ON MINIMAL APPROXIMATIONS OF MODULES

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Let R be a ring and consider the category $\operatorname{Mod} R$ of (right) R-modules. Given a class $\mathcal{C}$ of R-modules, a morphism $M \to N$ in $\operatorname{Mod} R$ is called a left $\mathcal{C}$-approximation of M provided that N belongs to $\mathcal{C}$ and the induced morphism $\operatorname{Hom}_R(N, C) \to \operatorname{Hom}_R(M, C)$ is surjective for every C in $\mathcal{C}$. This concept has been introduced by Auslander and Smalø [2], and independently by Enochs [5]. Approximations arise naturally in various situations, for instance in representation theory of finite dimensional algebras, in particular in tilting theory, or in commutative algebra. In this note we address the problem of finding minimal approximations. In general, a left $\mathcal{C}$-approximation $M \to N$ of M is not uniquely determined by M, but it happens quite often that there exists an approximation $\phi\colon M \to N$ which is left minimal in the sense that any endomorphism ψ of N satisfying $\psi \circ \phi = \phi$ is an isomorphism. Note that such a minimal approximation of M is unique up to isomorphism.

This paper is divided into four sections. In Section 1 we characterize the morphisms $\phi\colon M \to N$ which admit a decomposition $\phi = (\phi', \phi'')\colon M \to N' \coprod N'' = N$ such that ϕ' is left minimal and $\phi'' = 0$. For example, such a decomposition always exists if N is pure-injective. In Section 2 we discuss criteria for a class $\mathcal{C}$ of R-modules such that every R-module admits a left $\mathcal{C}$-approximation. In Section 3 we introduce a new class of modules M which we call product-complete since the class $\operatorname{Add} M$ of direct summands of coproducts of copies of M is closed under taking products. We prove that an R-module M is product-complete if and only if every R-module admits a (minimal) left $\operatorname{Add} M$-approximation. In fact, we provide a host of characterizations of such modules which indicate that product-complete modules are particulary well-behaved. For example, every endofinite module is product-complete. More precisely, we prove in Section 4 that a module M is endofinite if and only if every direct summand of M is product-complete.

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1. Left minimal morphisms

Let R be a an associative ring with identity. We consider the category $\operatorname{Mod} R$ of (right) R-modules and denote by $\operatorname{mod} R$ the full subcategory of finitely presented R-modules. A morphism $\phi\colon M \to N$ in any category is said to be *left minimal* if any endomorphism ψ of N satisfying $\psi \circ \phi = \phi$ is an isomorphism. A concept which

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is closely related is that of an injective envelope. Recall that a monomorphism $\phi\colon M \to N$ in any abelian category is an *injective envelope* of M provided that N is an injective object and every morphism $N \to N'$ is a monomorphism if the composition with ϕ is a monomorphism.

Lemma 1.1. *A monomorphism $\phi\colon M \to N$ into an injective object N is an injective envelope if and only if ϕ is left minimal.*

Proof. Straightforward. $\square$

The following result gives the precise relation between left minimal morphisms and injective envelopes.

Proposition 1.2. *Let $\phi\colon M \to N$ be a morphism in some additive category $\mathcal{A}$ and suppose that idempotents in* $\operatorname{End}(N)$ *split. Then the following are equivalent:*

(1) *There exists a decomposition $\phi = (\phi', \phi'')\colon M \to N' \coprod N'' = N$ such that ϕ' is left minimal and $\phi'' = 0$.*
(2) *There exists an abelian category $\mathcal{B}$ and an additive functor $F\colon \mathcal{A} \to \mathcal{B}$ such that*
    (a) *$F(N)$ is injective;*
    (b) *F induces an isomorphism* $\operatorname{Hom}(M \coprod N, N) \to \operatorname{Hom}(F(M \coprod N), F(N))$;
    (c) *the image* $\operatorname{Im} F(\phi)$ *has an injective envelope.*

A decomposition $\phi = (\phi', \phi'')\colon M \to N' \coprod N'' = N$ of a morphism ϕ such that ϕ' is left minimal and $\phi'' = 0$ is unique up to isomorphism. The morphism ϕ' is sometimes called the *minimal version* of ϕ.

Proof. We apply the characterization of injective envelopes from Lemma 1.1.

(1) $\Rightarrow$ (2) Let $S = \operatorname{End}(N)^{\mathrm{op}}$ and consider the functor

$$F\colon \mathcal{A} \longrightarrow \mathcal{B} = (\operatorname{Mod} S)^{\mathrm{op}}, \quad X \mapsto \operatorname{Hom}(X, N).$$

It is clear that (a) and (b) are satisfied. A decomposition $\phi = (\phi', \phi'')\colon M \to N' \coprod N'' = N$ of ϕ such that ϕ' is left minimal and $\phi'' = 0$ induces a morphism $\operatorname{Im} F(\phi) \to F(N')$ in $\mathcal{B}$ which is an injective envelope by Lemma 1.1.

(2) $\Rightarrow$ (1) Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor and suppose that $\varepsilon\colon \operatorname{Im} F(\phi) \to Q$ is an injective envelope. Consider the induced morphism $\psi\colon \operatorname{Im} F(\phi) \to F(N)$. There are morphisms $\alpha\colon Q \to F(N)$ and $\beta\colon F(N) \to Q$ satisfying $\alpha \circ \varepsilon = \psi$ and $\beta \circ \psi = \varepsilon$ since Q and $F(N)$ are injective. We may assume that $\beta \circ \alpha = \mathrm{id}_Q$ since ε is an injective envelope. Thus $\alpha \circ \beta$ is an idempotent in $\operatorname{End}(F(N))$ and we obtain therefore a decomposition $N = N' \coprod N''$ in $\mathcal{A}$ with $F(N') = \operatorname{Im} \alpha$. It is now easily checked that in the induced decomposition $\phi = (\phi', \phi'')\colon M \to N' \coprod N'' = N$ of ϕ the morphism ϕ' is left minimal and $\phi'' = 0$. $\square$

Our main application of the preceding result depends on the well-known fact that the fully faithful functor

$$\operatorname{Mod} R \longrightarrow D(R) = (\operatorname{mod} R^{\mathrm{op}}, \mathrm{Ab}), \quad M \mapsto M \otimes_R -$$

from $\operatorname{Mod} R$ into the category of additive functors $\operatorname{mod} R^{\mathrm{op}} \to \mathrm{Ab}$ identifies the pure-injective R-modules with the injective objects in $D(R)$. Recall that a map

$M \to N$ in $\operatorname{Mod} R$ is a *pure monomorphism* if $M \otimes_R X \to N \otimes_R X$ a monomorphism for every X in $\operatorname{mod} R^{\mathrm{op}}$, and M is *pure-injective* if every pure monomorphism $M \to N$ splits.

Corollary 1.3. *Let $\phi \colon M \to N$ be a morphism in $\operatorname{Mod} R$ and suppose that N is pure-injective. Then there exists a decomposition*

$$\phi = (\phi', \phi'') \colon M \longrightarrow N' \coprod N'' = N$$

such that ϕ' is left minimal and $\phi'' = 0$.

Proof. The Grothendieck category $D(R)$ has injective envelopes, and the assertion is therefore a consequence of Proposition 1.2. □

Another application of Proposition 1.2 shows that any morphism in a Krull-Schmidt category has a minimal version.

Corollary 1.4. *Let $\phi \colon M \to N$ be a morphism in some additive category and suppose that idempotents in $\operatorname{End}(N)$ split. If N is a finite coproduct of indecomposable objects with local endomorphism ring, then there exists a decomposition*

$$\phi = (\phi', \phi'') \colon M \longrightarrow N' \coprod N'' = N$$

such that ϕ' is left minimal and $\phi'' = 0$.

Proof. We consider as in the proof of Proposition 1.2 the functor

$$F \colon \mathcal{A} \longrightarrow \mathcal{B} = (\operatorname{Mod} S)^{\mathrm{op}}, \quad X \mapsto \operatorname{Hom}(X, N)$$

where $S = \operatorname{End}(N)^{\mathrm{op}}$. The assumption on N implies that S is a semi-perfect ring, and therefore every finitely generated S-module has a projective cover in $\operatorname{Mod} S$. By duality, it follows that $\operatorname{Im} F(\phi)$ has an injective envelope in $\mathcal{B}$, and the assertion is now a consequence of Proposition 1.2. □

Remark 1.5. We leave it to the reader to formulate the dual statements of Proposition 1.2 and Corollary 1.4. For instance, the dual of Corollary 1.4 generalizes the well-known fact that any finitely generated module over a semi-perfect ring has a projective cover.

2. Left approximations

Let $\mathcal{A}$ be any category and $\mathcal{C}$ be any class of objects in $\mathcal{A}$. Following [2], a morphism $M \to N$ in $\mathcal{A}$ is called a *left $\mathcal{C}$-approximation* of M provided that N belongs to $\mathcal{C}$ and the induced morphism $\operatorname{Hom}(N, C) \to \operatorname{Hom}(M, C)$ is surjective for every C in $\mathcal{C}$. If every object in $\mathcal{A}$ has a left $\mathcal{C}$-approximation, then $\mathcal{C}$ is said to be *covariantly finite* in $\mathcal{A}$. Note that this concept has been introduced independently in [5].

We begin our discussion with a criterion for a class of modules to be covariantly finite. Suppose that $\mathcal{C}$ is a class of R-modules which is closed under direct summands. It is easily seen that $\mathcal{C}$ needs to be closed under taking products if $\mathcal{C}$ is covariantly finite in $\operatorname{Mod} R$.

Lemma 2.1. *Let $\mathcal{C}$ be any class of R-modules which is closed under direct summands. If $\mathcal{C}$ is covariantly finite in* $\operatorname{Mod} R$*, then $\mathcal{C}$ is closed under taking products.*

Proof. Let $M = \prod_{i \in I} M_i$ be a product of modules in $\mathcal{C}$ and let $\phi\colon M \to N$ be a left $\mathcal{C}$-approxiamation. Every projection $M \to M_i$ factors through ϕ and therefore ϕ is a split monomorphism. It follows that M belongs to $\mathcal{C}$ since $\mathcal{C}$ is closed under direct summands by assumption. □

In [13], it has been shown that also the converse is true if $\mathcal{C}$ is closed under pure submodules.

Lemma 2.2. *Let $\mathcal{C}$ be any class of R-modules which is closed under pure submodules. If $\mathcal{C}$ is closed under taking products, then $\mathcal{C}$ is covariantly finite in* $\operatorname{Mod} R$*.*

Proof. See Corollary 3.5 (c) in [13]. □

The next result is essentially a reformulation of Corollary 1.3.

Theorem 2.3. *Let $\mathcal{C}$ be any class of modules which is closed under direct summands. Let M be an R-module and suppose that M has a left $\mathcal{C}$-approximation which is pure-injective. Then M has a minimal left $\mathcal{C}$-approximation.*

Proof. Apply Corollary 1.3. □

We present an interesting consequence.

Corollary 2.4. *Let $\mathcal{C}$ be any class of pure-injective R-modules which is closed under direct summands. Then the following are equivalent:*

(1) *Every R-module has a left $\mathcal{C}$-approximation.*
(2) *Every R-module has a minimal left $\mathcal{C}$-approximation.*
(3) *$\mathcal{C}$ is closed under taking products.*

Proof. (1) $\Rightarrow$ (2) Use Theorem 2.3.

(2) $\Rightarrow$ (3) Use Lemma 2.1.

(3) $\Rightarrow$ (1) Let $\mathcal{D}$ be the class of pure submodules of modules in $\mathcal{C}$. Then every R-module M has a left $\mathcal{D}$-approximation $M \to N$ by Lemma 2.2. Suppose that N is the pure submodule of $N' \in \mathcal{C}$. It is clear that the composition $M \to N'$ is a left $\mathcal{C}$-approximation. □

Example 2.5. Let $\mathcal{C}$ be the class of all pure-injective R-modules. Then a morphism $\phi\colon M \to N$ is a minimal left $\mathcal{C}$-approximation if and only if ϕ is a pure-injective envelope of M (i.e. N is pure-injective and any map $\psi\colon N \to N'$ is a pure monomorphism if and only if $\psi \circ \phi$ is a pure monomorphism). Indeed, the canonical map $\phi_M\colon M \to D^2M$ (where $D = \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$) is a left $\mathcal{C}$-approximation since D^2M is pure-injective and ϕ_M is a pure monomorphism. Therefore a minimal left $\mathcal{C}$-approximation $M \to N$ is a pure monomorphism, hence a pure-injective envelope. Conversely, if $\phi\colon M \to N$ is a pure-injective envelope, then any endomorphism ψ of N satisfying $\phi = \psi \circ \phi$ needs to be a pure-monomorphism, hence a split monomorphism, and therefore an isomorphism. The fact that every module has a pure-injective envelope is a classical result of Kiełpiński [10].

3. Product-complete modules

Given any R-module M we denote by $\operatorname{Add} M$ the full subcategory of R-modules which are direct summands of coproducts of copies of M. Note that $\operatorname{Add} M$ is the smallest full subcategory of $\operatorname{Mod} R$ which contains M and is closed under forming arbitrary coproducts and direct summands. Our main result is a characterization of the fact that $\operatorname{Add} M$ is closed under taking products.

Theorem 3.1. *The following are equivalent for an R-module M:*

(1) $\operatorname{Add} M$ *is closed under taking products.*
(2) *Every product of copies of M is a direct summand of a coproduct of copies of M.*
(3) *Every product of copies of M is a coproduct of (indecomposable) direct summands of M.*
(4) *Every R-module has a (minimal) left* $\operatorname{Add} M$*-approximation.*

We call a module M *product-complete* if M satisfies the equivalent conditions of the preceding theorem.

Recall that a module M is Σ*-pure-injective* if every coproduct of copies of M is pure-injective. We shall work with the following characterization of a Σ-pure-injective module.

Lemma 3.2. *A module M is Σ-pure-injective if and only if there exists a cardinal κ such that every product of copies of M is a pure submodule of a coproduct of modules having cardinality at most κ.*

Proof. See [9]. □

We shall also need the following property of a Σ-pure-injective module.

Lemma 3.3. *Any Σ-pure-injective module is a coproduct of indecomposable modules with local endomorphism ring.*

Proof. See [9, 16, 17]. □

Proof of Theorem 3.1. (1) ⇒ (2) Clear.

(2) ⇒ (3) M is Σ-pure-injective by Lemma 3.2. It follows from Lemma 3.3 in combination with the Krull-Remak-Schmidt-Azumaya Theorem that every direct summand of a coproduct of copies of M decomposes into indecomposable modules which are direct summands of M. Thus (2) implies (3).

(3) ⇒ (4) M is Σ-pure-injective by Lemma 3.2. Thus $\operatorname{Add} M$ consists of pure-injective modules and is closed under taking products. It follows from Corollary 2.4 that every R-module has a minimal left $\operatorname{Add} M$-approximation.

(4) ⇒ (1) Use Lemma 2.1. □

We give an example of a Σ-pure-injective module which is not product-complete.

Example 3.4. Let $R = \mathbb{Z}$ and let $M = \mathbb{Z}_{p^\infty}$ be a Prüfer group. This module is Σ-pure-injective but not product-complete. However $M \coprod \mathbb{Q}$ is product-complete.

We list some basic properties of product-complete modules.

(1) The class of product-complete modules is closed under finite coproducts.

(2) Let I be any non-empty set. Then a module M is product-complete if and only if $M^{(I)}$ is product-complete.

(3) If M is product-complete, then M^I is product-complete for every set I.

We have the following categorical characterization of product-complete modules.

Proposition 3.5. *An R-module M is product-complete if and only if there exists a (locally noetherian) Grothendieck category $\mathcal{A}$ and a functor $F\colon \mathcal{A} \to \operatorname{Mod} R$ which commutes with products and coproducts, such that F induces an equivalence between the full subcategory $\operatorname{Inj}\mathcal{A}$ of injective objects in $\mathcal{A}$ and $\operatorname{Add} M$.*

Proof. We use the functor category $D(R) = (\operatorname{mod} R^{\mathrm{op}}, \mathrm{Ab})$. Suppose that M is product-complete and consider the full subcategory $\mathcal{T}$ of all objects X in $D(R)$ with $\operatorname{Hom}(X, M \otimes_R -) = 0$. This is a localizing subcategory and we can therefore form the quotient category $\mathcal{A} = D(R)/\mathcal{T}$, e.g. see [6]. The composition of the embedding $\operatorname{Mod} R \to D(R)$, $N \mapsto N \otimes_R -$, with the quotient functor $D(R) \to \mathcal{A}$ identifies $\operatorname{Add} M$ with $\operatorname{Inj}\mathcal{A}$ since $M \otimes_R -$ is an injective cogenerator for $\mathcal{A}$, and therefore $\mathcal{A}$ is locally noetherian since $\operatorname{Inj}\mathcal{A}$ is closed under forming coproducts [14]. The composition of the right adjoint of the quotient functor with the functor $D(R) \to \operatorname{Mod} R$, $X \mapsto X(R)$, gives the functor $F\colon \mathcal{A} \to \operatorname{Mod} R$ which has the desired properties. Conversely, suppose there exists a functor $F\colon \mathcal{A} \to \operatorname{Mod} R$ which commutes with products and coproducts, such that F induces an equivalence $\operatorname{Inj}\mathcal{A} \to \operatorname{Add} M$. It follows that $\operatorname{Add} M$ is closed under taking products since $\operatorname{Inj}\mathcal{A}$ is closed under taking products, and therefore $\operatorname{Add} M$ is product-complete. □

Corollary 3.6. *If M is product-complete, then $\operatorname{Add} M$ is closed under taking direct limits.*

Proof. If $\mathcal{A}$ is a locally noetherian category, then $\operatorname{Inj}\mathcal{A}$ is closed under taking direct limits [6, p.358]. The assertion is therefore a consequence of the construction given in the preceding proof. □

In [18], Ziegler introduced a topology on the set $\operatorname{Ind} R$ of isomorphism classes of indecomposable pure-injective R-modules, and it is also possible to characterize the product-complete modules in terms of this topology.

Proposition 3.7. *A module M is product-complete if and only if M is Σ-pure-injective and the indecomposable direct summands of M form a Ziegler-closed subset of $\operatorname{Ind} R$.*

Proof. In [11, Proposition 8.7] it is shown that for every Σ-pure-injective module M the indecomposable direct summands of products of copies of M form a Ziegler-closed set. Using the characterization (2) in Theorem 3.1 the assertion follows. □

The argument in the preceding proof leads to a new characterization of Σ-pure-injective modules.

Corollary 3.8. *A module M is Σ-pure-injective if and only if there exists a non-empty set I such that M^I is product-complete.*

Proof. One direction is clear. Suppose therefore that M is Σ-pure-injective. The argument given in the proof of Proposition 3.7 shows that we can choose a non-empty set I such that the indecomposable direct sumands of M^I form a Ziegler-closed set. Thus M^I is product-complete by Proposition 3.7. □

We now characterize the finitely generated modules which are product-complete.

Proposition 3.9. *A finitely generated module M is product-complete if and only if $S = \mathrm{End}_R(M)$ is left coherent and right perfect, and M is finitely presented as S^{op}-module.*

Proof. The functors $\mathrm{Hom}_R(M, -)$ and $- \otimes_S M$ induce mutually inverse equivalences between $\mathrm{Add}\, M$ and the full subcategory $\mathrm{Proj}\, S$ of projective S-modules since M is finitely generated. In particular, the canonical map $\varepsilon_N\colon \mathrm{Hom}_R(M, N) \otimes_S M \to N$ is an isomorphism for all N in $\mathrm{Add}\, M$. Now suppose that M is product-complete. Since $M^I \in \mathrm{Add}\, M$, we get that $S^I \in \mathrm{Proj}\, S$, for every set I. A classical result of Chase says that S is right perfect and left coherent [3, Theorem 3.3]. Moreover, since ε_{M^I} is an isomorphism for every set I, due to our above observation, the canonical map $S^I \otimes_S M \to M^I$ is an isomorphism, for every set I. This is well-known to be equivalent to the fact that M is finitely presented as S^{op}-module; see, e.g., [15, Lemma I.13.2]. To prove the converse, we need to check that $M^I \in \mathrm{Add}\, M$ for every set I. We have $S^I \in \mathrm{Proj}\, S$ since S is left coherent and right perfect, and $S^I \otimes_S M \simeq M^I$ since M is finitely presented over S^{op}. Thus $M^I \in \mathrm{Add}\, M$, and therefore M is product-complete. □

Remark 3.10. (1) When $M = R$ in the above proposition, we get that every R-module has a minimal projective left approximation if and only if R is right perfect and left coherent [1, Proposition 3.5].

(2) The hypothesis that M is finitely generated cannot be omitted in the above proposition. Indeed, if R is a right noetherian ring and M is a coproduct of a representative set of indecomposable injective R-modules, then $\mathrm{Add}\, M = \mathrm{Inj}\, R$, so that M is product-complete. However, in general, M has an infinite number of direct summands and so $S = \mathrm{End}_R(M)$ is not right perfect.

4. Endofinite modules

Recall that an R-module M is *endofinite* if it has finite length as a module over its endomorphism ring $\mathrm{End}_R(M)^{\mathrm{op}}$. This class of modules has been studied by various authors. Crawley-Boevey noticed their importance in representation theory of finite dimensional algebras, and he introduced their name [4]. In this section we discuss the relation between endofinite and product-complete modules. The basic result is the following.

Theorem 4.1. *A module M is endofinite if and only if every direct summand of M is product-complete.*

Proof. Suppose first that M is indecomposable. It is a well-known result due to Gruson [8] and Garavaglia [7] that M is endofinite if and only if $\mathrm{Add}\, M$ is closed under taking products. We refer to [12, Corollary 10.5] for a proof since

[8] contains no proof and [7] has never been published. Now suppose that M is an arbitrary module. We use the fact that M is endofinite if and only if there are indecomposable endofinite modules $M_1, \ldots, M_n$ and sets $I_1, \ldots, I_n$ such that $M \simeq \coprod_{i=1}^n M_i^{(I_i)}$; see [4, Proposition 4.5]. The fact that every endofinite module is product-complete is now a consequence of the basic properties of product-complete modules listed after Theorem 3.1, and the fact that every indecomposable endofinite module is product-complete. Conversely, let M be a module such that every direct summand is product-complete. It follows from Proposition 3.7 that the indecomposable direct summands of M form a Ziegler-closed subset $\mathbf{U}$ of $\operatorname{Ind} R$ having the property that every subset $\mathbf{V} \subseteq \mathbf{U}$ is also Ziegler-closed. However, $\operatorname{Ind} R$ is a quasi-compact space [18, Theorem 4.9] and therefore $\mathbf{U}$ needs to be finite. Thus M is endofinite since each $N \in \mathbf{U}$ is endofinite and M is a coproduct of modules in $\mathbf{U}$ by Theorem 3.1. This completes the proof. □

We fix an R-module M with $S = \operatorname{End}_R(M)^{\mathrm{op}}$. Recall that an additve subgroup of M is of *finite definition* if it arises as the kernel of a morphism $M \to M \otimes_R X$, $m \mapsto m \otimes x$, for some finitely presented R^{op}-module X, and some element $x \in X$. Note that a subgroup of finite definition is a S-submodule of M. If M is Σ-pure-injective, then any finitely generated S-submodule of M is a subgroup of finite definition. We shall now prove that every subgroup of finite definition is a finitely generated S-module provided that M is product-complete.

Proposition 4.2. *A module M is product-complete if and only if the following conditions hold:*

(1) *M has the descending chain condition on subgroups of finite definition.*
(2) *Every subgroup of finite definition is a finitely generated $\operatorname{End}_R(M)^{\mathrm{op}}$-module.*

Proof. We use again the embedding $\operatorname{Mod} R \to D(R)$, $N \mapsto T_N = N \otimes_R -$. Identifying $M = \operatorname{Hom}(T_R, T_M)$, any additive subgroup U is of finite definition if and only if $U = \operatorname{Hom}(T_R/F, T_M)$ for some finitely generated subfunctor $F \subseteq T_R$. Note that U is finitely generated as S-module if and only if there exists a map $\phi \colon T_R/F \to T_{M^n}$ for some $n \in \mathbb{N}$ such that every map $T_R/F \to T_M$ factors through ϕ. Now suppose that M is product-complete. Then M is Σ-pure-injective and it is well-known that (1) holds [9, 16]. To check (2) let $U = \operatorname{Hom}(T_R/F, T_M)$ be a subgroup of finite definition. There exists a map $\phi \colon T_R/F \to T_{M^I}$ for some set I such that every map $T_R/F \to T_M$ factors through ϕ. We find a set J such that M^I is a direct summand of $M^{(J)}$, and the map $T_R/F \to T_{M^I} \to T_{M^{(J)}}$ factors through $T_{M^{(J')}}$ for some finite subset $J' \subseteq J$ since F/T_R is finitely generated. Denoting by $\phi' \colon T_R/F \to T_{M^{(J')}}$ the corresponding map it is clear that every map $T_R/F \to T_M$ factors through ϕ'. Thus U is a finitely generated S-module and (2) holds. Using similar arguments it is not hard to check that M is product-complete if (1) and (2) hold. This is left to the reader. □

Lemma 4.3. *M is endofinite if and only if M is Σ-pure-injective and a noetherian module over S.*

Proof. It is well-known that M is Σ-pure-injective if and only if M has the dcc on subgroups of finite definition [9, 16]. Thus every endofinite module is Σ-pure-injective since every subgroup of finite definition of M is a S-submodule. Conversely, if M is Σ-pure-injective, then every finitely generated S-submodule is a subgroup of finite definition; e.g. see [4, Lemma 4.1]. It follows that M is artinian over S since every S-submodule of M is finitely generated if M is noetherian over S. Thus M has finite endolength. □

Corollary 4.4. *Let M be an R-module and suppose that $\operatorname{End}_R(M)$ is left noetherian. Then M is endofinite if and only if M is product-complete.*

Proof. If M is product-complete, then M is finitely generated over $S = \operatorname{End}_R(M)^{\mathrm{op}}$ by Proposition 4.2. Thus M is noetherian over S and the assertion follows from Lemma 4.3. □

The semi-simple modules form another class of modules where endofinite and product-complete modules coincide.

Proposition 4.5. *For a semi-simple R-module M the following are equivalent:*

(1) *M is endofinite.*
(2) *M is product-complete.*
(3) *Every product of copies of M is semi-simple.*
(4) *The ring $R/\operatorname{ann} M$ is semi-simple.*

Proof. (1) ⇒ (2) Use Theorem 4.1.

(2) ⇒ (3) Clear.

(3) ⇒ (4) Consider the canonical map $R \to M^M$ which induces a monomorphism $R/\operatorname{ann} M \to M^M$. Any submodule of a semi-simple module is semi-simple, and therefore $R/\operatorname{ann} M$ is a semi-simple ring.

(4) ⇒ (1) Any module over a semi-simple ring is endofinite. □

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