FINITE VERSUS INFINITE DIMENSIONAL REPRESENTATIONS – A NEW DEFINITION OF TAMENESS

HENNING KRAUSE

Let Λ be a finite dimensional algebra over some algebraically closed field k. In this note I discuss the relationship between finite and infinite dimensional modules over Λ. This discussion is based on the following three fundamental concepts:

- fp-idempotent ideals in the category $\operatorname{mod}\Lambda$ of finite dimensional Λ-modules
- endofinite modules in the category $\operatorname{Mod}\Lambda$ of all Λ-modules
- coherent functors $\operatorname{Mod}\Gamma \to \operatorname{Mod}\Lambda$ between two module categories

In order to illustrate the use of these concepts I present a new definition of tame representation type for Λ which only involves the category of finite dimensional Λ-modules.

This note follows closely the talk that I gave in Bielefeld in Summer 1998. I tried to cover a number of new results from my Habilitationsschrift [12]. In this thesis I introduced fp-idempotent ideals and used them to give a new tameness definition. My motivation was the following question raised by Ringel [14, p.144]: *Is there a definition of tameness which only involves finite dimensional modules, and avoids any reference to algebraic geometry?* The approach presented here which leads to a positive answer is based on an analysis of certain infinite dimensional modules. These are the endofinite modules which Crawley-Boevey used for his definition of generic tameness [2]. The main ingredients of my analysis can be summarized as follows:

- the Ziegler spectrum, which contains all indecomposable endofinite modules and puts some structure on the set of these modules;
- the fp-idempotent ideals, which represent the infinite dimensional endofinite modules inside the category of finite dimensional modules;
- the coherent functors, which transfer the structure of the collection of all endofinite modules from one algebra to another.

In the first three sections of this paper I will cover these aspects. Section 4 is devoted to giving two new characaterizations of tameness, and Section 5 illustrates the relationship between finite and infinite dimensional modules for tame algebras.

A final word about proofs: There are a number of proofs given in the first three sections, with the exception of some deeper results where I need to refer to the literature. The results in Section 4 are presented with complete proofs, whereas no proofs are given in the final Section 5.

1. FP-IDEMPOTENT IDEALS

We fix a finite dimensional algebra Λ over some algebraically closed field k. The category of (right) Λ-modules is denoted by $\operatorname{Mod}\Lambda$, and $\operatorname{mod}\Lambda$ denotes the category of finite dimensional Λ-modules. In this section we introduce fp-idempotent ideals in the category $\operatorname{mod}\Lambda$ and discuss their relationship with a number of concepts which are defined in the category $\operatorname{Mod}\Lambda$ of all Λ-modules.

Dedicated to Professor Herbert Kupisch on the occasion of his 70th birthday.

A basic tool is the category $(\operatorname{mod}\Lambda, \mathrm{Ab})$ of additive functors $\operatorname{mod}\Lambda \to \mathrm{Ab}$ into the category Ab of abelian groups. Note that $(\operatorname{mod}\Lambda, \mathrm{Ab})$ is an abelian category. The natural transformations between functors form the morphisms in this category, and (co)kernels, (co)products, etc. are defined pointwise. A functor $F\colon \operatorname{mod}\Lambda \to \mathrm{Ab}$ is *finitely presented* if there exists an exact sequence

$$(*) \qquad \operatorname{Hom}_\Lambda(Y,-) \longrightarrow \operatorname{Hom}_\Lambda(X,-) \longrightarrow F \longrightarrow 0,$$

and $\operatorname{fp}(\operatorname{mod}\Lambda, \mathrm{Ab})$ denotes the full subcategory of finitely presented functors. Using the fact that $\operatorname{mod}\Lambda$ has cokernels, it is not difficult to show that $\operatorname{fp}(\operatorname{mod}\Lambda, \mathrm{Ab})$ is an abelian category. A functor $F\colon \operatorname{Mod}\Lambda \to \mathrm{Ab}$ is called *coherent* if there exists an exact sequence of the form $(*)$ with X and Y in $\operatorname{mod}\Lambda$. It is clear that $F \mapsto F|_{\operatorname{mod}\Lambda}$ defines an equivalence between the category $\operatorname{Coh}\Lambda$ of coherent functors and the category $\operatorname{fp}(\operatorname{mod}\Lambda, \mathrm{Ab})$.

Viewing $\operatorname{mod}\Lambda$ as a ring with several objects, we recall that an *ideal* $\mathfrak{I}$ in $\operatorname{mod}\Lambda$ consists of subgroups $\mathfrak{I}(X,Y)$ in $\operatorname{Hom}_\Lambda(X,Y)$ for every pair of objects X, Y in $\operatorname{mod}\Lambda$ such that for all ϕ in $\mathfrak{I}(X,Y)$ and all maps $\alpha\colon X' \to X$ and $\beta\colon Y \to Y'$ in $\operatorname{mod}\Lambda$ the composition $\beta \circ \phi \circ \alpha$ belongs to $\mathfrak{I}(X', Y')$. Note that an ideal $\mathfrak{I}$ in $\operatorname{mod}\Lambda$ is idempotent, i.e. $\mathfrak{I}^2 = \mathfrak{I}$, if and only if the class of functors in $(\operatorname{mod}\Lambda, \mathrm{Ab})$ vanishing on $\mathfrak{I}$ is closed under extensions. This motivates the following definition.

Definition 1.1. An ideal $\mathfrak{I}$ in $\operatorname{mod}\Lambda$ is called *fp-idempotent* if the class of finitely presented functors in $(\operatorname{mod}\Lambda, \mathrm{Ab})$ vanishing on $\mathfrak{I}$ is closed under extensions.

Observe that any idempotent ideal is fp-idempotent; however the converse is usually not true. There are three other concepts equivalent to fp-idempotent ideals:

- *Definable subcategories* of $\operatorname{Mod}\Lambda$. These are full subcategories of $\operatorname{Mod}\Lambda$ of the form $\{M \in \operatorname{Mod}\Lambda \mid F_i(M) = 0 \text{ for all } i \in I\}$ for a family $(F_i)_{i\in I}$ of functors in $\operatorname{Coh}\Lambda$.
- *Closed subsets* of the set $\operatorname{Ind}\Lambda$ of isomorphism classes of indecomposable pure-injective Λ-modules. These are subsets of the form $\mathcal{X} \cap \operatorname{Ind}\Lambda$ for some definable subcategory $\mathcal{X}$ of $\operatorname{Mod}\Lambda$.
- *Serre subcategories* of $\operatorname{Coh}\Lambda$. These are full subcategories of $\operatorname{Coh}\Lambda$ which are closed under forming subobjects, quotient objects, and extensions.

In order to formulate the correspondence between these concepts we introduce the following notation. Let $\mathcal{X}$ be a class of Λ-modules. We denote by $[\mathcal{X}]$ the ideal of maps in $\operatorname{mod}\Lambda$ which factor through a finite coproduct of modules in $\mathcal{X}$. The class of Λ-modules which are products of modules in $\mathcal{X}$ is denoted by $\prod \mathcal{X}$.

Fundamental correspondence. *There are bijections between*

- *the set of definable subcategories* $\mathcal{X}$ *of* $\operatorname{Mod}\Lambda$*,*
- *the set of closed subsets* $\mathbf{U}$ *of* $\operatorname{Ind}\Lambda$*,*
- *the set of Serre subcategories* $\mathcal{S}$ *of* $\operatorname{Coh}\Lambda$*,*
- *the set of fp-idempotent ideals* $\mathfrak{I}$ *in* $\operatorname{mod}\Lambda$*.*

These bijections are defined as follows:

$$\mathcal{X} \mapsto \begin{cases} \mathbf{U} = \mathcal{X} \cap \operatorname{Ind}\Lambda \\ \mathcal{S} = \{F \in \operatorname{Coh}\Lambda \mid F(M) = 0 \textit{ for all } M \in \mathcal{X}\} \\ \mathfrak{I} = [\mathcal{X}] \end{cases}$$

$$\mathbf{U} \mapsto \begin{cases} \mathcal{X} = \{M \in \operatorname{Mod}\Lambda \mid M \textit{ is a pure submodule of some } N \in \prod \mathbf{U}\} \\ \mathcal{S} = \{F \in \operatorname{Coh}\Lambda \mid F(M) = 0 \textit{ for all } M \in \mathbf{U}\} \\ \mathfrak{I} = [\prod \mathbf{U}] \end{cases}$$

$$\mathcal{S} \mapsto \begin{cases} \mathscr{X} = \{M \in \operatorname{Mod}\Lambda \mid F(M) = 0 \textit{ for all } F \in \mathcal{S}\} \\ \mathbf{U} = \{M \in \operatorname{Ind}\Lambda \mid F(M) = 0 \textit{ for all } F \in \mathcal{S}\} \\ \mathfrak{I} = \{\phi \in \operatorname{mod}\Lambda \mid F(\phi) = 0 \textit{ for all } F \in \mathcal{S}\} \end{cases}$$

$$\mathfrak{I} \mapsto \begin{cases} \mathscr{X} = \{M \in \operatorname{Mod}\Lambda \mid F(M) = 0 \textit{ for all } F \in \operatorname{Coh}\Lambda \textit{ with } F(\mathfrak{I}) = 0\} \\ \mathbf{U} = \{M \in \operatorname{Ind}\Lambda \mid F(M) = 0 \textit{ for all } F \in \operatorname{Coh}\Lambda \textit{ with } F(\mathfrak{I}) = 0\} \\ \mathcal{S} = \{F \in \operatorname{Coh}\Lambda \mid F(\phi) = 0 \textit{ for all } \phi \in \mathfrak{I}\} \end{cases}$$

The correspondence is based on the work of several mathematicians. Ziegler introduced the closed subsets of $\operatorname{Ind}\Lambda$ in model-theoretic terms and noticed that they form the closed sets of a quasi-compact space [15]; see also [8] for an algebraic argument. The correspondence between closed subsets of $\operatorname{Ind}\Lambda$ and Serre subcategories of $\operatorname{Coh}\Lambda$ has been established by Herzog in [6]; see also [8]. The definable subcategories were introduced by Crawley-Boevey [4]. Finally, fp-idempotent ideals were introduced in [12]. There one also finds a complete proof for the above correspondence.

The set $\operatorname{Ind}\Lambda$ together with Ziegler's topology is often called the *Ziegler spectrum* of Λ. We shall denote for every subset $\mathbf{U}$ of $\operatorname{Ind}\Lambda$ by

$$\overline{\mathbf{U}} = \bigcap_{\mathbf{U} \subseteq \mathbf{V} \text{ closed}} \mathbf{V}$$

the *closure* of $\mathbf{U}$ which is the smallest closed subset of $\operatorname{Ind}\Lambda$ containing $\mathbf{U}$. Note that $\mathbf{U}$ is closed with respect to Ziegler's topology if and only if $\mathbf{U} = \overline{\mathbf{U}}$.

2. Endofinite modules

The *endolength* $\operatorname{endol}(M)$ of a Λ-module M is the length of M viewed as a module over its endomorphism ring $\operatorname{End}_\Lambda(M)$. Following Crawley-Boevey, a Λ-module M is said to be *endofinite* provided that $\operatorname{endol}(M)$ is finite. It follows a list of basic properties of endofinite modules.

(E1) *If M is finite dimensional, then* $\operatorname{endol}(M) \leq \dim_k(M)$*, and equality holds if M is indecomposable.*

Proof. The endomorphism ring $\operatorname{End}_\Lambda(M)$ is a k-algebra and therefore the length of M is bounded by $\dim_k(M)$. If M is indecomposable then $\operatorname{endol}(M) = \dim_k(M)$, since $\operatorname{End}_\Lambda(M)$ is local and k is algebraically closed. □

(E2) *An endofinite module M is pure-injective and has a decomposition $M = \coprod_{i \in I} M_i$ into indecomposable endofinite modules with local endomorphism rings. Conversely, such a coproduct is endofinite if and only if there are only finitely many isomorphism classes involved.*

Proof. See Proposition 4.3 in [3]. □

We denote the set of isomorphism classes of indecomposable endofinite modules as follows:

$$\operatorname{Ind}_n \Lambda = \{M \in \operatorname{Ind}\Lambda \mid \operatorname{endol}(M) = n\},$$

$$\operatorname{ind}_n \Lambda = \{M \in \operatorname{ind}\Lambda \mid \operatorname{endol}(M) = n\},$$

where $\operatorname{ind}\Lambda = \operatorname{Ind}\Lambda \cap \operatorname{mod}\Lambda$. Given a Λ-module M, we denote by $\operatorname{Add} M$ the full subcategory of Λ-modules which are direct factors of coproducts of copies of M.

(E3) *Let* M *be an endofinite* Λ*-module and denote by* $M_1,\dots,M_n$ *the isomorphism classes of indecomposable modules which occur in a decomposition* $M=\coprod_{i\in I}M_i$ *into indecomposables. Then* $\operatorname{Add}M$ *is a definable subcategory of* $\operatorname{Mod}\Lambda$ *and* $\{M_1,\dots,M_n\}=\operatorname{Add}M\cap\operatorname{Ind}\Lambda$ *is the corresponding closed subset of* $\operatorname{Ind}\Lambda$.

Proof. A full subcategory of $\operatorname{Mod}\Lambda$ is definable if and only if it is closed under products, direct limits, and pure submodules [4, 3.2]; see also [12, Theorem 2.1]. Using this result, it is proved in [12, Theorem 6.16] that $\operatorname{Add}M$ is a definable subcategory if M is endofinite. It follows from (E2) that $\operatorname{Add}M\cap\operatorname{Ind}\Lambda=\{M_1,\dots,M_n\}$. ☐

(E4) *Let* M *be an endofinite module. Then* $[M]$ *is the fp-idempotent ideal corresponding to the definable subcategory* $\operatorname{Add}M$*. Moreover, the ideal* $[M]$ *is nilpotent if and only if* M *has no finite dimensional indecomposable direct factor. In this case* $[M]^{n+1}=0$ *where* $n=\operatorname{endol}(M)$.

Proof. Let $\phi\colon X\to Y$ be a map which factors through a coproduct $\coprod_{i\in I}M_i$. If X is finitely generated, then ϕ factors through $\coprod_{i\in J}M_i$ for some finite subset $J\subseteq I$. Therefore $[\operatorname{Add}M]=[M]$. The second part of the assertion follows directly from the fact that $(\operatorname{rad}\operatorname{End}_\Lambda(M))^n=0$ by Nakayama's lemma, where $n=\operatorname{endol}(M)$. ☐

(E5) *The set* $\{M\in\operatorname{Ind}\Lambda\mid\operatorname{endol}(M)\le n\}$ *is closed for all* $n\in\mathbb{N}$.

Proof. See [6, p.554]; see also [9, Proposition 5.4]. ☐

(E6) *Let* $M\in\operatorname{Ind}\Lambda$*. Then* $\{M\}$ *is open if and only if* $M\in\operatorname{ind}\Lambda$.

Proof. This is proved, using Auslander-Reiten theory, in [13, Proposition 13.1]. ☐

(E7) *Let* $U\subseteq\operatorname{ind}\Lambda$*. Then the closure* $\overline{\mathbf{U}}$ *of* $\mathbf{U}$ *corresponds to the fp-idempotent ideal* $[\mathbf{U}]$*, and* $\overline{\mathbf{U}}\setminus\mathbf{U}\subseteq\operatorname{Ind}\Lambda\setminus\operatorname{ind}\Lambda$*. Moreover,* $\overline{\mathbf{U}}=\mathbf{U}$ *if and only if* $\mathbf{U}$ *is finite.*

Proof. Clearly, $[\mathbf{U}]$ is an idempotent ideal and therefore fp-idempotent. The corresponding closed subset of $\operatorname{Ind}\Lambda$ is certainly the smallest containing $\mathbf{U}$ which is $\overline{\mathbf{U}}$. Any module in $\overline{\mathbf{U}}\setminus\mathbf{U}$ is infinite dimensional by (E6). If $\mathbf{U}$ is finite, then $\overline{\mathbf{U}}=\mathbf{U}$ since $\{M\}$ is closed for every $M\in\operatorname{ind}\Lambda$ by (E1) and (E3). Conversely, $\overline{\mathbf{U}}=\mathbf{U}$ implies that $\bigcup_{M\in\mathbf{U}}\{M\}$ is an open covering of a quasi-compact space by (E6) since $\operatorname{Ind}\Lambda$ is quasi-compact. Therefore $\mathbf{U}$ needs to be finite. ☐

3. Coherent functors

In this section we study functors $\operatorname{Mod}\Gamma\to\operatorname{Mod}\Lambda$ between two module categories. We assume that Λ and Γ are arbitrary rings. This is essential for the applications which will follow. Note that all concepts which have been defined in the previous sections still make sense if one replaces finite dimensional by finitely presented modules. A functor $F\colon\operatorname{Mod}\Gamma\to\operatorname{Mod}\Lambda$ is called *coherent* if the composition $\widehat{F}=\operatorname{Hom}_\Lambda(\Lambda,-)\circ F$ with the forgetful functor is coherent, i.e., there exists an exact sequence

$$(**)\qquad \operatorname{Hom}_\Gamma(Y,-)\longrightarrow\operatorname{Hom}_\Gamma(X,-)\longrightarrow\widehat{F}\longrightarrow 0$$

with X and Y finitely presented. We denote by $\operatorname{Coh}(\Gamma,\Lambda)$ the category of coherent functors $\operatorname{Mod}\Gamma\to\operatorname{Mod}\Lambda$. For example, $\operatorname{Coh}(\Gamma,\mathbb{Z})=\operatorname{Coh}\Gamma$. Note that for any $F\in\operatorname{Coh}(\Gamma,\Lambda)$, the Λ-action on $\widehat{F}(\Gamma)$ induces a ring homomorphism

$$\phi_F\colon\Lambda\longrightarrow\operatorname{End}_{\mathbb{Z}}(\widehat{F}(\Gamma))^{\mathrm{op}}\cong\operatorname{End}(\widehat{F})^{\mathrm{op}}.$$

It follows a list of basic properties of coherent functors.

(C1) *The functor*

$$\operatorname{Coh}(\Gamma,\Lambda) \longrightarrow \{(G,\psi) \mid G \in \operatorname{Coh}\Gamma, \psi \in \operatorname{Hom}(\Lambda, \operatorname{End}(G)^{\mathrm{op}})\}, \quad F \mapsto (\widehat{F}, \phi_F),$$

is an equivalence.

Proof. We provide an inverse for this functor. Let $G \in \operatorname{Coh}\Gamma$ and $\psi\colon \Lambda \to \operatorname{End}(G)^{\mathrm{op}}$ be a ring homomorphism. The map ψ defines a Λ-action on $G(M)$ for all $M \in \operatorname{Mod}\Gamma$ and this gives rise to a coherent functor $G_\psi\colon \operatorname{Mod}\Gamma \to \operatorname{Mod}\Lambda$. Clearly, $\widehat{F}_{\phi_F} = F$ and $(\widehat{G_\psi}, \phi_{(G_\psi)}) = (G, \psi)$. ☐

(C2) *A functor is coherent if and only if it preserves direct limits and products.*

Proof. One implication is easy. In fact, a representable functor $\operatorname{Hom}_\Gamma(X,-)$ preserves direct limits and products provided that X is a finitely presented Γ-module. From this follows that every coherent functor preserves direct limits and products. For the converse, see Corollary 12.2 in [11]. ☐

(C3) *Let ${}_\Gamma B_\Lambda$ be a bimodule. Then* $\operatorname{Mod}\Gamma \to \operatorname{Mod}\Lambda$, $M \mapsto M \otimes_\Gamma B$, *is coherent if and only if B is a finitely presented Γ-module.*

Proof. Apply (C2). The tensor functor $-\otimes_\Gamma B$ always preserves direct limits; it preserves products if and only if B is a finitely presented Γ-module. ☐

(C4) *A coherent functor sends pure-injective modules to pure-injective modules, and pure-exact sequences to pure-exact sequences.*

Proof. The assertion follows immediately from (C2) if one uses the appropriate description of pure-injectivity and pure-exactness: a module M is pure-injective if and only if for every set I the summation map $M^{(I)} \to M$ factors through the canonical map $M^{(I)} \to M^I$, and a sequence $0 \to L \to M \to N \to 0$ is pure-exact if and only if it is a direct limit of split exact sequences $0 \to L_i \to M_i \to N_i \to 0$; see [7]. ☐

(C5) *Let $F\colon$* $\operatorname{Mod}\Gamma \to \operatorname{Mod}\Lambda$ *be coherent and $\mathcal{X}$ be a definable subcategory of* $\operatorname{Mod}\Lambda$. *Then* $F^{-1}(\mathcal{X}) = \{M \in \operatorname{Mod}\Gamma \mid F(M) \in \mathcal{X}\}$ *is a definable subcategory of* $\operatorname{Mod}\Gamma$.

Proof. It follows directly from (C2) that a composition of coherent functors is again coherent. If $\mathcal{X}$ is defined by the vanishing of the family $(F_i)_{i\in I}$ in $\operatorname{Coh}\Lambda$, then $F^{-1}(\mathcal{X})$ is defined by the vanishing of the family $(F_i \circ F)_{i\in I}$ in $\operatorname{Coh}\Gamma$ and is therefore definable. ☐

(C6) *Let $F\colon$* $\operatorname{Mod}\Gamma \to \operatorname{Mod}\Lambda$ *be coherent and* $\mathbf{U}$ *be a closed subset of* $\operatorname{Ind}\Gamma$. *Then the indecomposable direct factors of modules in* $\{F(M) \mid M \in \mathbf{U}\}$ *form a closed subset of* $\operatorname{Ind}\Lambda$.

Proof. See Theorem 7.8 in [10]. ☐

(C7) *Let $F\colon$* $\operatorname{Mod}\Gamma \to \operatorname{Mod}\Lambda$ *be coherent and suppose that $F(M)$ is indecomposable for all* $M \in \operatorname{Ind}\Gamma$. *Then* $\operatorname{Ind}\Gamma \to \operatorname{Ind}\Lambda$, $M \mapsto F(M)$, *is a continuous and closed map.*

Proof. Combine (C4) – (C6). ☐

Given a coherent functor $F\colon \operatorname{Mod}\Gamma \to \operatorname{Mod}\Lambda$, we denote by n_F the minimal number of elements which generate the Γ-module X in a presentation $(**)$ of $\operatorname{Hom}_\Lambda(\Lambda, -)\circ F$.

(C8) *Let* $F\colon \operatorname{Mod}\Gamma \to \operatorname{Mod}\Lambda$ *be a coherent functor. Then* $\operatorname{endol}(F(M)) \leq n_F \cdot \operatorname{endol}(M)$ *for all* $M \in \operatorname{Mod}\Gamma$.

Proof. The length of $F(M)$ over $\operatorname{End}_\Gamma(M)$ is bounded by $n_F \cdot \operatorname{endol}(M)$. Using the ring homomorphism $\operatorname{End}_\Gamma(M) \to \operatorname{End}_\Lambda(F(M))$, it follows that $\operatorname{endol}(F(M)) \leq n_F \cdot \operatorname{endol}(M)$. □

(C9) *Let* $F\colon \operatorname{Mod}\Gamma \to \operatorname{Mod}\Lambda$ *be a coherent functor. If* Γ *and* Λ *are* k*-algebras and* F *is* k*-linear, then* $\dim_k(F(M)) \leq n_F \cdot \dim_k(M)$ *for all* $M \in \operatorname{Mod}\Gamma$.

Proof. Adapt the argument of (C8). □

4. Tameness

We fix again a finite dimensional algebra Λ over some algebraically closed field k. Let us recall Drozd's definition of tameness. A *one-parameter family* of Λ-modules of dimension n is the set of Λ-modules

$$\{k[T]/(T-\lambda)\otimes_{k[T]} B \mid \lambda \in k\}$$

where B is a $k[T]$-Λ-bimodule which is free of rank n over $k[T]$. The algebra is said to be of *tame representation type* provided that there is for every $n \in \mathbb{N}$ a finite number of such one-parameter families such that every indecomposable Λ-module of dimension n is isomorphic to a module in one of these families [5, 1]. The minimal number of one-parameter families which is needed to parametrize all but finitely many modules in $\operatorname{ind}_n\Lambda$ is denoted by $\mu_\Lambda(n)$. We give now an alternative description of tameness which is based on the topology defined on $\operatorname{Ind}\Lambda$.

Theorem 4.1. *A finite dimensional algebra* Λ *is of tame representation type if and only if* $\overline{\operatorname{ind}_n\Lambda}\setminus\operatorname{ind}_n\Lambda$ *is finite for all* $n \in \mathbb{N}$*. Moreover, in this case*

$$\mu_\Lambda(n) = \operatorname{card}(\overline{\operatorname{ind}_n\Lambda}\setminus\operatorname{ind}_n\Lambda)$$

for all $n \in \mathbb{N}$.

We need the following easy reformulation of the classical tameness definition.

Lemma 4.2. *The algebra* Λ *is tame if and only if there exists for every* $n \in \mathbb{N}$ *a finite product* $\Gamma = k[T]\times\ldots\times k[T]$ *of polynomial rings and a* k*-linear exact and coherent functor* $F\colon \operatorname{Mod}\Gamma \to \operatorname{Mod}\Lambda$ *such that* $\operatorname{ind}_n\Lambda \subseteq F(\operatorname{ind}_1\Gamma)$.

Proof. Straightforward. □

The next lemma presents the information about $\operatorname{Ind} k[T]$ which is sufficient for our purpose.

Lemma 4.3. $\operatorname{Ind}_1 k[T] = \operatorname{ind}_1 k[T] \cup \{k(T)\} = \overline{\operatorname{ind}_1 k[T]}$.

Proof. The Ziegler spectrum of a Dedekind domain is computed in [15, Example 9.5]. □

Proof of Theorem 4.1. Suppose first that Λ has tame representation type and let $n \in \mathbb{N}$. Consider the corresponding coherent functor $F\colon \operatorname{Mod}\Gamma \to \operatorname{Mod}\Lambda$ such that $\operatorname{ind}_n\Lambda \subseteq F(\operatorname{ind}_1\Gamma)$ and $\Gamma = k[T]\times\ldots\times k[T]$ which exists by Lemma 4.2. Note that $\overline{\operatorname{ind}_1\Gamma} = \operatorname{ind}_1\Gamma \cup \{Q_1, \ldots Q_r\}$ by Lemma 4.3, where r denotes the number of factors of Γ. It follows from (C6) that every module in $\overline{\operatorname{ind}_n\Lambda}\setminus\operatorname{ind}_n\Lambda$ is isomorphic to a direct factor of $F(Q_i)$ for some i. However, each $F(Q_i)$ is endofinite by (C8) and has therefore only finitely many non-isomorphic direct factors by (E2). We conclude that $\overline{\operatorname{ind}_n \Lambda} \setminus \operatorname{ind}_n \Lambda$ is finite.

To prove the converse we assume that the algebra Λ is not tame. We apply the Tame and Wild Theorem in a form which is due to Crawley-Boevey [2]. In fact, there exists a k-linear *representation embedding* $F\colon \operatorname{Mod} k\langle X, Y\rangle \to \operatorname{Mod} \Lambda$ which as far as we are concerned means that F is coherent and induces an injective map $\operatorname{Ind} k\langle X, Y\rangle \to \operatorname{Ind} \Lambda$. Now consider for every pair $\alpha, \beta \in k$ the $k\langle X, Y\rangle$-module

$$M_{\alpha,\beta} = k\langle X, Y\rangle/(X-\alpha, Y-\beta)$$

and let Q_α be the endofinite $k\langle X, Y\rangle$-module whose underlying space is $k(T)$ with X acting by multiplication with α and Y acting by multiplication with T. Using Lemma 4.3, it is not hard to see that for each $\alpha \in k$ the closure of $\mathbf{U}_\alpha = \{M_{\alpha,\beta} \mid \beta \in k\}$ is $\mathbf{U}_\alpha \cup \{Q_\alpha\}$, and therefore $\mathbf{Q} = \{Q_\alpha \mid \alpha \in k\} \subseteq \overline{\mathbf{U}}$ with $\mathbf{U} = \bigcup_{\alpha \in k} \mathbf{U}_\alpha \subseteq \operatorname{Ind}_1 k\langle X, Y\rangle$. Now let $n_F \in \mathbb{N}$ such that $\operatorname{endol}(F(M)) \leq n_F \cdot \operatorname{endol}(M)$ for all $M \in \operatorname{Mod} k\langle X, Y\rangle$ which exists by (C8). It follows that

$$F(\mathbf{U}) \subseteq \operatorname{ind}_1 \Lambda \cup \ldots \cup \operatorname{ind}_{n_F} \Lambda$$

since F sends finite dimensional to finite dimensional modules by (C9). This implies

$$F(\mathbf{Q}) \subseteq F(\overline{\mathbf{U}}) = \overline{F(\mathbf{U})} \subseteq \overline{\operatorname{ind}_1 \Lambda} \cup \ldots \cup \overline{\operatorname{ind}_{n_F} \Lambda}$$

since the map $\operatorname{Ind} k\langle X, Y\rangle \to \operatorname{Ind} \Lambda$ is continuous and closed by (C7). Moreover,

$$F(\mathbf{Q}) \subseteq F(\overline{\mathbf{U}} \setminus \mathbf{U}) = \overline{F(\mathbf{U})} \setminus F(\mathbf{U})$$

which is contained in $\operatorname{Ind} \Lambda \setminus \operatorname{ind} \Lambda$ by (E7). We conclude that $\overline{\operatorname{ind}_n \Lambda} \setminus \operatorname{ind}_n \Lambda$ is infinite for some $n \leq n_F$.

It remains to show that

$$\mu_\Lambda(n) = \operatorname{card}(\overline{\operatorname{ind}_n \Lambda} \setminus \operatorname{ind}_n \Lambda)$$

for all $n \in \mathbb{N}$. In [9, Corollary 9.7] it is shown that an indecomposable module M belongs to $\overline{\operatorname{ind}_n \Lambda} \setminus \operatorname{ind}_n \Lambda$ if and only if M is infinite dimensional and $\operatorname{endol}(M)$ divides n. There are precisely $\mu_\Lambda(n)$ indecomposable modules with this property by [2, Theorem 5.6], and this gives the equality. ☐

We continue with a description of the set $\overline{\operatorname{ind}_n \Lambda} \setminus \operatorname{ind}_n \Lambda$ in terms of fp-idempotent ideals in $\operatorname{mod} \Lambda$. To this end let

$$\operatorname{fpnil}_n \Lambda = \{\mathfrak{I} \subseteq \operatorname{mod} \Lambda \mid \mathfrak{I} \text{ is fp-idempotent, nilpotent, and } 0 \neq \mathfrak{I} \subseteq [\operatorname{ind}_n \Lambda]\}.$$

Proposition 4.4. *Let* $n \in \mathbb{N}$. *Then* $\operatorname{fpnil}_n \Lambda$ *is finite if and only if* $\overline{\operatorname{ind}_n \Lambda} \setminus \operatorname{ind}_n \Lambda$ *is finite. Moreover, the map*

$$\overline{\operatorname{ind}_n \Lambda} \setminus \operatorname{ind}_n \Lambda \longrightarrow \{\mathfrak{I} \in \operatorname{fpnil}_n \Lambda \mid \mathfrak{I} \supseteq \mathfrak{J} \in \operatorname{fpnil}_n \Lambda \textit{ implies } \mathfrak{I} = \mathfrak{J}\}, \quad M \mapsto [M],$$

is bijective.

Proof. Let $\mathbf{U}_n = \operatorname{ind}_n \Lambda$. We use the bijection between closed subsets of $\operatorname{Ind} \Lambda$ and fp-idempotent ideals in $\operatorname{mod} \Lambda$. This correspondence sends $\overline{\mathbf{U}_n}$ to $[\mathbf{U}_n]$ by (E7), and induces therefore an injective map

$$\overline{\mathbf{U}_n} \longrightarrow \{\mathfrak{I} \subseteq \operatorname{mod} \Lambda \mid \mathfrak{I} \subseteq [\mathbf{U}_n] \text{ is fp-idempotent}\}, \quad M \mapsto [M],$$

because every module in $\overline{\mathbf{U}_n}$ is endofinite by (E5), hence a closed singleton by (E3). It follows from (E4) and (E7) that $[M]$ is nilpotent for every $M \in \overline{\mathbf{U}_n} \setminus \mathbf{U}_n$, and it is clear that the image of the restriction $\overline{\mathbf{U}_n} \setminus \mathbf{U}_n \to \operatorname{fpnil}_n \Lambda$ is precisely the set of minimal elements in $\operatorname{fpnil}_n \Lambda$. In particular

$$\operatorname{card}(\overline{\mathbf{U}_n} \setminus \mathbf{U}_n) \leq \operatorname{card}(\operatorname{fpnil}_n \Lambda).$$

On the other hand, every $\mathfrak{I} \in \operatorname{fpnil}_n \Lambda$ corresponds to a closed subset of $\overline{\mathbf{U}_n} \setminus \mathbf{U}_n$, and therefore $\operatorname{fpnil}_n \Lambda$ is finite if $\overline{\mathbf{U}_n} \setminus \mathbf{U}_n$ is finite. $\square$

Combining Theorem 4.1 and Proposition 4.4, one obtains the following alternative description of tameness which only involves finite dimensional modules.

Corollary 4.5. *The algebra Λ is of tame representation type if and only if for every $n \in \mathbb{N}$ there are only finitely many non-zero fp-idempotent and nilpotent ideals in* $\operatorname{mod}\Lambda$ *which are contained in* $[\operatorname{ind}_n \Lambda]$. *Moreover, in this case $\mu_\Lambda(n)$ equals the number of minimal elements in the set of these ideals.*

5. One-parameter families and generic modules

The infinite dimensional modules occuring in the preceding characterizations of tame representation type are generic modules. Recall that an indecomposable Λ-module is *generic* if it is of finite endolength but of infinite length over Λ; see [2]. In this section we describe explicitly the relation between one-parameter families and generic modules over algebras of tame representation type. The results presented here are mainly due to Crawley-Boevey; those results which involve the Ziegler spectrum are due to the author of this note. We do not give proofs but refer to Section 5 in [2] and Section 9 in [9]. Throughout this section Λ is a finite dimensional algebra over some algebraically closed field.

Theorem 5.1. *Let Λ be of tame representation type and let $n \in \mathbb{N}$. Then there are one-parameter families $\mathbf{U}_1, \dots, \mathbf{U}_{\mu_\Lambda(n)}$ of n-dimensional indecomposable Λ-modules having the following properties:*

1. $\operatorname{ind}_n \Lambda \setminus (\mathbf{U}_1 \cup \ldots \cup \mathbf{U}_{\mu_\Lambda(n)})$ *is finite.*
2. *The closure $\overline{\mathbf{U}_i}$ contains a unique generic Λ-module G_i for every i.*
3. *$G_i = G_j$ if and only if $i = j$.*
4. $\overline{\operatorname{ind}_n \Lambda} \setminus \operatorname{ind}_n \Lambda = \{G_1, \dots, G_{\mu_\Lambda(n)}\}$.
5. *A generic Λ-module belongs to $\{G_1, \dots, G_{\mu_\Lambda(n)}\}$ if and only if its endolength divides n.*

In [1], Crawley-Boevey has shown that over a tame algebra almost all indecomposable modules of a fixed dimension belong to homogeneous tubes of the Auslander-Reiten quiver of Λ. Recall that a family $\mathbf{T} = (M_i)_{i \in \mathbb{N}}$ of Λ-modules forms a *homogeneous tube* if there are maps $M_i \to M_{i+1}$ and $M_{i+1} \to M_i$ for every $i \in \mathbb{N}$ which induce almost split sequences $0 \to M_i \to M_{i-1} \coprod M_{i+1} \to M_i \to 0$ for every $i \in \mathbb{N}$ where $M_0 = 0$. We say that $\mathbf{T}$ is *generic* if $\overline{\mathbf{T}} = \mathbf{T} \cup \{\varinjlim M_i, \varprojlim M_i, G\}$ for some generic module G.

Theorem 5.2. *Let Λ be of tame representation type and let $n \in \mathbb{N}$. Then all but finitely many n-dimensional indecomposable Λ-modules belong to generic homogeneous tubes.*

The occurence of homogeneous tubes is a characteristic phenomenon for tame algebras. This can be made precise as follows.

Theorem 5.3. *An algebra Λ is of tame representation type if and only if every generic Λ-module belongs to the closure of a* (*generic*) *homogeneous tube.*

References

[1] W.W. Crawley-Boevey, On tame algebras and bocses, Proc. London Math. Soc. **56** (1988), 451–483.

[2] W.W. Crawley-Boevey, Tame algebras and generic modules, Proc. London Math. Soc. **63** (1991), 241–264.

[3] W.W. Crawley-Boevey, Modules of finite length over their endomorphism ring, in: Representations of algebras and related topics, eds. S. Brenner and H. Tachikawa, London Math. Soc. Lec. Note Series **168** (1992), 127–184.

[4] W.W. Crawley-Boevey, Infinite dimensional modules in the representation theory of finite dimensional algebras, preprint (1996).

[5] Yu.A. Drozd, Tame and wild matrix problems, Representations and quadratic forms, Institute of Mathematics, Academy of Sciences, Ukrainian SSR, Kiev (1979), 39–74, Amer. Math. Soc. Transl. **128** (1986), 31–55.

[6] I. Herzog, The Ziegler spectrum of a locally coherent Grothendieck category, Proc. London Math. Soc. **74** (1997), 503–558.

[7] C.U. Jensen and H. Lenzing, Model theoretic algebra, Gordon and Breach, New York (1989).

[8] H. Krause, The spectrum of a locally coherent category, J. Pure Appl. Algebra **114** (1997), 259–271.

[9] H. Krause, Generic modules over artin algebras, Proc. London Math. Soc. **76** (1998), 276–306.

[10] H. Krause, Exactly definable categories, J. Algebra **201** (1998), 456–492.

[11] H. Krause, Functors on locally finitely presented categories, Colloq. Math. **75** (1998), 105–131.

[12] H. Krause, The spectrum of a module category, Habilitationsschrift, Universität Bielefeld (1998).

[13] M. Prest, Model theory and modules, London Math. Soc. Lec. Note Series **130** (1988).

[14] C.M. Ringel, Recent advances in the representation theory of finite dimensional algebras, in: Representations of finite groups and finite dimensional algebras, Birkhäuser-Verlag, Basel (1991), 141–192.

[15] M. Ziegler, Model theory of modules, Ann. of Pure and Appl. Logic **26** (1984), 149–213.

Fakultät für Mathematik, Universität Bielefeld, 33501 Bielefeld, Germany
E-mail address: `henning@mathematik.uni-bielefeld.de`