COHERENT FUNCTORS IN STABLE HOMOTOPY THEORY

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Let $\mathcal{S}$ be a compactly generated triangulated category, for example the stable homotopy category of CW-spectra. We call a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ into the category of abelian groups *coherent* if there exists an exact sequence

$$\mathrm{Hom}(D,-) \longrightarrow \mathrm{Hom}(C,-) \longrightarrow F \longrightarrow 0$$

such that C and D are compact objects in $\mathcal{S}$ (an object X in $\mathcal{S}$ is *compact* if the representable functor $\mathrm{Hom}(X,-)$ preserves arbitrary coproducts).

The concept of a coherent functor has been introduced explicitly for abelian categories by Auslander [1], but it is also implicit in the work of Freyd on stable homotopy [9]. In this paper we characterize coherent functors in a number of ways and use them to study a wider class of functors $\mathcal{S} \to \mathrm{Ab}$ which share a weak exactness property. Another purpose of this paper is to investigate certain subcategories of $\mathcal{S}$ which are defined in terms of coherent functors.

In the category $\mathrm{Mod}\,\Lambda$ of modules over an associative ring Λ, the analogue of a compact object is a finitely presented module. This fact can be made precise (cf. the Appendix), and one has in this context the following classical result: a functor $F\colon \mathrm{Mod}\,\Lambda \to \mathrm{Ab}$ is coherent precisely if F preserves products and filtered colimits. There is no obvious way to formulate such a characterization for compactly generated triangulated categories because filtered colimits rarely exist in triangulated categories. Nevertheless, we are able to characterize the coherent functors as follows.

Theorem A. *For a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ the following conditions are equivalent:*

(1) *F is coherent.*
(2) *F preserves products and sends every homology colimit to a colimit.*
(3) *F preserves products and coproducts, and F is short exact.*

In the presence of Brown representability (for homology theories), there is a further equivalent condition:

(4) *F preserves products and minimal weak filtered colimits of compact objects.*

We call a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ *short exact* if for every triangle $X \to Y \to Z \to \Sigma X$ the sequence $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is exact, provided that $0 \to \mathrm{Hom}(C,X) \to \mathrm{Hom}(C,Y) \to \mathrm{Hom}(C,Z) \to 0$ is exact for each compact C. This seems to be an interesting exactness property. To explain this, let us introduce the following notation. The full subcategory of compact objects in $\mathcal{S}$ is denoted by $\mathcal{F}$ and $(\mathcal{F}^{\mathrm{op}},\mathrm{Ab})$ denotes the category of additive functors $\mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ into the category of abelian groups. For every object X in $\mathcal{S}$ consider the functor

$$H_X = \mathrm{Hom}(-,X)|_{\mathcal{F}}\colon \mathcal{F}^{\mathrm{op}} \longrightarrow \mathrm{Ab}.$$

This is an example of an exact functor. Recall that a functor from a triangulated category to the category of abelian groups is *exact* if it sends triangles to exact sequences. We call a triangle

$$(*) \qquad X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

in $\mathcal{S}$ *pure* if the induced sequence $0 \to H_X \to H_Y \to H_Z \to 0$ is exact, and a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ is by definition short exact if for every pure triangle $(*)$ the sequence $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is exact.

For the stable homotopy category, exact and coproduct preserving functors have been characterized by Brown and Adams [1]; they are precisely the functors $F\colon \mathcal{S} \to \mathrm{Ab}$ that are 'represented' by an object Y in $\mathcal{S}$ in the sense that

$$F(X) \cong \mathrm{Hom}(S, X \wedge Y)$$

for all X in $\mathcal{S}$ (where S denotes the sphere spectrum). One can also use the tensor-product

$$(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \times (\mathcal{F}, \mathrm{Ab}) \longrightarrow \mathrm{Ab}, \quad (H, G) \to H \otimes_{\mathcal{F}} G$$

to study functors $\mathcal{S} \to \mathrm{Ab}$. Recall that the tensor functor $- \otimes_{\mathcal{F}} G$ is determined by the fact that it preserves colimits and $H_X \otimes_{\mathcal{F}} G \cong G(X)$ for all X in $\mathcal{F}$. It turns out that $F\colon \mathcal{S} \to \mathrm{Ab}$ is exact and preserves coproducts if and only if there is a functorial isomorphism

$$F(X) \cong H_X \otimes_{\mathcal{F}} G$$

for some exact functor $G\colon \mathcal{F} \to \mathrm{Ab}$. The following result characterizes the functors which are 'represented' by an arbitrary functor $G\colon \mathcal{F} \to \mathrm{Ab}$.

Theorem B. *For a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ the following conditions are equivalent:*

(1) *F is short exact and preserves coproducts.*
(2) *There exist an additive functor $G\colon \mathcal{F} \to \mathrm{Ab}$ and a functorial isomorphism $F(X) \cong H_X \otimes_{\mathcal{F}} G$ for all X in $\mathcal{S}$.*
(3) *There exist a filtered diagram $(F_i)_{i\in\mathcal{I}}$ of coherent functors and a functorial isomorphism $F(X) \cong \mathrm{colim}_i F_i(X)$ for all X in $\mathcal{S}$.*

Let us consider the collection of all coherent functors $\mathcal{S} \to \mathrm{Ab}$ which we denote by $\mathrm{Coh}\,\mathcal{S}$. In fact, $\mathrm{Coh}\,\mathcal{S}$ is an abelian category if we take as maps the natural transformations. This category has been studied by Freyd in [9]. Here, we exhibit an interesting closure operation which is defined in terms of coherent functors. Given a class $\mathcal{C}$ of objects in $\mathcal{S}$, we define

$$\mathrm{Def}\,\mathcal{C} = \{X \in \mathcal{S} \mid F \in \mathrm{Coh}\,\mathcal{S} \text{ and } F(Y) = 0 \text{ for all } Y \in \mathcal{C} \text{ implies } F(X) = 0\}.$$

For example, Freyd's Generating Hypothesis [9] for the stable homotopy category could be reformulated as follows.

Generating Hypothesis (Freyd). $\mathrm{Def}\{S^n \mid n \in \mathbb{Z}\} = \mathcal{S}$.

There is an explicit construction which produces all objects in $\mathrm{Def}\,\mathcal{C}$, at least if we assume Brown representability. We call an object X the *reduced product* of a family of objects $(X_i)_{i\in I}$ in $\mathcal{S}$ with respect to a filter $\mathcal{U}$ on the set I, if

$$H_X \cong \mathrm{colim}_{J\in\mathcal{U}} \prod_{i\in J} H_{X_i}$$

where the filtered colimit is taken over the canonical projections $\prod_{i\in J_1} H_{X_i} \to \prod_{i\in J_2} H_{X_i}$ which are induced by the inclusions $J_2 \subseteq J_1$ of subsets $J_1, J_2 \in \mathcal{U}$. Note that a reduced product always exists; it is unique up to isomorphism and denoted by $\prod_{i\in I} X_i/\mathcal{U}$.

Theorem C. *Suppose that Brown representability holds for $\mathcal{S}$, and let $\mathcal{C}$ be a class of objects in $\mathcal{S}$. Then an object X in $\mathcal{S}$ belongs to $\mathrm{Def}\,\mathcal{C}$ if and only if there is a pure triangle $X \to Y \to Z \to \Sigma X$ such that Y is a reduced product of objects in $\mathcal{C}$.*

We say that a full subcategory $\mathcal{C}$ of $\mathcal{S}$ is *definable* if $\mathcal{C} = \operatorname{Def}\mathcal{C}$, equivalently if $\mathcal{C} = \{X \in \mathcal{S} \mid F_i(X) = 0 \text{ for all } i \in I\}$ for some family $(F_i)_{i\in I}$ of coherent functors. This concept has its origin in model theory of modules; in this context a definable subcategory corresponds to a complete theory of modules [23, 8]. There are three other concepts equivalent to definable subcategories:

• *Ziegler-closed subsets* of the set $\operatorname{Sp}\mathcal{S}$ of isomorphism classes of indecomposable pure-injective objects in $\mathcal{S}$. Recall that X in $\mathcal{S}$ is *pure-injective* if for every pure triangle $X \to Y \to Z \to \Sigma X$ the first map is a section. A subset of $\operatorname{Sp}\mathcal{S}$ is Ziegler-closed if it is of the form $\mathcal{C} \cap \operatorname{Sp}\mathcal{S}$ for some definable subcategory $\mathcal{C}$ of $\mathcal{S}$.

• *Serre subcategories* of $\operatorname{Coh}\mathcal{S}$. These are full subcategories of $\operatorname{Coh}\mathcal{S}$ which are closed under forming subobjects, quotient objects, and extensions.

• *Cohomological ideals* in $\mathcal{F}$. These are ideals of maps in $\mathcal{F}$ which are of the form $\{\phi \in \mathcal{F} \mid F(\phi) = 0\}$ for some exact functor $F\colon \mathcal{F} \to \mathrm{Ab}$. For example, given X in $\mathcal{S}$, the *annihilator*

$$\operatorname{Ann} X = \{\phi \in \mathcal{F} \mid \operatorname{Hom}(\phi, X) = 0\}$$

is cohomological.

Fundamental Correspondence. *There are bijections between*

- *the set of definable subcategories $\mathcal{C}$ of $\mathcal{S}$,*
- *the set of Ziegler-closed subsets $\mathbf{U}$ of $\operatorname{Sp}\mathcal{S}$,*
- *the set of Serre subcategories $\mathcal{T}$ of $\operatorname{Coh}\mathcal{S}$,*
- *the set of cohomological ideals $\mathfrak{I}$ in $\mathcal{F}$.*

These bijections are defined as follows:

$$\mathcal{C} \mapsto \begin{cases} \mathbf{U} = \mathcal{C} \cap \operatorname{Sp}\mathcal{S} \\ \mathcal{T} = \{F \in \operatorname{Coh}\mathcal{S} \mid F(X) = 0 \textit{ for all } X \in \mathcal{C}\} \\ \mathfrak{I} = \bigcap_{X\in\mathcal{C}} \operatorname{Ann} X \end{cases}$$

$$\mathbf{U} \mapsto \begin{cases} \mathcal{C} = \{X \in \mathcal{S} \mid \textit{there are } Y_i \in \mathbf{U} \textit{ and a pure triangle } X \to \prod_i Y_i \to Z \to \Sigma X\} \\ \mathcal{T} = \{F \in \operatorname{Coh}\mathcal{S} \mid F(X) = 0 \textit{ for all } X \in \mathbf{U}\} \\ \mathfrak{I} = \bigcap_{X\in\mathbf{U}} \operatorname{Ann} X \end{cases}$$

$$\mathcal{T} \mapsto \begin{cases} \mathcal{C} = \{X \in \mathcal{S} \mid F(X) = 0 \textit{ for all } F \in \mathcal{T}\} \\ \mathbf{U} = \{X \in \operatorname{Sp}\mathcal{S} \mid F(X) = 0 \textit{ for all } F \in \mathcal{T}\} \\ \mathfrak{I} = \{\phi \in \mathcal{F} \mid \operatorname{Im}\operatorname{Hom}(\phi, -) \in \mathcal{T}\} \end{cases}$$

$$\mathfrak{I} \mapsto \begin{cases} \mathcal{C} = \{X \in \mathcal{S} \mid \mathfrak{I} \subseteq \operatorname{Ann} X\} \\ \mathbf{U} = \{X \in \operatorname{Sp}\mathcal{S} \mid \mathfrak{I} \subseteq \operatorname{Ann} X\} \\ \mathcal{T} = \{F \in \operatorname{Coh}\mathcal{S} \mid F = \operatorname{Im}\operatorname{Hom}(\phi, -) \textit{ for some } \phi \in \mathfrak{I}\} \end{cases}$$

This correspondence is the analogue of a correspondence for module categories which is based on work of several mathematicians [23, 12, 8, 15]. For instance, Ziegler introduced the closed subsets of indecomposable pure-injective modules in model-theoretic terms [23]. In our setting, one obtains a topology on $\operatorname{Sp}\mathcal{S}$ by taking the Ziegler-closed subsets as closed subsets [15]. Examples of definable subcategories arise quite naturally. Take for instance a localization functor $L\colon \mathcal{S} \to \mathcal{S}$ which is smashing, i.e. L preserves coproducts. Then the L-local objects form a definable subcategory [17]. Or take an endofinite object X in $\mathcal{S}$ (in the sense of [18]). Then the direct factors of coproducts of copies of X form a definable subcategory.

1. The functor category

Purity and phantoms. We fix a triangulated category $\mathcal{S}$ and make the following additional assumptions:

- $\mathcal{S}$ has arbitrary coproducts;
- the isomorphism classes of compact objects in $\mathcal{S}$ form a set;
- $\operatorname{Hom}(C, X) = 0$ for all compact C implies $X = 0$ for every object X in $\mathcal{S}$.

A triangulated category satisfying these conditions is called *compactly generated*. The full subcategory of compact objects in $\mathcal{S}$ is always denoted by $\mathcal{F}$. Recall that X in $\mathcal{S}$ is *compact* if the representable functor $\operatorname{Hom}(X, -)$ preserves arbitrary coproducts. Our basic tool is the category of additive functors $\mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ which we denote by $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. The *restricted Yoneda functor*

$$\mathcal{S} \longrightarrow (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto H_X = \operatorname{Hom}(-, X)|_{\mathcal{F}},$$

relates the triangulated structure of $\mathcal{S}$ to the abelian structure of $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. The functor identifies the full subcategory of pure-projective objects in $\mathcal{S}$ with the full subcategory of projective objects in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$, and it identifies the full subcategory of pure-injective objects in $\mathcal{S}$ with the full subcategory of injective objects in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. We recall briefly the relevant definitions and refer to [17] for more details.

Definition 1.1. Let $\mathcal{S}$ be a compactly generated triangulated category.

(1) A map $X \to Y$ in $\mathcal{S}$ is a *pure monomorphism* if the map $H_X \to H_Y$ is a monomorphism. An object X in $\mathcal{S}$ is *pure-injective* if every pure monomorphism $X \to Y$ is a split monomorphism.
(2) A map $Y \to Z$ in $\mathcal{S}$ is a *pure epimorphism* if the map $H_Y \to H_Z$ is an epimorphism. An object Z in $\mathcal{S}$ is *pure-projective* if every pure epimorphism $Y \to Z$ is a split epimorphism.
(3) A triangle $X \to Y \to Z \to \Sigma X$ is *pure* if the sequence $0 \to H_X \to H_Y \to H_Z \to 0$ is exact.

Note that purity is closely related to properties of phantom maps; see for example [4] and [6]. Recall that a map $X \to Y$ in $\mathcal{S}$ is a *phantom map* if the induced map $H_X \to H_Y$ is zero. For instance, an object X in $\mathcal{S}$ is pure-injective if and only if there are no non-zero phantom maps ending in X. Dually, X is pure-projective if and only if there are no non-zero phantom maps starting in X. Finally, a triangle $X \to Y \to Z \to \Sigma X$ is pure if and only if the map $Z \to \Sigma X$ is phantom.

One can prove easily that an object in $\mathcal{S}$ is pure-projective if and only if it is a direct factor of a coproduct of compact objects. The following lemma describes some essential properties of pure-projective objects. This is well-known, see for example [4], but we include the proof for the convenience of the reader.

Lemma 1.2. *Let $\mathcal{S}$ be a compactly generated triangulated category, and let P be a projective object in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. Then there exists, up to isomorphism, a unique object X in $\mathcal{S}$ such that $P \cong H_X$. Moreover, the map*

$$\operatorname{Hom}(X, Y) \longrightarrow \operatorname{Hom}(H_X, H_Y), \quad \alpha \mapsto H_\alpha,$$

is an isomorphism for all Y in $\mathcal{S}$.

Proof. Every projective P is a direct factor of some coproduct $\coprod_{i \in I} H_{C_i}$ of representable functors with $C_i \in \mathcal{F}$ for all $i \in I$. Assume first that $P = \coprod_{i \in I} H_{C_i}$. Then one takes $X = \coprod_{i \in I} C_i$ and the isomorphism $\operatorname{Hom}(X, Y) \cong \operatorname{Hom}(H_X, H_Y)$ is an immediate consequence of Yoneda's lemma. The general case reduces to the first. In fact, if P is a proper direct factor of $\coprod_{i\in I} H_{C_i}$, then we get a corresponding idempotent in $\operatorname{End}(\coprod_{i\in I} C_i)$ which gives an object X in $\mathcal{S}$ satisfying $P \cong H_X$ since idempotents in $\mathcal{S}$ split. □

The next lemma describes some properties of pure-injective objects. The proof is essentially an application of Brown's representability theorem.

Lemma 1.3. *Let $\mathcal{S}$ be a compactly generated triangulated category, and let I be an injective object in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. Then there exists, up to isomorphism, a unique object Y in $\mathcal{S}$ such that $I \cong H_Y$. Moreover, the map*

$$\operatorname{Hom}(X, Y) \longrightarrow \operatorname{Hom}(H_X, H_Y), \quad \alpha \mapsto H_\alpha,$$

is an isomorphism for all X in $\mathcal{S}$.

Proof. See Lemma 1.7 in [17]. □

Injective envelopes. We shall also need to use the fact that $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ is a *Grothendieck category*, which as far as we are concerned means that it has injective envelopes [11]. The definition of an injective envelope can be reformulated as follows.

Lemma 1.4. *A monomorphism $\alpha\colon X \to Y$ is an injective envelope of X if and only if Y is an injective object and every endomorphism $\beta\colon Y \to Y$ satisfying $\beta \circ \alpha = \alpha$ is an isomorphism.*

Brown representability. Sometimes we shall use an additional assumption on the category $\mathcal{S}$. To this end recall that a functor from a triangulated category to the category of abelian groups is *exact* if it sends triangles to exact sequences. For example, every functor of the form H_X is exact. In some cases also the converse is true. More precisely, one says that *Brown representability* holds for $\mathcal{S}$, if

- every exact functor $\mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ is isomorphic to H_X for some object X in $\mathcal{S}$, and
- every natural transformation $H_X \to H_Y$ is of the form H_α for some map $\alpha\colon X \to Y$.

A classical theorem due to Brown and Adams states that Brown representability holds for the stable homotopy category [1]. More recently, Beligiannis, Christensen, Keller, and Neeman studied the problem when Brown representability holds [4, 5].

Flat functors. Recall that there exists a *tensor product*

$$(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \times (\mathcal{F}, \mathrm{Ab}) \longrightarrow \mathrm{Ab}, \quad (F, G) \to F \otimes_{\mathcal{F}} G$$

where for any functor $F\colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$, the tensor functor $F \otimes_{\mathcal{F}} -$ is determined by the fact that it preserves colimits and $F \otimes_{\mathcal{F}} \operatorname{Hom}(X, -) \cong F(X)$ for all X in $\mathcal{F}$; see for example [20]. A functor $F\colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ is *flat* if the tensor functor $F \otimes_{\mathcal{F}} -$ is exact. The following well-known characterization will be needed.

Lemma 1.5. *For a functor $F\colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ the following are equivalent:*

1. *F is flat.*
2. *F is an exact functor.*
3. *F is a filtered colimit of representable functors.*

Proof. For the characterization of flatness via condition (2), see Lemma 2.7 in [17], for (3), see Theorem 3.2 in [21]. □

Remark 1.6. The statement of Lemma 1.5 remains true if $\mathcal{F}$ is replaced by any skeletally small triangulated category.

Finitely presented functors. Some of our constructions involve finitely presented functors. Let us recall that a functor $F\colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ is *finitely presented* if there exists an exact sequence

$$\mathrm{Hom}(-,C) \longrightarrow \mathrm{Hom}(-,D) \longrightarrow F \longrightarrow 0$$

with C and D in $\mathcal{F}$. We wish to distinguish bewteen finitely presented and coherent functors. Both are by definition cokernels of maps between representable functors. However, for coherent functors $\mathcal{S} \to \mathrm{Ab}$ we restrict ourselves to representable functors which are represented by compact objects.

The concept of a finitely presented functor generalizes the concept of a finitely presented module, and we shall use a few basic facts about finitely presented functors which are well-known in the context of modules over a ring. For instance, every additive functor $F\colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ is a filtered colimit of finitely presented functors. The following characterization is another example.

Lemma 1.7. *For an additive functor $F\colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ the following are equivalent:*

(1) *F is finitely presented.*
(2) *The representable functor $\mathrm{Hom}(F,-)$ preserves filtered colimits.*
(3) *The tensor functor $F \otimes_{\mathcal{F}} -$ preserves products.*
(4) *The map $F \otimes_{\mathcal{F}} (\prod_i \mathrm{Hom}(C_i,-)) \to \prod_i (F \otimes_{\mathcal{F}} \mathrm{Hom}(C_i,-))$ is an isomorphism for every family $(C_i)_{i\in I}$ in $\mathcal{F}$.*

Proof. Adapt the proof for modules over a ring (cf. [22]). □

2. Weak colimits

A *diagram* in a category $\mathcal{C}$ is a functor $\mathcal{I} \to \mathcal{C}$, $i \mapsto X_i$, from a small category $\mathcal{I}$ to $\mathcal{C}$. We denote such a diagram by $(X_i)_{i\in\mathcal{I}}$ and call a family of maps $\mu_i\colon X_i \to X$ $(i \in \mathcal{I})$ a *cone* if $\mu_j \circ X_\lambda = \mu_i$ for every map $\lambda\colon i \to j$ in $\mathcal{I}$.

Definition 2.1. Let $\mu_i\colon X_i \to X$ $(i \in \mathcal{I})$ be a cone of a diagram $(X_i)_{i\in\mathcal{I}}$.

(1) The cone is a *weak colimit* of the diagram $(X_i)_{i\in\mathcal{I}}$ if for every cone $\nu_i\colon X_i \to Y$ $(i \in \mathcal{I})$ there exists a map $\alpha\colon X \to Y$ such that $\alpha \circ \mu_i = \nu_i$ for each $i \in \mathcal{I}$.
(2) The cone is *minimal* if every endomorphism $\alpha\colon X \to X$ satisfying $\alpha \circ \mu_i = \mu_i$ for each $i \in \mathcal{I}$ is an isomorphism.

If we require the factorization $\alpha\colon X \to Y$ in the definition of a weak colimit to be unique, this is the definition of a *colimit* which we denote by $\mathrm{colim}_{i\in\mathcal{I}} X_i$. Note that every colimit is a minimal weak colimit. A minimal weak colimit of a diagram $(X_i)_{i\in\mathcal{I}}$ is unique up to a (non-unique) isomorphism. Our terminology is borrowed from Auslander [3]. He calls a map $\alpha\colon X \to Y$ *left minimal* if every endomorphism $\beta\colon Y \to Y$ satisfying $\beta \circ \alpha = \alpha$ is an isomorphism. Viewing a cone of a diagram $\mathcal{I} \to \mathcal{C}$ as a map in the category of all functors $\mathcal{I} \to \mathcal{C}$, it is clear that this map is left minimal if and only if the cone is minimal.

Definition 2.2. Let $(X_i)_{i\in\mathcal{I}}$ be a diagram in a compactly generated triangulated category $\mathcal{S}$. A cone $X_i \to X$ $(i \in \mathcal{I})$ is called a *homology colimit* of the diagram $(X_i)_{i\in\mathcal{I}}$ if the induced map

$$\mathrm{colim}_{i\in\mathcal{I}} \mathrm{Hom}(C,X_i) \longrightarrow \mathrm{Hom}(C,X)$$

is an isomorphism for every compact object C in $\mathcal{S}$.

Note that a homology colimit of a diagram $(X_i)_{i\in\mathcal{I}}$ is minimal and therefore unique up to a (non-unique) isomorphism; it is denoted by $\mathrm{hcolim}_{i\in\mathcal{I}} X_i$. Our terminology is justified by the following observation.

Proposition 2.3. *A cone $X_i \to X$ $(i \in \mathcal{I})$ is a homology colimit if and only if for every exact and coproduct preserving functor $H\colon \mathcal{S} \to \mathrm{Ab}$ the induced map $\operatorname{colim}_i H(X_i) \to H(X)$ is an isomorphism.*

Proof. One direction is clear. Therefore suppose that the cone $X_i \to X$ $(i \in \mathcal{I})$ is a homology colimit and fix an exact and coproduct preserving functor $H\colon \mathcal{S} \to \mathrm{Ab}$. The restriction $H|_{\mathcal{F}}$ is exact and therefore a filtered colimit of representable functors by Lemma 1.5, that is

$$\operatorname{colim}_{j\in\mathcal{J}} \operatorname{Hom}(C_j, -)|_{\mathcal{F}} \cong H|_{\mathcal{F}}.$$

We obtain a filtered diagram of representable functors $(\operatorname{Hom}(C_j, -))_{j\in\mathcal{J}}$ and a compatible set of maps $\operatorname{Hom}(C_j, -) \to H$ $(j \in \mathcal{J})$, using Yoneda's lemma. This induces a functorial isomorphism $\operatorname{colim}_{j\in\mathcal{J}} \operatorname{Hom}(C_j, X) \cong H(X)$ for all X in $\mathcal{S}$ because both sides are exact, agree on $\mathcal{F}$, and preserve coproducts (cf. [17, Proposition 3.2]). We obtain the following commutative diagram:

$$\begin{array}{ccccc}
\operatorname{colim}_i H(X_i) & \cong & \operatorname{colim}_i \operatorname{colim}_j \operatorname{Hom}(C_j, X_i) & \cong & \operatorname{colim}_j \operatorname{colim}_i \operatorname{Hom}(C_j, X_i) \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\
H(X) & \cong & \operatorname{colim}_j \operatorname{Hom}(C_j, X) & = & \operatorname{colim}_j \operatorname{Hom}(C_j, X)
\end{array}$$

The map γ is the colimit of isomorphisms by our assumption on the cone, and we conclude that α is an isomorphism. This completes the proof. □

In [19], Margolis discusses weak colimits for the stable homotopy category, and there is also a more recent treatment in [13]. However, the definitions of a minimal weak colimit in [19] and [13] are more restrictive than the one given here. Note that our Proposition 2.3 generalizes Proposition 2.2.2 in [13].

Given a diagram $(X_i)_{i\in\mathcal{I}}$ in a compactly generated triangulated category, a weak colimit always exists. In fact, a weak colimit can be computed by taking the cofiber of an appropriate map

$$\coprod_{\lambda\colon i\to j} X_i \longrightarrow \coprod_k X_k$$

where $\lambda\colon i \to j$ runs through all maps and k runs through all objects in $\mathcal{I}$. The following result is essentially due to Margolis [19], and closely related to Theorem 4.2.3 in [13].

Proposition 2.4. *Let $\mathcal{S}$ be a compactly generated triangulated category and suppose that Brown representability holds. Then every filtered diagram of pure-projective objects in $\mathcal{S}$ has a homology colimit which is also a minimal weak colimit.*

Proof. Let $(X_i)_{i\in\mathcal{I}}$ be a filtered diagram of pure-projective objects. The functor $\operatorname{colim}_i H_{X_i}$ is exact and therefore isomorphic to H_X for some X in $\mathcal{S}$ since we assume Brown representability. Using Lemma 1.2 and the fact that each X_i is pure-projective, we get a family of maps $X_i \to X$ $(i \in \mathcal{I})$ which is a cone for $(X_i)_{i\in\mathcal{I}}$. Moreover, this cone is a homology colimit by construction. In order to show that the cone is a weak colimit, let $X_i \to Y$ $(i \in \mathcal{I})$ be another cone. Using Brown representability again, the induced map $H_X \cong \operatorname{colim}_i H_{X_i} \to H_Y$ is of the form H_α for some $\alpha\colon X \to Y$. The map α is compatible with the structural maps $X_i \to X$ and $X_i \to Y$ by Lemma 1.2, and therefore $X_i \to X$ $(i \in \mathcal{I})$ is a weak colimit. □

Next we collect a few basic facts about the existence of minimal weak colimits for arbitrary diagrams.

Lemma 2.5. *Let $(X_i)_{i\in\mathcal{I}}$ be a diagram in an additive category $\mathcal{C}$ and suppose that idempotents in $\mathcal{C}$ split. Let $X_i \to X$ $(i \in \mathcal{I})$ be a weak colimit and denote by M the image of the induced map $\operatorname{Hom}(X,X) \to \prod_i \operatorname{Hom}(X_i,X)$. Then the following conditions are equivalent:*

(1) *The $\operatorname{End}(X)$-module M has a projective cover.*
(2) *The diagram $(X_i)_{i\in\mathcal{I}}$ has a minimal weak colimit.*

Moreover, in this case $X_i \to X$ $(i \in \mathcal{I})$ is minimal if and only if the canonical map $\operatorname{Hom}(X,X) \to M$ is a projective cover.

Proof. The proof is straightforward if one observes that an epimorphism $\pi\colon P \to M$ with P projective is a projective cover of M if and only if every endomorphism $\varepsilon\colon P \to P$ satisfying $\pi \circ \varepsilon = \pi$ is an isomorphism. □

Let X be an object in an additive category and suppose that idempotents split. Then every finitely generated $\operatorname{End}(X)$-module has a projective cover if and only if X decomposes into finitely many indecomposable objects with local endomorphism rings. Using this elementary fact, one can prove the following.

Proposition 2.6. *Every finite diagram of compact objects in the category of p-local spectra has a minimal weak colimit.*

Proof. Every compact p-local spectrum X decomposes into finitely many indecomposable objects with local endomorphism rings (cf. [10]). The assertions is therefore a consequence of Lemma 2.5 because every finitely generated $\operatorname{End}(X)$-module has a projective cover. □

Another method to produce minimal weak colimits is to construct appropriate injective envelopes.

Proposition 2.7. *A diagram in a compactly generated triangulated category has a minimal weak colimit provided there exist a weak colimit which is pure-injective.*

Proof. Let $\mu_i\colon X_i \to X$ $(i \in \mathcal{I})$ be a cone of some diagram and suppose that X is pure-injective. It follows from Lemma 1.3 and the characterization of injective envelopes in Lemma 1.4 that the cone is minimal if and only if for $\mu\colon \coprod_i X_i \to X$ the induced map $\operatorname{Im} H_\mu \to H_X$ is an injective envelope. Now suppose that the above cone is a weak colimit. Taking an injective envelope $\operatorname{Im} H_\mu \to H_Y$ produces a new cone $\mu_i\colon X_i \to Y$ $(i \in \mathcal{I})$ which is a minimal weak colimit. □

We end this section with a characterization of pure triangles. Let us call a triangle $X \to Y \to Z \to \Sigma X$ *split* if the identity map id_Z factors through $Y \to Z$, equivalently, if id_X factors through $X \to Y$.

Lemma 2.8. *The following are equivalent for a triangle $\delta\colon X \to Y \to Z \to \Sigma X$ in a compactly generated triangulated category:*

(1) *δ is pure.*
(2) *δ is a homology colimit of a filtered diagram of split triangles of compact objects.*
(3) *δ is a homology colimit of a filtered diagram of split triangles.*

Proof. (1) $\Rightarrow$ (2) Given an object X in $\mathcal{S}$, the functor $H_X\colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ is exact and therefore flat by Lemma 1.5. A well-known consequence of this is the fact that the category $\mathcal{I}_X$ whose objects are the maps $X_i \to X$ with X_i compact and whose maps are the obvious commuting triangles forms a small filtered category with $\operatorname{hcolim}_{i\in\mathcal{I}_X} X_i = X$

(cf. [21, Theorem 3.2]). Analogously, one shows that for any map $\beta\colon Y \to Z$ in $\mathcal{S}$ the category $\mathcal{I}_\beta$ whose objects are the commuting squares

$$\begin{array}{ccc} Y_i & \xrightarrow{\beta_i} & Z_i \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\beta} & Z \end{array}$$

with Y_i and Z_i compact and whose maps $i \to j$ are the obvious commuting squares

$$\begin{array}{ccc} Y_i & \xrightarrow{\beta_i} & Z_i \\ \downarrow & & \downarrow \\ Y_j & \xrightarrow{\beta_j} & Z_j \end{array}$$

form a small filtered category with $\operatorname{hcolim}_{i\in\mathcal{I}_\beta} \beta_i = \beta$.

Now suppose that the triangle $\delta\colon X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ is pure. It follows that the commuting squares $i \in \mathcal{I}_\beta$ with β_i a split epimorphism form a cofinal subcategory of $\mathcal{I}_\beta$ which we denote by $\mathcal{I}$. In fact, every commuting square $i \in \mathcal{I}_\beta$ fits into a commutative diagram of the form

$$\begin{array}{ccc} Y_i & \xrightarrow{\beta_i} & Z_i \\ \downarrow & & \| \\ Y_i \coprod Z_i & \xrightarrow{\beta_i'} & Z_i \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\beta} & Z \end{array}$$

where the second component of β_i' is the identity since the map $Z_i \to Z$ factors through β. We obtain a filtered diagram $(\delta_i)_{i\in\mathcal{I}}$ of split triangles $\delta_i\colon X_i \xrightarrow{\alpha_i} Y_i \xrightarrow{\beta_i} Z_i \xrightarrow{\gamma_i} \Sigma X_i$ and one checks easily that the commuting squares corresponding to the α_i and γ_i with $i \in \mathcal{I}$ form cofinal subcategories of $\mathcal{I}_\alpha$ and $\mathcal{I}_\gamma$, respectively. We conclude that $\operatorname{hcolim}_{i\in\mathcal{I}} \alpha_i = \alpha$, $\operatorname{hcolim}_{i\in\mathcal{I}} \beta_i = \beta$, and $\operatorname{hcolim}_{i\in\mathcal{I}} \gamma_i = \gamma$. Thus $\delta = \operatorname{hcolim}_{i\in\mathcal{I}} \delta_i$.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) Suppose that $\delta = \operatorname{hcolim}_i \delta_i$ and that each δ_i is split. A split triangle $\delta_i\colon X_i \to Y_i \to Z_i \to \Sigma X_i$ induces an exact sequence $0 \to H_{X_i} \to H_{Y_i} \to H_{Z_i} \to 0$. Taking filtered homology colimits preserves exactness and we get therefore an exact sequence $0 \to H_X \to H_Y \to H_Z \to 0$. Thus δ is pure. □

3. Weak limits

The concept of a (minimal) weak limit is the obvious analogue of a (minimal) weak colimit which one obtains by reversing all the arrows in Definition 2.1. In this section we investigate the existence of minimal weak limits. We need the following lemma.

Lemma 3.1. *Let $X \xrightarrow{\alpha} I \xrightarrow{\beta} J \xrightarrow{\gamma} \Sigma X$ be a triangle in $\mathcal{S}$ and suppose that the induced map $\operatorname{Im} H_\beta \to H_J$ is an injective envelope. Then every endomorphism $\varepsilon\colon X \to X$ satisfying $\alpha \circ \varepsilon = \alpha$ is an isomorphism.*

Proof. Choose a map $\phi\colon J \to J$ which completes the following commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\alpha} & I & \xrightarrow{\beta} & J & \xrightarrow{\gamma} & \Sigma X \\ \downarrow{\scriptstyle\varepsilon} & & \| & & & & \downarrow{\scriptstyle\Sigma\varepsilon} \\ X & \xrightarrow{\alpha} & I & \xrightarrow{\beta} & J & \xrightarrow{\gamma} & \Sigma X \end{array}$$

The assumption on ε implies that H_ϕ keeps $\operatorname{Im} H_\beta$ fixed, and therefore H_ϕ is an isomorphism by Lemma 1.4. Thus ϕ is an isomorphism, and we conclude that ε is an isomorphism. ☐

We are now in a position to prove an existence criterion for minimal weak limits.

Theorem 3.2. *Let $\mathcal{S}$ be a compactly generated triangulated category. Then every diagram of pure-injective objects in $\mathcal{S}$ has a minimal weak limit.*

Proof. The proof uses the fact that the category $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ has injective envelopes. Let $(X_i)_{i\in\mathcal{I}}$ be a diagram of pure-injective objects in $\mathcal{S}$ and let $F = \lim_i H_{X_i}$ be the corresponding limit in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. There exists a minimal injective copresentation of F which is of the form

$$0 \longrightarrow F \longrightarrow H_I \xrightarrow{H_\beta} H_J$$

by Lemma 1.3. We complete β to a triangle $X \xrightarrow{\alpha} I \xrightarrow{\beta} J \xrightarrow{\gamma} \Sigma X$. The map H_α induces a map $H_X \to F$ which we compose with the structural maps $F \to H_{X_i}$ to obtain a family of maps $\mu_i\colon X \to X_i$, using Lemma 1.3 and the fact that each X_i is pure-injective. We claim that $\mu_i\colon X \to X_i$ $(i \in \mathcal{I})$ is a minimal weak limit of the diagram $(X_i)_{i\in\mathcal{I}}$. Observe first that $X_\lambda \circ \mu_i = \mu_j$ for every map $\lambda\colon i \to j$ in $\mathcal{I}$ since $H_{X_\lambda} \circ H_{\mu_i} = H_{\mu_j}$, again by Lemma 1.3. Now suppose there is another family $\nu_i\colon Y \to X_i$ $(i \in \mathcal{I})$ of maps satisfying $X_\lambda \circ \nu_i = \nu_j$ for every map $\lambda\colon i \to j$ in $\mathcal{I}$. The family $H_Y \to H_{X_i}$ $(i \in \mathcal{I})$ induces a map $H_Y \to F$ which we compose with $F \to H_I$ to get a map $Y \to I$. The composition of this map with β is zero and therefore $Y \to I$ factors through α via some map $Y \to X$. Using again the pure-injectivity of the X_i, it is easy to check that $Y \to X$ is compatible with the strucural maps μ_i and ν_i.

It remains to show that the family $X \to X_i$ $(i \in \mathcal{I})$ is minimal. Every endomorphism $\varepsilon\colon X \to X$ which is compatible with the μ_i induces a map H_ε which is compatible with the map $H_X \to F$. Therefore $\alpha \circ \varepsilon = \alpha$, and Lemma 3.1 implies that ε is an isomorphism. This shows that the weak limit is minimal and the proof is complete. ☐

4. Extending functors

It is often useful to extend a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ to a functor $\hat{F}\colon (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ab}$ such that $\hat{F}(H_X) = F(X)$ for all X in $\mathcal{S}$. We consider a number of conditions on F which translate into properties of the functor $\hat{F}$.

- (E) F is short exact, that is, for every pure triangle $X \to Y \to Z \to \Sigma X$ in $\mathcal{S}$ the sequence $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is exact.
- (Π) $F(\prod_i X_i) \cong \prod_i F(X_i)$ for every family $(X_i)_{i\in I}$ of pure-injective objects in $\mathcal{S}$.
- (Σ) $\coprod_i F(X_i) \cong F(\coprod_i X_i)$ for every family $(X_i)_{i\in I}$ of pure-projective objects in $\mathcal{S}$.

It is sometimes convenient to work with the following variant of (Π) and (Σ), respectively.

- (Π′) $F(\prod_i X_i) \cong \prod_i F(X_i)$ for every family $(X_i)_{i\in I}$ of compact objects in $\mathcal{S}$.
- (Σ′) $\coprod_i F(X_i) \cong F(\coprod_i X_i)$ for every family $(X_i)_{i\in I}$ of compact objects in $\mathcal{S}$.

Note that (Σ') and (Σ) are equivalent since every pure-projective object is a direct factor of a coproduct of compact objects. It turns out that condition (E) is sufficient to construct a functor $\hat{F}$ which extends F.

Proposition 4.1. *Let $\mathcal{S}$ be a compactly generated triangulated category and let $F\colon \mathcal{S} \to \mathrm{Ab}$ be a short exact functor. Then there exists, up to isomorphism, a unique functor $\hat{F}\colon (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ab}$ which is left exact and extends F, that is, $\hat{F}(H_X) = F(X)$ for all X in $\mathcal{S}$. Moreover,*

(1) *if F satisfies (Π), then $\hat{F}$ preserves products, and*
(2) *if F satisfies (Σ), then $\hat{F}$ preserves filtered colimits.*

Proof. Choose for every object X in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ an injective copresentation

$$0 \longrightarrow X \longrightarrow H_I \xrightarrow{H_\alpha} H_J.$$

This is possible by Lemma 1.3. Now one defines $\hat{F}(X) = \operatorname{Ker} F(\alpha)$ and checks easily that this can be extended to maps in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ and that it is well-defined. Condition (E) implies that $\hat{F}(H_X) = F(X)$ for all X in $\mathcal{S}$. In fact, we can choose for X pure triangles

$$X \longrightarrow I_0 \longrightarrow X_1 \longrightarrow \Sigma X \quad \text{and} \quad X_1 \longrightarrow I_1 \longrightarrow X_2 \longrightarrow \Sigma X_1$$

with I_i pure-injective. This gives an injective copresentation $0 \to H_X \to H_{I_0} \to H_{I_1}$ and $F(X) = \hat{F}(H_X)$ follows since $0 \to F(X) \to F(I_0) \to F(I_1)$ is exact. Clearly, $\hat{F}$ is left exact by construction. Moreover, any left exact functor $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ab}$ is uniquely determined by by its restriction to the full subcategory of injective objects.

Suppose now that (Π) holds. This condition says that the restriction of $\hat{F}$ to the full subcategory of injectives in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ preserves products. Let $(X_i)_{i\in I}$ be a family of arbitrary objects in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ and choose injective copresentations $0 \to X_i \to I_i \to J_i$. We get the following commutative diagram with exact rows since $\hat{F}$ is left exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \hat{F}(\prod_i X_i) & \longrightarrow & \hat{F}(\prod_i I_i) & \longrightarrow & \hat{F}(\prod_i J_i) \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ 0 & \longrightarrow & \prod_i \hat{F}(X_i) & \longrightarrow & \prod_i \hat{F}(I_i) & \longrightarrow & \prod_i \hat{F}(J_i) \end{array}$$

The maps β and γ are isomorphisms and it follows that α is an isomorphism. Thus $\hat{F}$ preserve products.

Finally suppose that (Σ) holds. We construct a new functor $\check{F}\colon (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ab}$ as follows. For a finitely presented functor X in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ choose a presentation

$$\operatorname{Hom}(-, A) \xrightarrow{\operatorname{Hom}(-,\alpha)} \operatorname{Hom}(-, B) \longrightarrow X \longrightarrow 0$$

and complete α to a triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A$. We get an exact sequence

$$0 \longrightarrow X \longrightarrow \operatorname{Hom}(-, C) \xrightarrow{\operatorname{Hom}(-,\gamma)} \operatorname{Hom}(-, \Sigma A).$$

Now define $\check{F}(X) = \operatorname{Ker} F(\gamma)$. Every object X in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ can be written as a filtered colimit of finitely presented functors. More precisely, the category $\mathcal{I}_X$ whose objects are the maps $X_i \to X$ with X_i finitely presented and whose maps are the obvious commuting triangles forms a small filtered category with $\operatorname{colim}_{i\in\mathcal{I}_X} X_i = X$. One defines $\check{F}(X) = \operatorname{colim}_i \check{F}(X_i)$ and checks easily that this definition can be extended to maps in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. Clearly, $\check{F}$ preserves filtered colimits. We claim that $\check{F}$ is left exact. To see this, fix an exact sequence $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ and write $\beta = \operatorname{colim}_i \beta_i$ as a filtered colimit of maps $\beta_i \colon Y_i \to Z_i$ between finitely presented objects. We get a filtered diagram of exact sequences $0 \to X_i \xrightarrow{\alpha_i} Y_i \xrightarrow{\beta_i} Z_i$ and $0 \to \check{F}(X_i) \xrightarrow{\check{F}(\alpha_i)} \check{F}(Y_i) \xrightarrow{\check{F}(\beta_i)} \check{F}(Z_i)$ is exact by construction. Taking filtered colimits preserves exactness and it follows that $0 \to \check{F}(X) \xrightarrow{\check{F}(\alpha)} \check{F}(Y) \xrightarrow{\check{F}(\beta)} \check{F}(Z)$ is exact.

Next we use (E) and (Σ) to show that $\check{F}(H_X) = F(X)$ for every X in $\mathcal{S}$. Condition (Σ) implies that this holds if X is pure-projective. Otherwise choose for $X = X_0$ pure triangles

$$X_2 \longrightarrow P_1 \longrightarrow X_1 \longrightarrow \Sigma X_2 \quad \text{and} \quad X_1 \longrightarrow P_0 \longrightarrow X_0 \longrightarrow \Sigma X_1$$

with P_i pure-projective. Each sequence $0 \to H_{X_{i+1}} \to H_{P_i} \to H_{X_i} \to 0$ is a filtered colimit of split exact sequences of the form $0 \to H_A \to H_B \to H_C \to 0$ with A, B, C compact, by Lemma 2.8. Thus $0 \to \check{F}(H_{X_{i+1}}) \to \check{F}(H_{P_i}) \to \check{F}(H_{X_i}) \to 0$ is exact and therefore $\check{F}(H_{P_1}) \to \check{F}(H_{P_0}) \to \check{F}(H_X) \to 0$ is exact. On the other hand, $F(P_1) \to F(P_0) \to F(X) \to 0$ is exact by (E), and therefore $\check{F}(H_X) = F(X)$. It follows that $\check{F}$ and $\hat{F}$ are isomorphic, and therefore $\hat{F}$ preserves filtered colimits. □

The preceding proposition has an analogue for right exact functors $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ab}$ extending a functor $F \colon \mathcal{S} \to \mathrm{Ab}$. The construction uses projective presentations instead of injective copresentations.

Proposition 4.2. *Let $\mathcal{S}$ be a compactly generated triangulated category and let $F \colon \mathcal{S} \to \mathrm{Ab}$ be a short exact functor. Then there exists, up to isomorphism, a unique functor $\check{F} \colon (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ab}$ which is right exact and extends F, that is, $\check{F}(H_X) = F(X)$ for all X in $\mathcal{S}$. Moreover, if F satisfies (Σ), then $\check{F}$ preserves coproducts and is isomorphic to $- \otimes_{\mathcal{F}} G$ where $G = F|_{\mathcal{F}}$.*

Proof. The proof is analogous to that of Proposition 4.1 and we leave the details to the reader. The last assertion about $- \otimes_{\mathcal{F}} G$ follows from the fact that $\check{F}(H_C) = F(C) = H_C \otimes_{\mathcal{F}} G$ for each $C \in \mathcal{F}$ and every $X \in (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ has a presentation

$$\coprod_j H_{D_j} \longrightarrow \coprod_i H_{C_i} \longrightarrow X \longrightarrow 0$$

with $C_i, D_j \in \mathcal{F}$ for all i, j. □

5. Coherent functors

We are now in a position to prove the first portion of our characterization of coherent functors.

Proposition 5.1. *Let $\mathcal{S}$ be a compactly generated triangulated category. For a functor $F \colon \mathcal{S} \to \mathrm{Ab}$ the following conditions are equivalent:*

1. *F is coherent.*
2. *F preserves products and sends every homology colimit to a colimit.*
3. *F preserves products and coproducts, and F is short exact.*
4. *F satisfies (E), (Π), and (Σ).*

Proof. (1) ⇒ (2) Each representable functor $\mathrm{Hom}(C, -)$ with C compact preserves products and sends every homology colimit to a colimit by the definition of a homology colimit. Clearly, this property is preserved if we pass to the cokernel of a map $\mathrm{Hom}(D, -) \to \mathrm{Hom}(C, -)$. Thus (2) holds for every coherent functor F.

(2) ⇒ (3) Suppose that F preserves homology colimits. It follows that F preserves coproducts because every coproduct in $\mathcal{S}$ is a homology colimit. Now suppose that

$\delta\colon X \to Y \to Z \to \Sigma X$ is a pure triangle. It has been shown in Lemma 2.8 that δ is a homology colimit of split triangles $\delta_i\colon X_i \to Y_i \to Z_i \to \Sigma X_i$. Clearly, each sequence $0 \to F(X_i) \to F(Y_i) \to F(Z_i) \to 0$ is exact since F is additive, and therefore the colimit $0 \to \operatorname{colim}_i F(X_i) \to \operatorname{colim}_i F(Y_i) \to \operatorname{colim}_i F(Z_i) \to 0$ is exact. However, this sequence is isomorphic to $0 \to F(X) \to F(Y) \to F(Z) \to 0$ by our assumption on F. This proves (3).

(3) $\Rightarrow$ (4) Clear.

(4) $\Rightarrow$ (1) We apply Proposition 4.1 to get a functor $\hat{F}\colon (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ab}$ which is left exact and extends F. Moreover, (Π) and (Σ) imply that $\hat{F}$ preserves products and filtered colimits. It follows that $\hat{F}$ preserves limits since every limit can be computed by taking kernels and products. Therefore the Adjoint Functor Theorem implies the existence of a left adjoint $G\colon \mathrm{Ab} \to (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ for $\hat{F}$. This gives for X in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ a functorial isomorphism

$$\hat{F}(X) \cong \mathrm{Hom}(\mathbb{Z}, \hat{F}(X)) \cong \mathrm{Hom}(G(\mathbb{Z}), X).$$

The criterion of Lemma 1.7 implies that $G(\mathbb{Z})$ is a finitely presented functor since $\hat{F}$ preserves filtered colimits. Choose a presentation

$$\mathrm{Hom}(-, A) \overset{\mathrm{Hom}(-,\alpha)}{\longrightarrow} \mathrm{Hom}(-, B) \longrightarrow G(\mathbb{Z}) \longrightarrow 0.$$

Applying $\mathrm{Hom}(-, H_X)$ gives an exact sequence

$$0 \longrightarrow \mathrm{Hom}(G(\mathbb{Z}), H_X) \longrightarrow \mathrm{Hom}(\mathrm{Hom}(-, B), H_X) \longrightarrow \mathrm{Hom}(\mathrm{Hom}(-, A), H_X)$$

which is isomorphic to

$$0 \longrightarrow F(X) \longrightarrow \mathrm{Hom}(B, X) \overset{\mathrm{Hom}(\alpha,X)}{\longrightarrow} \mathrm{Hom}(A, X)$$

thanks to Yoneda's lemma and the isomorphism $F(X) \cong \mathrm{Hom}(G(\mathbb{Z}), H_X)$. This sequence is functorial in X, and if we complete $\alpha\colon A \to B$ to a triangle $A \to B \to C \to \Sigma A$, we get the desired presentation

$$\mathrm{Hom}(\Sigma A, -) \longrightarrow \mathrm{Hom}(C, -) \longrightarrow F \longrightarrow 0$$

which shows that F is coherent. ∎

Proposition 5.2. *Let $\mathcal{S}$ be a compactly generated triangulated category. For a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ the following conditions are equivalent:*

(1) *F is coherent.*
(2) *F satisfies* (E), (Π'), *and* (Σ').

Proof. (1) $\Rightarrow$ (2) is shown in Proposition 5.1. Therefore suppose that F satisfies (E), (Π'), and (Σ'). Using conditions (E) and (Σ'), we can apply Proposition 4.2 and extend F to a functor $\check{F}\colon (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ab}$ which is isomorphic to $- \otimes_{\mathcal{F}} G$ for $G = F|_{\mathcal{F}}$. In particular, $F(X) \cong H_X \otimes_{\mathcal{F}} G$ for all $X \in \mathcal{S}$. We claim that G is finitely presented. In fact, this follows from Lemma 1.7 and condition (Π') since we have for every family $(C_i)_{i\in I}$ in $\mathcal{F}$

$$(\prod_i H_{C_i}) \otimes_{\mathcal{F}} G \cong H_{\prod_i C_i} \otimes_{\mathcal{F}} G \cong F(\prod_i C_i) \cong \prod_i F(C_i) \cong \prod_i (H_{C_i} \otimes_{\mathcal{F}} G).$$

Tensoring a presentation $\mathrm{Hom}(D, -) \to \mathrm{Hom}(C, -) \to G \to 0$ with H_X for $X \in \mathcal{S}$ gives an exact sequence

$$H_X \otimes_{\mathcal{F}} \mathrm{Hom}(D, -) \longrightarrow H_X \otimes_{\mathcal{F}} \mathrm{Hom}(C, -) \longrightarrow H_X \otimes_{\mathcal{F}} G \longrightarrow 0$$

which is isomorphic to

$$\mathrm{Hom}(D,X) \longrightarrow \mathrm{Hom}(C,X) \longrightarrow F(X) \longrightarrow 0.$$

This sequence is functorial in X and therefore F is coherent. This completes the proof. □

The next proposition completes our characterization of coherent functors.

Proposition 5.3. *Let $\mathcal{S}$ be a compactly generated triangulated category and suppose that Brown representability holds. For a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ the following conditions are equivalent:*

1. *F is coherent.*
2. *F preserves products of families $(X_i)_{i\in I}$ and minimal weak colimits of filtered diagrams $(X_j)_{j\in\mathcal{J}}$ provided that each X_i and each X_j is a direct factor of a coproduct of compact objects.*
3. *F preserves products of families $(X_i)_{i\in I}$ and minimal weak colimits of filtered diagrams $(X_j)_{j\in\mathcal{J}}$ provided that each X_i and each X_j is a compact object.*
4. *F satisfies* (E), (Π'), *and* (Σ').

Proof. (1) ⇒ (2) A coherent functor preserves products and sends homology colimits to colimits by Proposition 5.1. Every minimal weak colimit of a filtered diagram of pure-projective objects in $\mathcal{S}$ is also a homology colimit by Proposition 2.4. Therefore (1) implies (2).

(2) ⇒ (3) Clear.

(3) ⇒ (4) We check that F satisfies the conditions (E) and (Σ').

(E) It has been shown in Lemma 2.8 that a pure triangle $\delta\colon X \to Y \to Z \to \Sigma X$ can be expressed as a homology colimit of a diagram $(\delta_i)_{i\in\mathcal{I}}$ of split triangles $\delta_i\colon X_i \to Y_i \to Z_i \to \Sigma X_i$ of compact objects. Note that δ is also a minimal weak colimit of the diagram $(\delta_i)_{i\in\mathcal{I}}$ by Proposition 2.4. Applying F gives a filtered diagram of exact sequences $0 \to F(X_i) \to F(Y_i) \to F(Z_i) \to 0$. The colimit of these exact sequences is again exact and isomorphic to the sequence $0 \to F(X) \to F(Y) \to F(Z) \to 0$, by our assumptions on the functor F.

(Σ') The coproduct of a family $(X_i)_{i\in I}$ is the filtered colimit of the finite coproducts $\coprod_{i\in J} X_i$ where J runs through all finite subsets of I. Note that $\coprod_{i\in J} X_i = \prod_{i\in J} X_i$ if J is finite. Thus (3) implies that F preserves coproducts of compact objects in $\mathcal{S}$.

(4) ⇒ (1) See Proposition 5.2. □

6. Short exact functors

In this section we study some properties of short exact functors. Recall that a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ is *short exact* if for every triangle $X \to Y \to Z \to \Sigma X$ the sequence $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is exact whenever $0 \to \mathrm{Hom}(C,X) \to \mathrm{Hom}(C,Y) \to \mathrm{Hom}(C,Z) \to 0$ is exact for every compact C.

Theorem 6.1. *For a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ the following conditions are equivalent:*

1. *F is short exact and preserves coproducts.*
2. *There exists an additive functor $G\colon \mathcal{F} \to \mathrm{Ab}$ and a functorial isomorphism $F(X) \cong H_X \otimes_{\mathcal{F}} G$ for all X in $\mathcal{S}$.*
3. *There exists a filtered diagram $(F_i)_{i\in\mathcal{I}}$ of coherent functors and a functorial isomorphism $F(X) \cong \mathrm{colim}_i\, F_i(X)$ for all X in $\mathcal{S}$.*

Proof. (1) ⇒ (2) Let $G = F|_{\mathcal{F}}$. Condition (1) implies that F extends to a functor $\check{F}\colon (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ab}$ which is isomorphic to $- \otimes_{\mathcal{F}} G$ by Proposition 4.2. Therefore

$$F(X) \cong \check{F}(H_X) \cong H_X \otimes_{\mathcal{F}} G$$

for all X in $\mathcal{S}$.

(2) ⇒ (3) Suppose that $F(X) \cong H_X \otimes_{\mathcal{F}} G$ for some functor $G\colon \mathcal{F} \to \mathrm{Ab}$. Writing $G = \mathrm{colim}_i\, G_i$ as a filtered colimit of finitely presented functors, we get a filtered diagram of coherent functors $F_i\colon \mathcal{S} \to \mathrm{Ab}$ if we define $F_i(X) = H_X \otimes_{\mathcal{F}} G_i$ for each i. This gives an isomorphism

$$F(X) \cong H_X \otimes_{\mathcal{F}} (\mathrm{colim}_i\, G_i) \cong \mathrm{colim}_i (H_X \otimes_{\mathcal{F}} G_i) = \mathrm{colim}_i\, F_i(X)$$

since $H_X \otimes_{\mathcal{F}} -$ preserves colimits.

(3) ⇒ (1) A coherent functor is short exact and preserves coproducts by Proposition 5.1. Taking filtered colimits preserves exactness and coproducts, and therefore a filtered colimit of coherent functors is short exact and preserves coproducts. □

A short exact functor kills phantom maps, and we have the converse if the functor is exact.

Theorem 6.2. *Let $F\colon \mathcal{S} \to \mathrm{Ab}$ be a functor.*

1. *If F is short exact, then $F(\alpha) = 0$ for every phantom map α.*
2. *Suppose F is exact. Then F is short exact if and only if $F(\alpha) = 0$ for every phantom map α.*

Proof. Apply Proposition 4.1. □

It is not true in general that a functor which kills phantom maps is short exact. Take for instance an object Z in $\mathcal{S}$ which is not pure-projective, and let $F(X) = \mathrm{Hom}(H_Z, H_X)$ for X in $\mathcal{S}$. Clearly, $F(\alpha) = 0$ for every phantom map α. However, if $X \to Y \to Z \to \Sigma X$ is a pure triangle with Y pure-projective, the sequence $0 \to F(X) \to F(Y) \to F(Z) \to 0$ cannot be exact.

7. Definable subcategories

In this section we use coherent functors to study certain subcatgories of $\mathcal{S}$.

Definition 7.1. We call a full subctegory $\mathcal{C}$ of $\mathcal{S}$ *definable* if it is of the form

$$\mathcal{C} = \{X \in \mathcal{S} \mid F_i(X) = 0 \text{ for all } i \in I\}$$

for some family $(F_i)_{i \in I}$ of coherent functors.

There are three other concepts equivalent to definable subcategories:

- Ziegler-closed subsets of the set $\mathrm{Sp}\, \mathcal{S}$ of indecomposable pure-injectives in $\mathcal{S}$,
- Serre subcategories of $\mathrm{Coh}\, \mathcal{S}$, and
- cohomological ideals in $\mathcal{F}$.

We refer to the introduction for precise definitions and the statement of the 'fundamental correspondence' which relates these concept to each other. Here, we use the functor category $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ to prove this correspondence. We start with some preparations.

Let $\mathrm{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ be the full subcategory formed by the finitely presented functors in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. Note that $\mathrm{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ is abelian since $\mathcal{F}$ has weak kernels. Given a functor $F\colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$, we define a functor $F^{\vee}\colon \mathcal{S} \to \mathrm{Ab}$ by

$$F^{\vee}(X) = \mathrm{Hom}(F, H_X).$$

Lemma 7.2. *The assignment $F \mapsto F^\vee$ induces an equivalence*

$$(\mathrm{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}))^{\mathrm{op}} \longrightarrow \mathrm{Coh}\,\mathcal{S}.$$

Proof. Let $F \in \mathrm{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ and fix a presentation

$$\mathrm{Hom}(-,A) \longrightarrow \mathrm{Hom}(-,B) \longrightarrow F \longrightarrow 0.$$

Completing the map $A \to B$ to a triangle $A \to B \to C \to \Sigma A$, it follows from Yoneda's lemma that we get a presentation

$$\mathrm{Hom}(\Sigma A,-) \longrightarrow \mathrm{Hom}(C,-) \longrightarrow F^\vee \longrightarrow 0.$$

Thus $F^\vee$ is coherent. It is now straightforward to write down an inverse for $F \mapsto F^\vee$. □

We denote by $\mathrm{Spec}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ the set of isomorphism classes of indecomposable injective objects in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. A subset of $\mathrm{Spec}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ is *closed* if it is of the form

$$\{X \in \mathrm{Spec}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \mid \mathrm{Hom}(F_i, X) = 0 \text{ for all } i \in I\}$$

for some family $(F_i)_{i\in I}$ of finitely presented functors $\mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$.

Proposition 7.3. *The assignments*

$$\mathbf{U} \mapsto \{F \in \mathrm{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \mid \mathrm{Hom}(F,X) = 0 \textit{ for all } X \in \mathbf{U}\} \textit{ and}$$

$$\mathcal{T} \mapsto \{X \in \mathrm{Spec}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \mid \mathrm{Hom}(F,X) = 0 \textit{ for all } F \in \mathcal{T}\}$$

induce mutually inverse bijections between the set of closed subsets of $\mathrm{Spec}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ *and the set of Serre subcategories of* $\mathrm{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$*.*

Proof. See Theorem 4.2 in [15]. □

Given an object X in $\mathcal{S}$, we consider the *annihilator*

$$\mathrm{Ann}\,X = \{\phi \in \mathcal{F} \mid \mathrm{Hom}(\phi, X) = 0\}.$$

Clearly, $\mathrm{Ann}\,X$ is a cohomological ideal in $\mathcal{F}$, and the converse is also true.

Proposition 7.4. *Every cohomological ideal in $\mathcal{F}$ is of the form $\mathrm{Ann}\,X$ for some pure-injective object X in $\mathcal{S}$.*

Proof. We fix a cohomological ideal $\mathfrak{I}$. By definition, there exists an exact functor $F\colon \mathcal{F} \to \mathrm{Ab}$ such that $\mathfrak{I} = \{\phi \in \mathcal{F} \mid F(\phi) = 0\}$. The functor $-\otimes_{\mathcal{F}} F\colon (\mathcal{F}^{\mathrm{op}}, Ab) \to \mathrm{Ab}$ is exact by Lemma 1.5, and we obtain therefore a Serre subcategory of $\mathrm{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ by taking

$$\mathcal{T} = \{G \in \mathrm{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \mid G \otimes_{\mathcal{F}} F = 0\}.$$

Now let I be the product of all $Y \in \mathrm{Spec}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ such that $\mathrm{Hom}(G,Y) = 0$ for all $G \in \mathcal{T}$. The correspondence in Proposition 7.3 implies

$$\mathcal{T} = \{G \in \mathrm{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \mid \mathrm{Hom}(G,I) = 0\},$$

and we find $X \in \mathcal{S}$ with $H_X \cong I$ by Lemma 1.3. Now let ϕ be an arbitrary map in $\mathcal{F}$ and put $G = \mathrm{Im}\,H_\phi$. We get

$$F(\phi) = 0 \;\Leftrightarrow\; G \otimes_{\mathcal{F}} F = 0 \;\Leftrightarrow\; \mathrm{Hom}(G, H_X) = 0 \;\Leftrightarrow\; \mathrm{Hom}(\phi, X) = 0 \;\Leftrightarrow\; \phi \in \mathrm{Ann}\,X.$$

Thus $\mathfrak{I}$ is of the form $\mathrm{Ann}\,X$. □

Proof of the Fundamental Correspondence. The assignment $X \mapsto H_X$ identifies the pure-injective objects in $\mathcal{S}$ with the injective objects in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ (cf. [17, Corollary 1.9]) and induces therefore a bijection $\mathrm{Sp}\,\mathcal{S} \to \mathrm{Spec}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ which identifies the Ziegler closed subsets of $\mathrm{Sp}\,\mathcal{S}$ with the closed subsets of $\mathrm{Spec}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ by Lemma 7.2. We conclude from Proposition 7.3 that

$$\mathbf{U} \mapsto \{F \in \mathrm{Coh}\,\mathcal{S} \mid F(X) = 0 \text{ for all } X \in \mathbf{U}\} \text{ and}$$

$$\mathcal{T} \mapsto \{X \in \mathrm{Sp}\,\mathcal{S} \mid F(X) = 0 \text{ for all } F \in \mathcal{T}\}$$

induce mutually inverse bijections between Ziegler-closed subsets of $\mathrm{Sp}\,\mathcal{S}$ and Serre subcategories of $\mathrm{Coh}\,\mathcal{S}$. It is an immediate consequence that

$$\mathcal{C} \mapsto \{F \in \mathrm{Coh}\,\mathcal{S} \mid F(X) = 0 \text{ for all } X \in \mathcal{C}\} \text{ and}$$

$$\mathcal{T} \mapsto \{X \in \mathcal{S} \mid F(X) = 0 \text{ for all } F \in \mathcal{T}\}$$

induce mutually inverse bijections between definable subcategories of $\mathcal{S}$ and Serre subcategories of $\mathrm{Coh}\,\mathcal{S}$. In other words: a definable subcategory $\mathcal{C}$ is already determined by $\mathcal{C} \cap \mathrm{Sp}\,\mathcal{S}$. In fact, each definable subcategory $\mathcal{C}$ can be reconstructed explicitly from the corresponding Ziegler-closed subset $\mathbf{U} = \mathcal{C} \cap \mathrm{Sp}\,\mathcal{S}$ since

$$\mathcal{C} = \{X \in \mathcal{S} \mid \text{there are } Y_i \in \mathbf{U} \text{ and a pure triangle } X \to \textstyle\prod_i Y_i \to Z \to \Sigma X\}.$$

This follows from Proposition 3.2 in [15].

Next we consider the cohomological ideals. Observe that a functor $F\colon \mathcal{S} \to \mathrm{Ab}$ is coherent precisely if $F = \mathrm{Im}\,\mathrm{Hom}(\phi, -)$ for some map $\phi\colon C \to D$ in $\mathcal{F}$. Clearly, $F(X) = 0$ for some X in $\mathcal{S}$ if and only if $\phi \in \mathrm{Ann}\,X$. Using the correspondence between definable subcategories and Serre subcategories of coherent functors, it follows that

$$\mathcal{C} \mapsto \bigcap_{X \in \mathcal{C}} \mathrm{Ann}\,X$$

induces an injective map from the set of definable subcategories of $\mathcal{S}$ into the set of cohomological ideals in $\mathcal{F}$. It remains to show that this map is surjective. To this end fix a cohomological ideal $\mathfrak{I}$ in $\mathcal{F}$. We have $\mathfrak{I} = \mathrm{Ann}\,Y$ for some $Y \in \mathcal{S}$ by Proposition 7.4. Thus $\mathcal{C} = \{X \in \mathcal{S} \mid \mathfrak{I} \subseteq \mathrm{Ann}\,X\}$ is a definable subcategory satisfying

$$\mathfrak{I} = \bigcap_{X \in \mathcal{C}} \mathrm{Ann}\,X.$$

This completes the proof of the correspondence between definable subcategories, Ziegler closed subsets, Serre subcategories and cohomological ideals. □

Given a class $\mathcal{C}$ of objects in $\mathcal{S}$, the definable subcategory generated by $\mathcal{C}$ is

$$\mathrm{Def}\,\mathcal{C} = \{X \in \mathcal{S} \mid F \in \mathrm{Coh}\,\mathcal{C} \text{ and } F(Y) = 0 \text{ for all } Y \in \mathcal{C} \text{ implies } F(X) = 0\}.$$

It remains to prove the following description of $\mathrm{Def}\,\mathcal{C}$ via reduced products which is formulated in Theorem C.

Theorem 7.5. *Suppose that Brown representability holds for $\mathcal{S}$, and let $\mathcal{C}$ be a class of objects in $\mathcal{S}$. Then an object X in $\mathcal{S}$ belongs to* $\mathrm{Def}\,\mathcal{C}$ *if and only if there is a pure triangle $X \to Y \to Z \to \Sigma X$ such that $Y = \prod_{i \in I} X_i/\mathcal{U}$ for some family $(X_i)_{i \in I}$ of objects in $\mathcal{C}$ and some filter $\mathcal{U}$ on I.*

Proof. We fix a class $\mathcal{C}$ of objects in $\mathcal{S}$ and put

$$\mathcal{T} = \{F \in \operatorname{Coh}\mathcal{S} \mid F(X) = 0 \text{ for all } X \in \mathcal{C}\}.$$

We use again the functor category $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. Recall that $F \in (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ is *fp-injective* if $\operatorname{Ext}^1(G, F) = 0$ for all $G \in \operatorname{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. A functor $F \colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ is fp-injective if and only if it is exact (cf. Lemma 2.7 in [17]) and therefore the restricted Yoneda functor

$$\mathcal{S} \longrightarrow (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto H_X = \operatorname{Hom}(-, X)|_{\mathcal{F}},$$

identifies the objects in $\mathcal{S}$ with the fp-injective objects in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ since we assume Brown representability. Now let $\mathcal{C}' = \{H_X \mid X \in \mathcal{C}\}$ and put

$$\mathcal{T}' = \{F \in \operatorname{fp}(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \mid \operatorname{Hom}(F, X) = 0 \text{ for all } X \in \mathcal{C}'\}.$$

Note that $\mathcal{T} = \{F^\vee \mid F \in \mathcal{T}'\}$ since $F^\vee(X) = \operatorname{Hom}(F, H_X)$ for $X \in \mathcal{S}$. A reduced product of a family $(X_i)_{i \in I}$ of objects in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ with respect to some filter $\mathcal{U}$ on I is by definition the filtered colimit $\operatorname{colim}_{J \in \mathcal{U}} \prod_{i \in J} X_i$ so that the restricted Yoneda functor preserves reduced products. It follows from Proposition 4.5 in [16] that an fp-injective object X in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ is a subobject of some reduced product of objects in $\mathcal{C}'$ if and only if $\operatorname{Hom}(F, X) = 0$ for all $F \in \mathcal{T}'$. Using again the restricted Yoneda functor, it follows that $X \in \mathcal{S}$ fits into a triangle $X \to Y \to Z \to \Sigma X$ such that Y is a reduced product of objects in $\mathcal{C}$ if and only if $F(X)$ for all $F \in \mathcal{T}$. This completes the proof since $\operatorname{Def}\mathcal{C} = \{X \in \mathcal{S} \mid F(X) = 0 \text{ for all } F \in \mathcal{T}\}$. ☐

Appendix: Finitely presented modules versus compact objects

In this appendix we explain the analogy between compact objects in a compactly generated triangulated category, and finitely presented modules in the category of modules over an associative ring.

Let $\mathcal{A}$ be an additive category and suppose that it has arbitrary products and coproducts. We make the following definitions:

- An object Q is *p-injective* if for every set I the summation map $\coprod_I Q \to Q$ factors through the canonical map $\coprod_I Q \to \prod_I Q$.
- A sequence of maps $X \to Y \to Z$ is *p-exact* if for every p-injective object Q in $\mathcal{A}$ the sequence $0 \to \operatorname{Hom}(Z, Q) \to \operatorname{Hom}(Y, Q) \to \operatorname{Hom}(X, Q) \to 0$ is exact.
- An object P is *p-projective* if for every p-exact sequence $X \to Y \to Z$ the sequence $0 \to \operatorname{Hom}(P, X) \to \operatorname{Hom}(P, Y) \to \operatorname{Hom}(P, Z) \to 0$ is exact.
- An object X is *compact* if the functor $\operatorname{Hom}(X, -)$ preserves coproducts.

If $\mathcal{A}$ is the category $\operatorname{Mod}\Lambda$ of modules over an associative ring Λ, then the above concept of p-exactness coincides with the concept of pure-exactness introduced by Cohn [7]. This follows essentially from the characterization of pure-injective modules via the summation map which is due to Jensen and Lenzing (cf. [14, Proposition 7.32]). In this context the compact p-projective objects are characterized as follows.

Proposition. *Let $\mathcal{A}$ be the category of modules over an associative ring. Then an object in $\mathcal{A}$ is compact and p-projective if and only if it is a finitely presented module.*

Proof. The assertion is an immediate consequence of the well-known fact that a module is pure-projective if and only if it is a direct factor of a coproduct of finitely presented modules. ☐

Now suppose that $\mathcal{A}$ is a compactly generated triangulated category and denote by $\mathcal{F}$ the full subcategory of compact objects. Then the p-injective objects are precisely the objects which are pure-injective in the sense of Definition 1.1 (cf. [17, Theorem 1.8]).

Therefore the functor $\mathcal{A} \to (\mathcal{F}^{\rm op}, {\rm Ab})$, $X \mapsto H_X$, identifies the p-injective objects in $\mathcal{A}$ with the injective objects in $(\mathcal{F}^{\rm op}, {\rm Ab})$ (cf. [17, Corollary 1.9]). Note that ${\rm Hom}(X, Q) \cong {\rm Hom}(H_X, H_Q)$ for all X in $\mathcal{A}$ and every p-injective object Q by Lemma 1.3. Therefore $X \mapsto H_X$ identifies the p-exact sequences $X \to Y \to Z$ with the exact sequences $0 \to H_X \to H_Y \to H_Z \to 0$ since the injective objects cogenerate $(\mathcal{F}^{\rm op}, {\rm Ab})$.

Proposition. *Let $\mathcal{A}$ be a compactly generated triangulated category. Then an object in $\mathcal{A}$ is compact and p-projective if and only if it is compact.*

Proof. We need to show that every compact object is p-projective. However, this is just a reformulation of the fact that for each p-exact sequence $X \to Y \to Z$ the sequence $0 \to H_X \to H_Y \to H_Z \to 0$ is exact. □

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