The representation dimension of a finite dimensional algebra

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A motivation from group theory

G = a finite group
k = a field of characteristic $p > 0$
B = a non-semisimple block of kG
$\mathrm{ll}\, B$ = the Loewy length of B
D = a defect group of B

Conjecture [Benson]. $\mathrm{ll}\, B > \mathrm{rank}_p\, D$.

Theorem [Rouquier 2003, Opperman 2006].

$$\mathrm{ll}\, B \geq \mathrm{rep.dim}\, B \geq \dim(\underline{\mathrm{mod}}\, B) + 2 > \mathrm{rank}_p\, D.$$

Finite dimensional representations

Λ = a finite dimensional algebra over a field k

$\operatorname{mod}\Lambda$ = the category of finite dimensional right Λ-modules

Krull-Remak-Schmidt. Every $M \in \operatorname{mod}\Lambda$ has an essentially unique finite decomposition

$$M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$$

such that all M_i are indecomposable.

Homological properties. $\operatorname{mod}\Lambda$ is an abelian category. Every object $M \in \operatorname{mod}\Lambda$ admits
– an epimorphism $P \twoheadrightarrow M$ with P projective,
– a monomorphism $M \rightarrowtail I$ with I injective.

Representation dimension

The *projective dimension* of $M \in \operatorname{mod}\Lambda$ is

$$\begin{aligned}\operatorname{pd} M = \min\{n \geq 0 \mid &\text{ there is a projective resolution} \\ &0 \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0\}.\end{aligned}$$

The *global dimension* of Λ is

$$\begin{aligned}\operatorname{gl.dim}\Lambda &= \sup\{\operatorname{pd} M \mid M \in \operatorname{mod}\Lambda\} \\ &= \operatorname{pd}\Lambda/\operatorname{rad}\Lambda.\end{aligned}$$

Definition [Auslander, 1971]**.** The *representation dimension* of Λ is

$$\begin{aligned}\operatorname{rep.dim}\Lambda = \min\{\operatorname{gl.dim}\operatorname{End}_\Lambda(G) \mid &\, G \text{ generates} \\ &\text{and cogenerates } \operatorname{mod}\Lambda\}.\end{aligned}$$

G *generates* $\operatorname{mod}\Lambda$ if every $M \in \operatorname{mod}\Lambda$ admits an epimorphism $G^n \twoheadrightarrow M$ (for some n).

G *cogenerates* $\operatorname{mod}\Lambda$ if every $M \in \operatorname{mod}\Lambda$ admits a monomorphism $M \rightarrowtail G^n$ (for some n).

Finite representation type

Λ has *finite representation type* if there are only finitely many isomorphism classes of indecomposable $M \in \operatorname{mod} \Lambda$.

Theorem [Auslander, 1971]**.**

(1) Λ is semi-simple iff $\operatorname{rep.dim} \Lambda = 0$.

(2) $\operatorname{rep.dim} \Lambda \neq 1$.

(3) Λ has finite rep. type iff $\operatorname{rep.dim} \Lambda \leq 2$.

Conclusion. The representation dimension provides a “reasonable way of measuring how far a finite dimensional algebra is from being representation finite” [Auslander, 1971].

More about finite representation type

Theorem [Auslander, 1971]**.** There are bijections (up to Morita equivalence):

$$\{\Lambda \text{ f.d. algebra} \mid \Lambda \text{ has finite rep. type}\}/_\sim$$

$$\updownarrow$$

$$\{\Gamma \text{ f.d. algebra} \mid \operatorname{gl.dim}\Gamma \leq 2, \operatorname{dom.dim}\Gamma \geq 2\}/_\sim$$

$$\Lambda \longmapsto \operatorname{End}_\Lambda\Big(\bigoplus_{M \text{ indec.}} M\Big)$$

$$\Gamma \longmapsto \operatorname{End}_\Gamma\Big(\bigoplus_{\substack{P \text{ indec.} \\ P \text{ proj. and inj.}}} P\Big)$$

$\operatorname{dom.dim}\Gamma \geq n$ if there exists an exact sequence

$$0 \to \Gamma \to I_0 \to I_1 \to \ldots \to I_{n-1}$$

with I_i injective and projective for all $i < n$.

The finitistic dimension conjecture

Question [Auslander, 1971]**.**

- Is $\operatorname{rep.dim} \Lambda < \infty$?

- Is $\operatorname{rep.dim} \Lambda > 3$ possible?

Conjecture [Bass, 1960]**.** The finitistic dimension

$$\operatorname{fin.dim} \Lambda = \sup\{\operatorname{pd} M \mid M \in \operatorname{mod} \Lambda, \operatorname{pd} M < \infty\}$$

is finite.

Theorem [Igusa/Todorov, 1991]**.** $\operatorname{rep.dim} \Lambda \leq 3$ implies $\operatorname{fin.dim} \Lambda < \infty$.

The representation dimension is finite

Theorem [Iyama, 2002]**.**

$$\operatorname{rep.dim}\Lambda < \infty.$$

Idea of proof [Ringel]:

- For every $M \in \operatorname{mod}\Lambda$ there exists $M' \in \operatorname{mod}\Lambda$ such that $\operatorname{End}_\Lambda(M \oplus M')$ is a quasi-hereditary algebra. Take
$$M' = \bigoplus_{i\geq 1} \delta^i M \text{ with } \delta M = (\operatorname{rad}\operatorname{End}_\Lambda(M))M.$$

- Take for M a generator and cogenerator (exists always).

- Use that $\operatorname{gl.dim}\Gamma < \infty$ for every quasi-hereditary algebra Γ.

Exterior algebras

Theorem [Rouquier, 2003]**.** For the exterior algebra $\Lambda(k^n)$ we have

$$\operatorname{rep.dim} \Lambda(k^n) = n + 1.$$

The proof uses dimensions of triangulated categories and a differential graded version of Koszul duality [Keller, 1994].

Beilinson algebras

Let k be a field and $n \geq 1$. The *Beilinson algebra*

$$\Lambda_n = k[Q_n]/I_n$$

is the quotient of the path algebra of the quiver

$$\begin{tikzcd} Q_n & 0 \arrow[r, bend left, "x_0"] \arrow[r, bend right, "x_n"'] \arrow[r, phantom, "\vdots"] & 1 \arrow[r, bend left, "x_0"] \arrow[r, bend right, "x_n"'] \arrow[r, phantom, "\vdots"] & 2 & \cdots & n-1 \arrow[r, bend left, "x_0"] \arrow[r, bend right, "x_n"'] \arrow[r, phantom, "\vdots"] & n \end{tikzcd}$$

modulo the ideal

$$I_n = \langle x_i x_j - x_j x_i \mid 0 \leq i, j \leq n \rangle.$$

Theorem [K./Kussin, 2005]**.** Let M generate $\operatorname{mod} \Lambda_n$. Then

$$\operatorname{gl.dim} \operatorname{End}_{\Lambda_n}(M) \geq n.$$

In particular $\operatorname{rep.dim} \Lambda_n \geq n$.

A derived equivalence

$\mathcal{A}$ = an abelian category
$\mathbf{D}^b(\mathcal{A})$ = the *bounded derived category* of $\mathcal{A}$

(The objects are bounded chain complexes in $\mathcal{A}$; the morphisms are chain maps plus formal inverses for all maps inducing a homology isomorphism.)

$\mathbf{P}^n$ = the projective n-space over the field k
coh $\mathbf{P}^n$ = the category of coherent sheaves on $\mathbf{P}^n$

$T = \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbf{P}^n}(i)$ is a *tilting sheaf* in coh $\mathbf{P}^n$ with

$$\mathrm{End}_{\mathcal{O}_{\mathbf{P}^n}}(T) \cong \Lambda_n.$$

Theorem [Beilinson, 1978]**.** The functor

$$\mathbf{R}\mathrm{Hom}_{\mathcal{O}_{\mathbf{P}^n}}(T,-)\colon \mathbf{D}^b(\mathrm{coh}\,\mathbf{P}^n) \xrightarrow{\sim} \mathbf{D}^b(\mathrm{mod}\,\Lambda_n)$$

is an equivalence of triangulated categories.

Dimensions of triangulated categories

$\mathcal{A}$ = an abelian category
$\mathcal{T} = \mathbf{D}^b(\mathcal{A})$ is a triangulated category

Exact sequences $0 \to X \to Y \to Z \to 0$ of chain maps induce *triangles* $X \to Y \to Z \to$ in $\mathbf{D}^b(\mathcal{A})$.

For $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2 \subseteq \mathcal{T}$ define:

$$\mathcal{X}_1 * \mathcal{X}_2 = \{X \in \mathcal{T} \mid \exists \text{ triangle } X_1 \to X \to X_2 \to \text{ with } X_i \in \mathcal{X}_i\}$$

$\langle \mathcal{X} \rangle$ = the closure of $\mathcal{X}$ under direct sums, direct summands, and shifts

$\langle \mathcal{X} \rangle_0 = 0$

$\langle \mathcal{X} \rangle_n = \langle \langle \mathcal{X} \rangle_{n-1} * \langle \mathcal{X} \rangle \rangle$ for $n \geq 1$

Definition [Bondal/Van den Bergh, 2003]**.**

$$\dim \mathcal{T} = \min\{n \geq 0 \mid \exists\, X \in \mathcal{T} \text{ with } \langle X \rangle_{n+1} = \mathcal{T}\}.$$

Example. $\operatorname{gl.dim} \Lambda = n$ implies

$$\mathbf{D}^b(\operatorname{mod} \Lambda) = \langle \Lambda \rangle_n.$$

Beilinson algebras Λ_n revisited

Proposition. Let M generate $\operatorname{mod}\Lambda$. Then

$$\operatorname{gl.dim}\operatorname{End}_\Lambda(M) \geq \dim \mathbf{D}^b(\operatorname{mod}\Lambda).$$

Proposition. Let X be a reduced projective scheme. Then

$$\dim \mathbf{D}^b(\operatorname{coh} X) \geq \dim X.$$

Corollary. Let M generate $\operatorname{mod}\Lambda_n$. Then

$$\begin{aligned}\operatorname{gl.dim}\operatorname{End}_{\Lambda_n}(M) &\geq \dim \mathbf{D}^b(\operatorname{mod}\Lambda_n)\\ &= \dim \mathbf{D}^b(\operatorname{coh}\mathbf{P}^n)\\ &\geq \dim \mathbf{P}^n\\ &= n.\end{aligned}$$

In particular $\operatorname{rep.dim}\Lambda_n \geq n$.

Remark. Recently, Iyama and Oppermann showed

$$\operatorname{rep.dim}\Lambda_n = n+2.$$

A general result with same proof

Theorem [K./Kussin, 2005]**.** Let X be a reduced projective scheme. If $T \in \mathbf{D}^b(\operatorname{coh} X)$ is a tilting complex, then

$$\operatorname{rep.dim} \operatorname{End}_{\mathcal{O}_X}(T) \geq \dim X.$$

More on dimensions of triangulated categories

We provide a general method for producing lower bounds. Here is the set-up:

$R = \bigoplus_{i\geq 0} R^i$ a graded comm. ring, R^0 artinian

$\operatorname{Proj} R =$ set of homogeneous prime ideals $\mathfrak{p} \not\supseteq R^+$

For a graded R-module M let

$$\operatorname{Supp}_R^+ M = \{\mathfrak{p} \in \operatorname{Proj} R \mid M_\mathfrak{p} \neq 0\}.$$

For $V \subseteq \operatorname{Proj} R$ let

$$\dim V = \sup\{d \mid \exists\ \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_d \text{ in } V\}.$$

A lower bound for $\dim\mathcal{T}$

Let $\mathcal{T}$ be an R-linear triangulated category. Thus R acts *centrally* on

$$\operatorname{Hom}^*_{\mathcal{T}}(X,Y)=\bigoplus_{i\in\mathbb{Z}}\operatorname{Hom}_{\mathcal{T}}(X,Y[i])$$

for all $X,Y\in\mathcal{T}$.

$$r\alpha=(-1)^{|r||\alpha|}\alpha r\quad(\alpha\in\mathcal{T},r\in R)$$

Call a graded R-module M *eventually noetherian* if $M^{\geq n}=\bigoplus_{i\geq n}M^i$ is noetherian for some $n\in\mathbb{Z}$.

Theorem [Bergh/Iyengar/K./Oppermann, 2008]**.** Let G be a generator of $\mathcal{T}$ such that the R-module $\operatorname{End}^*_{\mathcal{T}}(G)$ is eventually noetherian. Then

$$\dim\mathcal{T}\geq\dim\operatorname{Supp}^+_R\operatorname{End}^*_{\mathcal{T}}(G).$$

Proof: Use the ghost lemma and Koszul objects.

The ghost lema and Koszul objects

Ghost lemma [Kelly, Carlsson, Christensen, Beligiannis, Rouquier, Avramov/Buchweitz/Iyengar, Bergh ...]**.** Let $\mathcal{T}$ be a triangulated category and $G, X \in \mathcal{T}$. Suppose there are morphisms

$$K_c \xrightarrow{\theta_c} \cdots \xrightarrow{\theta_2} K_1 \xrightarrow{\theta_1} K_0$$

such that

(1) $\operatorname{Hom}^n_{\mathcal{T}}(G, \theta_i) = 0$ for $n \gg 0$ and all i,
(2) $\operatorname{Hom}^n_{\mathcal{T}}(X, \theta_c \ldots \theta_1) \neq 0$ for inf. many $n \geq 0$.

Then $X \notin \langle G \rangle_c$.

Koszul objects. Let $c = \dim \operatorname{Supp}^+_R \operatorname{End}^*_{\mathcal{T}}(G)$ and $K_0 = G$. There are $r_1, \ldots, r_c \in R^+$ which induce exact triangles $K_i \overset{\theta_i}{\to} K_{i-1} \overset{r_i}{\to} K_{i-1} \to$ with

$$K/\mathbf{r} = K_c \notin \langle G \rangle_c.$$

An application for finite groups

$G =$ a finite p-group
$k =$ a field of characteristic p
$H^*(G,k) = \mathrm{Ext}^*_{kG}(k,k)$ the group cohomology

Then $\underline{\mathrm{mod}}\, kG$ is $H^*(G,k)$-linear and k is a generator. The Tate cohomology $\underline{\mathrm{End}}^*_{kG}(k)$ is eventually noetherian. Thus

$$\dim(\underline{\mathrm{mod}}\, kG) \geq \dim \operatorname{Proj} H^*(G,k) = \mathrm{rank}_p\, G - 1.$$

Benson's conjecture

G = a finite group
k = a field of characteristic $p > 0$
B = a non-semisimple block of kG
$\mathrm{ll}\, B$ = the Loewy length of B
D = a defect group of B

Theorem [Rouquier 2003, Opperman 2006]

$$\begin{aligned} \mathrm{ll}\, B &\geq \mathrm{rep.\,dim}\, B && \text{(Auslander, 1971)}\\ &\geq \dim(\underline{\mathrm{mod}}\, B) + 2 && \text{(Rouquier, 2003)}\\ &= \dim(\underline{\mathrm{mod}}\, kD) + 2 && \text{(induction/restriction)}\\ &> \mathrm{rank}_p\, D && \text{(previous slide)} \end{aligned}$$

Alternative argument: $\mathrm{HH}^*(B)$ acts on $\underline{\mathrm{mod}}\, B$ and $\underline{\mathrm{End}}^*_B(B/\mathrm{rad}\, B)$ is eventually noetherian. Thus

$$\begin{aligned} \dim(\underline{\mathrm{mod}}\, B) &\geq \dim \mathrm{Supp}^+\, \underline{\mathrm{End}}^*_B(B/\mathrm{rad}\, B)\\ &= \dim \mathrm{Proj}\, \mathrm{HH}^*(B)\\ &= \mathrm{rank}_p\, D - 1. \end{aligned}$$

Exterior algebras revisited

Let $\Lambda = \Lambda(k^n)$. Then $\mathsf{HH}^*(\Lambda)$ acts on $\underline{\mathsf{mod}}\,\Lambda$ and $\underline{\mathsf{End}}^*_\Lambda(k)$ is eventually noetherian. Thus

$$\begin{aligned} n+1 &= \mathsf{ll}\,\Lambda \\ &\geq \mathsf{rep.\,dim}\,\Lambda \\ &\geq \mathsf{dim}(\underline{\mathsf{mod}}\,\Lambda) + 2 \\ &\geq \mathsf{dim}\,\mathsf{Supp}^+\,\underline{\mathsf{End}}^*_\Lambda(k) + 2 \\ &= \mathsf{dim}\,\mathsf{Proj}\,\mathsf{HH}^*(\Lambda) + 2 \\ &= n+1. \end{aligned}$$