \circ \psi$  is an isomorphism, since  $\beta_{\mathcal{S}}$  is right minimal. In particular,  $GY_{\mathcal{T}} = Y_{\mathcal{S}} \amalg W$  for some object  $W$ . It follows that

$$GX_{\mathcal{T}} \xrightarrow{G\alpha_{\mathcal{T}}} GY_{\mathcal{T}} \xrightarrow{G\beta_{\mathcal{T}}} GZ_{\mathcal{T}} \xrightarrow{\gamma_{\mathcal{T}}} GX_{\mathcal{T}}[1]$$

is the coproduct of  $X_S \xrightarrow{\alpha_S} Y_S \xrightarrow{\beta_S} Z_S \xrightarrow{\gamma_S} X_S[1]$  and the triangle  $W \xrightarrow{\text{id}} W \rightarrow 0 \rightarrow W[1]$ .  $\square$

Now suppose that  $\Lambda$  is an artin algebra. We fix an indecomposable compact object  $Z$  in  $\mathbf{K}(\text{Inj } \Lambda)$ , and we want to compute the Auslander-Reiten triangle  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ . We apply Happel's functor

$$H: \mathbf{K}^c(\text{Inj } \Lambda) \xrightarrow{\sim} \mathbf{D}^b(\text{mod } \Lambda) \rightarrow \underline{\text{mod}} \hat{\Lambda}$$

and obtain an indecomposable non-projective  $\hat{\Lambda}$ -module  $Z' = HZ$ . For instance, if  $Z = \mathbf{i}N$  is the injective resolution of an indecomposable  $\Lambda$ -module  $N$ , then  $H\mathbf{i}N = N$  where  $N$  is viewed as a  $\hat{\Lambda}$ -module via the canonical algebra homomorphism  $\hat{\Lambda} \rightarrow \Lambda$ . Now take the almost split sequence  $0 \rightarrow D \text{Tr } Z' \rightarrow Y' \rightarrow Z' \rightarrow 0$  in  $\text{Mod } \hat{\Lambda}$ . This gives rise to an Auslander-Reiten triangle  $D \text{Tr } Z' \rightarrow Y' \rightarrow Z' \rightarrow D \text{Tr } Z'[1]$  in  $\underline{\text{Mod}} \hat{\Lambda}$ . We apply the composite

$$\underline{\text{Mod}} \hat{\Lambda} \xrightarrow{\sim} \mathbf{K}_{\text{ac}}(\text{Inj } \hat{\Lambda}) \xrightarrow{\text{Hom}_{\hat{\Lambda}}(\Lambda, -)} \mathbf{K}(\text{Inj } \Lambda).$$

It follows from Proposition 8.1 that the result is a coproduct of the Auslander-Reiten triangle  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$  and a split exact triangle.

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