

# THE TELESCOPE CONJECTURE FOR HEREDITARY RINGS VIA EXT-ORTHOGONAL PAIRS

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ABSTRACT. For the module category of a hereditary ring, the Ext-orthogonal pairs of subcategories are studied. For each Ext-orthogonal pair that is generated by a single module, a 5-term exact sequence is constructed. The pairs of finite type are characterized and two consequences for the class of hereditary rings are established: homological epimorphisms and universal localizations coincide, and the telescope conjecture for the derived category holds true.

## 1. INTRODUCTION

In this paper, we prove the telescope conjecture for the derived category of any hereditary ring. To achieve this, we study Ext-orthogonal pairs of subcategories for hereditary module categories.

The telescope conjecture for the derived category of a module category is also called smashing conjecture. It is the analogue of the telescope conjecture from stable homotopy theory which is due to Bousfield and Ravenel [5, 22]. In each case one deals with a compactly generated triangulated category. The conjecture then claims that a localizing subcategory is generated by compact objects provided it is smashing, that is, the localizing subcategory arises as the kernel of a localization functor that preserves arbitrary coproducts [18].

In this general form, the telescope conjecture seems to be wide open. For the stable homotopy category, we refer to the work of Mahowald, Ravenel, and Shick [17] for more details. For the derived category of a module category, only two results seem to be known so far. Neeman proved the conjecture for the derived category of a commutative noetherian ring [19], essentially by classifying all localizing subcategories. On the other hand, Keller gave an explicit example of a commutative ring where the conjecture does not hold [13]. In fact, an analysis of Keller's argument [14] shows that there are such examples having global dimension 2. The approach for hereditary rings presented here is completely different from Neeman's. In particular, we are working in a non-commutative setting and without using any noetherian assumption.

A useful concept for proving the telescope conjecture in our context is the notion of an Ext-orthogonal pair. This concept seems to be new, but it is based on the notion of a perpendicular category which is one of the fundamental tools for studying hereditary categories arising in representation theory [27, 11].

Given any abelian category  $\mathcal{A}$ , we call a pair  $(\mathcal{X}, \mathcal{Y})$  of full subcategories *Ext-orthogonal* if  $\mathcal{X}$  and  $\mathcal{Y}$  are orthogonal to each other with respect to the bifunctor  $\coprod_{n \geq 0} \text{Ext}_{\mathcal{A}}^n(-, -)$ .

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Version from October 6, 2008.

The second author is supported by the Research Council of Norway through the Storforsk-project "Homological and geometric methods in algebra".

This concept is the analogue of a *torsion pair* and a *cotorsion pair* where one considers instead the bifunctors  $\mathrm{Hom}_{\mathcal{A}}(-, -)$  and  $\prod_{n>0} \mathrm{Ext}_{\mathcal{A}}^n(-, -)$  respectively [8, 24].

Torsion and cotorsion pairs are most interesting when they are *complete*. For a torsion pair this means that each object  $M$  in  $\mathcal{A}$  admits a short exact sequence  $0 \rightarrow X_M \rightarrow M \rightarrow Y^M \rightarrow 0$  with  $X_M \in \mathcal{X}$  and  $Y^M \in \mathcal{Y}$ . In the second case this means that each object  $M$  admits short exact sequences  $0 \rightarrow Y_M \rightarrow X_M \rightarrow M \rightarrow 0$  and  $0 \rightarrow M \rightarrow Y^M \rightarrow X^M \rightarrow 0$  with  $X_M, X^M \in \mathcal{X}$  and  $Y_M, Y^M \in \mathcal{Y}$ .

It turns out that there is also a reasonable notion of completeness for Ext-orthogonal pairs. In that case each object  $M$  in  $\mathcal{A}$  admits a 5-term exact sequence

$$0 \rightarrow Y_M \rightarrow X_M \rightarrow M \rightarrow Y^M \rightarrow X^M \rightarrow 0$$

with  $X_M, X^M \in \mathcal{X}$  and  $Y_M, Y^M \in \mathcal{Y}$ .

In this work we study Ext-orthogonal pairs for the module category of a hereditary ring. The assumption on the ring implies a close connection between the module category and its derived category. It is this connection which we exploit in both directions. We use Bousfield localization functors which exist for the derived category to establish the completeness of certain Ext-orthogonal pairs for the module category; see §2. On the other hand, we are able to prove the telescope conjecture for the derived category by showing first a similar result for Ext-orthogonal pairs; see §5 and §7.

At a first glance the telescope conjecture seems to be a rather abstract statement about unbounded derived categories. However in the context of a fixed hereditary ring, it turns out that smashing localizing subcategories are parametrized by a number of finite structures which play a role in various situations. We mention here extension closed abelian subcategories of finitely presented modules and homological ring epimorphisms; see §8. Note that Ext-orthogonal pairs provide a useful link between these structures.

The telescope conjecture and its proof are related to interesting recent work by some other authors. In [28], Schofield describes for any hereditary ring its universal localizations in terms of appropriate subcategories of finitely presented modules. This is a consequence of the present work since we show that homological epimorphisms and universal localizations coincide for any hereditary ring; see §6. In [21], Nicolás and Saorín establish for a differential graded algebra a correspondence between recollements for its derived category and differential graded homological epimorphisms. This correspondence specializes for a hereditary ring to the above mentioned bijection between smashing localizing subcategories and homological epimorphisms.

Specific examples of Ext-orthogonal pairs arise in the representation theory of finite dimensional algebras via perpendicular categories; see §4. Note that a perpendicular category is just one half of an Ext-orthogonal pair. Schofield introduced perpendicular categories for representations of quivers [27] and this fits into our set-up because the path algebra of any quiver is hereditary. In fact, the concept of a perpendicular category is fundamental for studying hereditary categories arising in representation theory [11]. It is therefore somewhat surprising that the 5-term exact sequence for a complete Ext-orthogonal pair seems to appear for the first time in this work.

The first author would like to thank Lidia Angeleri Hügel and Manolo Saorín for some helpful discussions concerning this work.

## 2. EXT-ORTHOGONAL PAIRS

Let  $\mathcal{A}$  be an abelian category. Given a pair of objects  $X, Y \in \mathcal{A}$ , set

$$\mathrm{Ext}_{\mathcal{A}}^*(X, Y) = \prod_{n \in \mathbb{Z}} \mathrm{Ext}_{\mathcal{A}}^n(X, Y).$$

For a subcategory  $\mathcal{C}$  of  $\mathcal{A}$  we consider its full Ext-orthogonal subcategories

$$\begin{aligned} {}^{\perp}\mathcal{C} &= \{X \in \mathcal{A} \mid \mathrm{Ext}_{\mathcal{A}}^*(X, C) = 0 \text{ for all } C \in \mathcal{C}\}, \\ \mathcal{C}^{\perp} &= \{Y \in \mathcal{A} \mid \mathrm{Ext}_{\mathcal{A}}^*(C, Y) = 0 \text{ for all } C \in \mathcal{C}\}. \end{aligned}$$

**Definition 2.1.** An *Ext-orthogonal pair* for  $\mathcal{A}$  is a pair  $(\mathcal{X}, \mathcal{Y})$  of full subcategories such that  $\mathcal{X}^{\perp} = \mathcal{Y}$  and  $\mathcal{X} = {}^{\perp}\mathcal{Y}$ . An Ext-orthogonal pair  $(\mathcal{X}, \mathcal{Y})$  is called *complete* if there exists for each object  $M \in \mathcal{A}$  an exact sequence

$$\varepsilon_M: 0 \rightarrow Y_M \rightarrow X_M \rightarrow M \rightarrow Y^M \rightarrow X^M \rightarrow 0$$

with  $X_M, X^M \in \mathcal{X}$  and  $Y_M, Y^M \in \mathcal{Y}$ . The pair  $(\mathcal{X}, \mathcal{Y})$  is *generated* by a subcategory  $\mathcal{C}$  of  $\mathcal{A}$  if  $\mathcal{Y} = \mathcal{C}^{\perp}$ .

The definition can be extended to the derived category  $\mathbf{D}(\mathcal{A})$  of  $\mathcal{A}$  if we put for each pair of complexes  $X, Y \in \mathbf{D}(\mathcal{A})$  and  $n \in \mathbb{Z}$

$$\mathrm{Ext}_{\mathcal{A}}^n(X, Y) = \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X, Y[n]).$$

Thus an *Ext-orthogonal pair* for  $\mathbf{D}(\mathcal{A})$  is a pair  $(\mathcal{X}, \mathcal{Y})$  of full subcategories of  $\mathbf{D}(\mathcal{A})$  such that  $\mathcal{X}^{\perp} = \mathcal{Y}$  and  $\mathcal{X} = {}^{\perp}\mathcal{Y}$ .

Recall that an *abelian subcategory* of  $\mathcal{A}$  is a full subcategory  $\mathcal{C}$  such that the category  $\mathcal{C}$  is abelian and the inclusion functor  $\mathcal{C} \rightarrow \mathcal{A}$  is exact. Suppose  $\mathcal{A}$  is *hereditary*, that is,  $\mathrm{Ext}_{\mathcal{A}}^n(-, -)$  vanishes for all  $n > 1$ . Then a simple calculation shows that for any subcategory  $\mathcal{C}$  of  $\mathcal{A}$ , the subcategories  $\mathcal{C}^{\perp}$  and  ${}^{\perp}\mathcal{C}$  are extension closed abelian subcategories; see [11, Proposition 1.1].

The following result establishes the completeness for certain Ext-orthogonal pairs. Recall that an abelian category is a *Grothendieck category* if it has a set of generators and admits colimits that are exact when taken over filtered categories.

**Theorem 2.2.** *Let  $\mathcal{A}$  be a hereditary abelian Grothendieck category and  $X$  an object in  $\mathcal{A}$ . Set  $\mathcal{Y} = X^{\perp}$  and let  $\mathcal{X}$  denote the smallest extension closed abelian subcategory of  $\mathcal{A}$  that is closed under taking coproducts and contains  $X$ . Then  $(\mathcal{X}, \mathcal{Y})$  is a complete Ext-orthogonal pair for  $\mathcal{A}$ . Thus there exists for each object  $M \in \mathcal{A}$  a natural exact sequence*

$$0 \rightarrow Y_M \rightarrow X_M \rightarrow M \rightarrow Y^M \rightarrow X^M \rightarrow 0$$

with  $X_M, X^M \in \mathcal{X}$  and  $Y_M, Y^M \in \mathcal{Y}$ . The sequence induces bijections  $\mathrm{Hom}_{\mathcal{A}}(X, X_M) \rightarrow \mathrm{Hom}_{\mathcal{A}}(X, M)$  and  $\mathrm{Hom}_{\mathcal{A}}(Y^M, Y) \rightarrow \mathrm{Hom}_{\mathcal{A}}(M, Y)$  for all  $X \in \mathcal{X}$  and  $Y \in \mathcal{Y}$ .

The proof uses derived categories and Bousfield localization functors. Thus we need to collect some basic facts about hereditary abelian categories and their derived categories.

**The derived category of a hereditary abelian category.** Let  $\mathcal{A}$  be a hereditary abelian category and let  $\mathbf{D}(\mathcal{A})$  denote its derived category. We assume that  $\mathcal{A}$  admits coproducts and that the coproduct of any set of exact sequences is again exact. Thus the category  $\mathbf{D}(\mathcal{A})$  admits coproducts, and for each integer  $n$  these coproducts are preserved by the functor  $H^n: \mathbf{D}(\mathcal{A}) \rightarrow \mathcal{A}$  which takes a complex to its cohomology in degree  $n$ .

It is well-known that each complex is quasi-isomorphic to its cohomology.

**Lemma 2.3.** *Given a complex  $X$  in  $\mathbf{D}(\mathcal{A})$ , there are (non-canonical) isomorphisms*

$$\prod_{n \in \mathbb{Z}} (H^n X)[-n] \cong X \cong \prod_{n \in \mathbb{Z}} (H^n X)[-n].$$

*Proof.* See for instance [15, §1.6]. □

A full subcategory  $\mathcal{C}$  of  $\mathbf{D}(\mathcal{A})$  is called *thick* if  $\mathcal{C}$  is a triangulated subcategory and closed under taking direct summands. A thick subcategory is *localizing* if it is closed under taking coproducts. Note that for each subcategory  $\mathcal{C}$  the subcategories  $\mathcal{C}^\perp$  and  ${}^\perp\mathcal{C}$  are thick.

To a subcategory  $\mathcal{C}$  of  $\mathbf{D}(\mathcal{A})$  we assign the full subcategory

$$H^0\mathcal{C} = \{M \in \mathcal{A} \mid M = H^0 X \text{ for some } X \in \mathcal{C}\},$$

and given a subcategory  $\mathcal{X}$  of  $\mathcal{A}$ , we define the full subcategory

$$\mathbf{D}_{\mathcal{X}}(\mathcal{A}) = \{X \in \mathbf{D}(\mathcal{A}) \mid H^n X \in \mathcal{X} \text{ for all } n \in \mathbb{Z}\}.$$

Both assignments induce mutually inverse bijections between appropriate subcategories. This is a useful fact which we recall from [6, Theorem 6.1].

**Proposition 2.4.** *The functor  $H^0: \mathbf{D}(\mathcal{A}) \rightarrow \mathcal{A}$  induces a bijection between the localizing subcategories of  $\mathbf{D}(\mathcal{A})$  and the extension closed abelian subcategories of  $\mathcal{A}$  that are closed under coproducts. The inverse map sends a subcategory  $\mathcal{X}$  of  $\mathcal{A}$  to  $\mathbf{D}_{\mathcal{X}}(\mathcal{A})$ .* □

*Remark 2.5.* The bijection in Proposition 2.4 has an analogue for thick subcategories. Given any abelian category  $\mathcal{B}$ , the functor  $H^0: \mathbf{D}^b(\mathcal{B}) \rightarrow \mathcal{B}$  induces a bijection between the thick subcategories of  $\mathbf{D}^b(\mathcal{B})$  and the extension closed abelian subcategories of  $\mathcal{B}$ .

Next we extend these maps to bijections between Ext-orthogonal pairs.

**Proposition 2.6.** *The functor  $H^0: \mathbf{D}(\mathcal{A}) \rightarrow \mathcal{A}$  induces a bijection between the Ext-orthogonal pairs for  $\mathbf{D}(\mathcal{A})$  and the Ext-orthogonal pairs for  $\mathcal{A}$ . The inverse map sends a pair  $(\mathcal{X}, \mathcal{Y})$  for  $\mathcal{A}$  to  $(\mathbf{D}_{\mathcal{X}}(\mathcal{A}), \mathbf{D}_{\mathcal{Y}}(\mathcal{A}))$ .*

*Proof.* First observe that for each pair of complexes  $X, Y \in \mathbf{D}(\mathcal{A})$ , we have  $\text{Ext}_{\mathcal{A}}^*(X, Y) = 0$  if and only if  $\text{Ext}_{\mathcal{A}}^*(H^p X, H^q Y) = 0$  for all  $p, q \in \mathbb{Z}$ . This is a consequence of Lemma 2.3. It follows that  $H^0$  and its inverse send Ext-orthogonal pairs to Ext-orthogonal pairs. Each Ext-orthogonal pair is determined by its first half, and therefore an application of Proposition 2.4 shows that both maps are mutually inverse. □

**Localization functors.** Let  $\mathcal{T}$  be a triangulated category. A *localization functor*  $L: \mathcal{T} \rightarrow \mathcal{T}$  is an exact functor that admits a natural transformation  $\eta: \text{Id}_{\mathcal{T}} \rightarrow L$  such that  $L\eta_X$  is an isomorphism and  $L\eta_X = \eta_{LX}$  for all objects  $X \in \mathcal{T}$ . Basic facts about localization functors one finds, for example, in [4, §3].

**Proposition 2.7.** *Let  $\mathcal{A}$  be a hereditary abelian category. For a full subcategory  $\mathcal{X}$  of  $\mathcal{A}$  the following are equivalent.*

- (1) *There exists a localization functor  $L: \mathbf{D}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A})$  such that  $\text{Ker } L = \mathbf{D}_{\mathcal{X}}(\mathcal{A})$ .*  
(2) *There exists a complete Ext-orthogonal pair  $(\mathcal{X}, \mathcal{Y})$  for  $\mathcal{A}$ .*

*Proof.* (1)  $\Rightarrow$  (2): The kernel  $\text{Ker } L$  and the essential image  $\text{Im } L$  of a localization functor  $L$  form an Ext-orthogonal pair for  $\mathbf{D}(\mathcal{A})$ ; see for instance [4, Lemma 3.3]. Then it follows from Proposition 2.6 that the pair  $(\mathcal{X}, \mathcal{Y}) = (H^0 \text{Ker } L, H^0 \text{Im } L)$  is Ext-orthogonal for  $\mathcal{A}$ .

The localization functor  $L$  comes equipped with a natural transformation  $\eta: \text{Id}_{\mathbf{D}(\mathcal{A})} \rightarrow L$ , and for each complex  $M$  we complete the morphism  $\eta_M: M \rightarrow LM$  to an exact triangle

$$\Gamma M \rightarrow M \rightarrow LM \rightarrow \Gamma M[1].$$

Note that  $\Gamma M \in \text{Ker } L$  and  $LM \in \text{Im } L$  since  $L\eta_M$  is an isomorphism and  $L$  is exact. Now suppose that  $M$  is concentrated in degree zero. Applying  $H^0$  to this triangle yields an exact sequence

$$0 \rightarrow Y_M \rightarrow X_M \rightarrow M \rightarrow Y^M \rightarrow X^M \rightarrow 0$$

with  $X_M, X^M \in \mathcal{X}$  and  $Y_M, Y^M \in \mathcal{Y}$ .

(2)  $\Rightarrow$  (1): Let  $(\mathcal{X}, \mathcal{Y})$  be an Ext-orthogonal pair for  $\mathcal{A}$ . This pair induces an Ext-orthogonal pair  $(\mathbf{D}_{\mathcal{X}}(\mathcal{A}), \mathbf{D}_{\mathcal{Y}}(\mathcal{A}))$  for  $\mathbf{D}(\mathcal{A})$  by Proposition 2.6. In order to construct a localization functor  $L: \mathbf{D}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A})$  such that  $\text{Ker } L = \mathbf{D}_{\mathcal{X}}(\mathcal{A})$ , it is sufficient to construct for each object  $M$  in  $\mathbf{D}(\mathcal{A})$  an exact triangle  $X \rightarrow M \rightarrow Y \rightarrow X[1]$  with  $X \in \mathbf{D}_{\mathcal{X}}(\mathcal{A})$  and  $Y \in \mathbf{D}_{\mathcal{Y}}(\mathcal{A})$ . Then one defines  $LM = Y$  and the morphism  $M \rightarrow Y$  induces a natural transformation  $\eta: \text{Id}_{\mathbf{D}(\mathcal{A})} \rightarrow L$  having the required properties. In view of Lemma 2.3 it is sufficient to assume that  $M$  is a complex concentrated in degree zero.

Suppose that  $M$  admits an approximation sequence

$$\varepsilon_M: 0 \rightarrow Y_M \rightarrow X_M \rightarrow M \rightarrow Y^M \rightarrow X^M \rightarrow 0$$

with  $X_M, X^M \in \mathcal{X}$  and  $Y_M, Y^M \in \mathcal{Y}$ . Let  $M'$  denote the image of  $X_M \rightarrow M$  and  $M''$  the image of  $M \rightarrow Y^M$ . Then  $\varepsilon_M$  induces the following three exact sequences

$$\begin{aligned} \alpha_M: 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0, \\ \beta_M: 0 \rightarrow Y_M \rightarrow X_M \rightarrow M' \rightarrow 0, \\ \gamma_M: 0 \rightarrow M'' \rightarrow Y^M \rightarrow X^M \rightarrow 0. \end{aligned}$$

In  $\mathbf{D}(\mathcal{A})$  these three exact sequence give rise to the following square

$$\begin{array}{ccc} X^M[-2] & \xrightarrow{\gamma_M} & M''[-1] \\ \downarrow 0 & & \downarrow \alpha_M \\ X_M & \xrightarrow{\beta_M} & M' \end{array}$$

which is commutative since  $\text{Hom}_{\mathbf{D}(\mathcal{A})}(U[-2], V) = 0$  for any  $U, V \in \mathcal{A}$ . An application of the octahedral axiom shows that this square can be extended as follows to a diagram

where each row and each column is an exact triangle.

$$\begin{array}{ccccccc}
X^M[-2] & \longrightarrow & M''[-1] & \longrightarrow & Y^M[-1] & \longrightarrow & X^M[-1] \\
\downarrow 0 & & \downarrow & & \downarrow 0 & & \downarrow 0 \\
X_M & \longrightarrow & M' & \longrightarrow & Y_M[1] & \longrightarrow & X_M[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_M \oplus X^M[-1] & \longrightarrow & M & \longrightarrow & Y_M[1] \oplus Y^M & \longrightarrow & X_M[1] \oplus X^M \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X^M[-1] & \longrightarrow & M'' & \longrightarrow & Y^M & \longrightarrow & X^M
\end{array}$$

The first and third column are split exact triangles, and this explains the objects appearing in the third row. In particular, this yields the desired exact triangle  $X \rightarrow M \rightarrow Y \rightarrow X[1]$  with  $X \in \mathbf{D}_{\mathcal{X}}(\mathcal{A})$  and  $Y \in \mathbf{D}_{\mathcal{Y}}(\mathcal{A})$ .  $\square$

Next we formulate the functorial properties of the 5-term exact sequence constructed in Proposition 2.7.

**Lemma 2.8.** *Let  $\mathcal{A}$  be an abelian category and  $(\mathcal{X}, \mathcal{Y})$  an Ext-orthogonal pair for  $\mathcal{A}$ . Suppose there is an exact sequence*

$$\varepsilon_M: 0 \rightarrow Y_M \rightarrow X_M \rightarrow M \rightarrow Y^M \rightarrow X^M \rightarrow 0$$

in  $\mathcal{A}$  with  $X_M, X^M \in \mathcal{X}$  and  $Y_M, Y^M \in \mathcal{Y}$ .

- (1) *The sequence  $\varepsilon_M$  induces for all  $X \in \mathcal{X}$  and  $Y \in \mathcal{Y}$  bijections  $\mathrm{Hom}_{\mathcal{A}}(X, X_M) \rightarrow \mathrm{Hom}_{\mathcal{A}}(X, M)$  and  $\mathrm{Hom}_{\mathcal{A}}(Y^M, Y) \rightarrow \mathrm{Hom}_{\mathcal{A}}(M, Y)$ .*
- (2) *Let  $\varepsilon_N: 0 \rightarrow Y_N \rightarrow X_N \rightarrow N \rightarrow Y^N \rightarrow X^N \rightarrow 0$  be an exact sequence in  $\mathcal{A}$  with  $X_N, X^N \in \mathcal{X}$  and  $Y_N, Y^N \in \mathcal{Y}$ . Then each morphism  $M \rightarrow N$  extends uniquely to a morphism  $\varepsilon_M \rightarrow \varepsilon_N$  of exact sequences.*
- (3) *Any exact sequence  $0 \rightarrow Y' \rightarrow X' \rightarrow M \rightarrow Y'' \rightarrow X'' \rightarrow 0$  in  $\mathcal{A}$  with  $X', X'' \in \mathcal{X}$  and  $Y', Y'' \in \mathcal{Y}$  is uniquely isomorphic to  $\varepsilon_M$ .*

*Proof.* We prove part (1). Then part (2) and (3) are immediate consequences.

Fix an object  $X \in \mathcal{X}$ . The map  $\mu: \mathrm{Hom}_{\mathcal{A}}(X, X_M) \rightarrow \mathrm{Hom}_{\mathcal{A}}(X, M)$  is injective because  $\mathrm{Hom}_{\mathcal{A}}(X, Y_M) = 0$ . Any morphism  $X \rightarrow M$  factors through the kernel  $M'$  of  $M \rightarrow Y^M$  since  $\mathrm{Hom}_{\mathcal{A}}(X, Y^M) = 0$ . The induced morphism  $X \rightarrow M'$  factors through  $X_M \rightarrow M'$  since  $\mathrm{Ext}_{\mathcal{A}}^1(X, Y_M) = 0$ . Thus  $\mu$  is surjective. The argument for the other map  $\mathrm{Hom}_{\mathcal{A}}(Y^M, Y) \rightarrow \mathrm{Hom}_{\mathcal{A}}(M, Y)$  is dual.  $\square$

**Ext-orthogonal pairs for Grothendieck categories.** We give the proof of Theorem 2.2. The basic idea is to establish a localization functor for  $\mathbf{D}(\mathcal{A})$  and to derive the exact approximation sequence in  $\mathcal{A}$  by taking the cohomology of some appropriate exact triangle.

*Proof of Theorem 2.2.* Let  $\mathcal{X}$  denote the smallest extension closed abelian subcategory of  $\mathcal{A}$  that contains  $X$  and is closed under coproducts. Then Proposition 2.4 implies that  $\mathbf{D}_{\mathcal{X}}(\mathcal{A})$  is the smallest localizing subcategory of  $\mathbf{D}(\mathcal{A})$  containing  $X$ . Thus there exists a localization functor  $L: \mathbf{D}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A})$  with  $\mathrm{Ker} L = \mathbf{D}_{\mathcal{X}}(\mathcal{A})$ . This is a result which goes back to Bousfield. In the context of derived categories we refer to [2, Theorem 5.7].

Now apply Proposition 2.7 to get the 5-term exact sequence for each object  $M$  in  $\mathcal{A}$ . The properties of this sequence follow from Lemma 2.8.  $\square$

*Remark 2.9.* We do not know an example of an Ext-orthogonal pair  $(\mathcal{X}, \mathcal{Y})$  for a hereditary abelian Grothendieck category such that the pair  $(\mathcal{X}, \mathcal{Y})$  is not complete.

Further examples of Ext-orthogonal pairs arise as follows. Let  $\mathcal{A}$  be any abelian Grothendieck category and  $\mathcal{X}$  a *localizing subcategory*. Thus  $\mathcal{X}$  is a full subcategory closed under taking coproducts such that for any exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  in  $\mathcal{A}$  we have  $M \in \mathcal{X}$  if and only if  $M', M'' \in \mathcal{X}$ . Set  $\mathcal{Y} = \mathcal{X}^\perp$  and let  $\mathcal{Y}_{\text{inj}}$  denote the full subcategory of injective objects in  $\mathcal{Y}$ . Then  $\mathcal{X} = {}^\perp \mathcal{Y}_{\text{inj}}$  and therefore  $(\mathcal{X}, \mathcal{Y})$  is an Ext-orthogonal pair for  $\mathcal{A}$ ; see [9, III.4] for details.

**Torsion and cotorsion pairs.** Let  $\mathcal{A}$  be an abelian category and  $(\mathcal{X}, \mathcal{Y})$  an Ext-orthogonal pair. We sketch an interpretation of the pair  $(\mathcal{X}, \mathcal{Y})$  in terms of torsion and cotorsion pairs. Here, a pair  $(\mathcal{U}, \mathcal{V})$  of full subcategories of  $\mathcal{A}$  is called *torsion pair* if  $\mathcal{U}$  and  $\mathcal{V}$  are orthogonal to each other with respect to  $\text{Hom}_{\mathcal{A}}(-, -)$ . Analogously, a pair of full subcategories is a *cotorsion pair* if both categories are orthogonal to each other with respect to  $\prod_{n>0} \text{Ext}_{\mathcal{A}}^n(-, -)$ .

The subcategory  $\mathcal{X}$  generates a torsion pair  $(\mathcal{X}_0, \mathcal{Y}_0)$  and a cotorsion pair  $(\mathcal{X}_1, \mathcal{Y}_1)$  for  $\mathcal{A}$ , if one defines the corresponding full subcategories of  $\mathcal{A}$  as follows:

$$\begin{aligned} \mathcal{Y}_0 &= \{Y \in \mathcal{A} \mid \text{Hom}_{\mathcal{A}}(X, Y) = 0 \text{ for all } X \in \mathcal{X}\}, \\ \mathcal{X}_0 &= \{X \in \mathcal{A} \mid \text{Hom}_{\mathcal{A}}(X, Y) = 0 \text{ for all } Y \in \mathcal{Y}_0\}, \\ \mathcal{Y}_1 &= \{Y \in \mathcal{A} \mid \text{Ext}_{\mathcal{A}}^n(X, Y) = 0 \text{ for all } X \in \mathcal{X}, n > 0\}, \\ \mathcal{X}_1 &= \{X \in \mathcal{A} \mid \text{Ext}_{\mathcal{A}}^n(X, Y) = 0 \text{ for all } Y \in \mathcal{Y}_1, n > 0\}. \end{aligned}$$

Note that  $\mathcal{X} = \mathcal{X}_0 \cap \mathcal{X}_1$  and  $\mathcal{Y} = \mathcal{Y}_0 \cap \mathcal{Y}_1$ . In particular, one recovers the pair  $(\mathcal{X}, \mathcal{Y})$  from  $(\mathcal{X}_0, \mathcal{Y}_0)$  and  $(\mathcal{X}_1, \mathcal{Y}_1)$ .

Suppose an object  $M \in \mathcal{A}$  admits an approximation sequence

$$\varepsilon_M: 0 \rightarrow Y_M \rightarrow X_M \rightarrow M \rightarrow Y^M \rightarrow X^M \rightarrow 0$$

with  $X_M, X^M \in \mathcal{X}$  and  $Y_M, Y^M \in \mathcal{Y}$ . We give the following interpretation of this sequence. Let  $M'$  denote the image of  $X_M \rightarrow M$  and  $M''$  the image of  $M \rightarrow Y^M$ . Then there are three short exact sequences:

$$\begin{aligned} \alpha_M: 0 &\rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0, \\ \beta_M: 0 &\rightarrow Y_M \rightarrow X_M \rightarrow M' \rightarrow 0, \\ \gamma_M: 0 &\rightarrow M'' \rightarrow Y^M \rightarrow X^M \rightarrow 0. \end{aligned}$$

The sequence  $\alpha_M$  is the approximation sequence of  $M$  with respect to the torsion pair  $(\mathcal{X}_0, \mathcal{Y}_0)$ , that is,  $M' \in \mathcal{X}_0$  and  $M'' \in \mathcal{Y}_0$ . On the other hand,  $\beta_M$  and  $\gamma_M$  are approximation sequences of  $M'$  and  $M''$  respectively, with respect to the cotorsion pair  $(\mathcal{X}_1, \mathcal{Y}_1)$ , that is,  $X_M, X^M \in \mathcal{X}_1$  and  $Y_M, Y^M \in \mathcal{Y}_1$ . Thus the 5-term exact sequence  $\varepsilon_M$  is obtained by splicing together three short exact approximation sequences.

Suppose that the Ext-orthogonal pair  $(\mathcal{X}, \mathcal{Y})$  is complete. Then the associated torsion pair  $(\mathcal{X}_0, \mathcal{Y}_0)$  has an explicit description: we have  $\mathcal{X}_0 = \text{Fac } \mathcal{X}$  and  $\mathcal{Y}_0 = \text{Sub } \mathcal{Y}$ , where

$$\text{Fac } \mathcal{X} = \{X/U \mid U \subseteq X, X \in \mathcal{X}\} \quad \text{and} \quad \text{Sub } \mathcal{Y} = \{U \mid U \subseteq Y, Y \in \mathcal{Y}\}.$$

## 3. HOMOLOGICAL EPIMORPHISMS

From now on we study Ext-orthogonal pairs for module categories. Thus we fix a ring  $A$  and denote by  $\text{Mod } A$  the category of (right)  $A$ -modules. The full subcategory formed by all finitely presented  $A$ -modules is denoted by  $\text{mod } A$ .

Most of our result require the ring  $A$  to be *hereditary*. This means the category of  $A$ -modules is hereditary, that is,  $\text{Ext}_A^n(-, -)$  vanishes for all  $n > 1$ .

Ext-orthogonal pairs for module categories over hereditary rings are closely related to homological epimorphisms. Recall that a ring homomorphism  $A \rightarrow B$  is a *homological epimorphism* if

$$B \otimes_A B \cong B \quad \text{and} \quad \text{Tor}_n^A(B, B) = 0 \quad \text{for all } n > 0,$$

equivalently, if restriction induces isomorphisms

$$\text{Ext}_B^*(X, Y) \xrightarrow{\sim} \text{Ext}_A^*(X, Y)$$

for all  $B$ -modules  $X, Y$ ; see [11] for details.

**Proposition 3.1.** *Let  $A$  be a hereditary ring and  $f: A \rightarrow B$  a homological epimorphism. Denote by  $\mathcal{Y}$  the category of  $A$ -modules which are restrictions of modules over  $B$ . Set  $\mathcal{X} = {}^\perp \mathcal{Y}$  and  $\mathcal{Y}^\perp = \mathcal{Z}$ . Then  $(\mathcal{X}, \mathcal{Y})$  and  $(\mathcal{Y}, \mathcal{Z})$  are complete Ext-orthogonal pairs for  $\text{Mod } A$  with  $\mathcal{Y} = (\text{Ker } f \oplus \text{Coker } f)^\perp$  and  $\mathcal{Z} = B^\perp$ .*

*Proof.* We wish to apply Theorem 2.2 which provides a construction for complete Ext-orthogonal pairs.

First observe that  $\mathcal{Y}$  is the smallest extension closed abelian subcategory of  $\text{Mod } A$  closed under coproducts and containing  $B$ . This yields  $\mathcal{Z} = B^\perp$ .

Next we show that  $\mathcal{Y} = (\text{Ker } f \oplus \text{Coker } f)^\perp$ . In fact, an  $A$ -module  $Y$  is the restriction of a  $B$ -module if and only if  $f$  induces an isomorphism  $\text{Hom}_A(B, Y) \rightarrow \text{Hom}_A(A, Y)$ . Using the assumptions on  $A$  and  $f$ , a simple calculation shows that this implies  $\mathcal{Y} = (\text{Ker } f \oplus \text{Coker } f)^\perp$ .

It remains to apply Theorem 2.2. Thus  $(\mathcal{X}, \mathcal{Y})$  and  $(\mathcal{Y}, \mathcal{Z})$  are complete Ext-orthogonal pairs.  $\square$

Next we use a theorem of Gabriel and de la Peña. It identifies the full subcategories of a module category  $\text{Mod } A$  that arise as the image of the restriction functor  $\text{Mod } B \rightarrow \text{Mod } A$  for a ring epimorphism  $A \rightarrow B$ .

**Proposition 3.2.** *Let  $A$  be a hereditary ring and  $\mathcal{Y}$  an extension closed abelian subcategory of  $\text{Mod } A$  that is closed under taking products and coproducts. Then there exists a homological epimorphism  $f: A \rightarrow B$  such that the restriction functor  $\text{Mod } B \rightarrow \text{Mod } A$  induces an equivalence  $\text{Mod } B \xrightarrow{\sim} \mathcal{Y}$ .*

*Proof.* It follows from [10, Theorem 1.2] that there exists an epimorphism  $f: A \rightarrow B$  such that the restriction functor  $\text{Mod } B \rightarrow \text{Mod } A$  induces an equivalence  $\text{Mod } B \xrightarrow{\sim} \mathcal{Y}$ . To be more specific, one constructs a left adjoint  $F: \text{Mod } A \rightarrow \mathcal{Y}$  for the inclusion  $\mathcal{Y} \rightarrow \text{Mod } A$ . Then  $FA$  is a small projective generator for  $\mathcal{Y}$ , because  $A$  has this property for  $\text{Mod } A$  and the inclusion of  $\mathcal{Y}$  is an exact functors that preserves coproducts. Thus one takes for  $f$  the induced map  $A \cong \text{End}_A(A) \rightarrow \text{End}_A(FA)$ .

We claim that restriction via  $f$  induces an isomorphism

$$\text{Ext}_B^n(X, Y) \xrightarrow{\sim} \text{Ext}_A^n(X, Y)$$

for all  $B$ -modules  $X, Y$  and all  $n \geq 0$ . This is clear for  $n = 0, 1$  since  $\mathcal{Y}$  is extension closed. On the other hand, the isomorphism for  $n = 1$  implies that  $\text{Ext}_B^1(X, -)$  is right exact since  $A$  is hereditary. It follows that  $\text{Ext}_B^n(-, -)$  vanishes for all  $n > 1$ .  $\square$

We will use the fact that each homological epimorphism  $A \rightarrow B$  induces a pair of localization functors  $\mathbf{D}(\text{Mod } B) \rightarrow \mathbf{D}(\text{Mod } A)$ .

**Lemma 3.3.** *Let  $A \rightarrow B$  be a homological epimorphism and denote by  $\mathcal{Y}$  the category of  $A$ -modules which are restrictions of modules over  $B$ .*

- (1) *The functor  $\mathbf{D}(\text{Mod } B) \rightarrow \mathbf{D}(\text{Mod } A)$  sending a complex  $X$  to  $X \otimes_A^L B$  is a localization functor with essential image equal to  $\mathbf{D}_{\mathcal{Y}}(\text{Mod } A)$ .*
- (2) *The functor  $\mathbf{D}(\text{Mod } B) \rightarrow \mathbf{D}(\text{Mod } A)$  sending a complex  $X$  to the cone of the natural morphism  $\mathbf{R}\text{Hom}_A(B, X) \rightarrow X$  is a localization functor with kernel equal to  $\mathbf{D}_{\mathcal{Y}}(\text{Mod } A)$ .*

*Proof.* Restriction along  $f: A \rightarrow B$  identifies  $\text{Mod } B$  with  $\mathcal{Y}$ . The functor induces an isomorphism

$$\text{Ext}_B^n(X, Y) \xrightarrow{\sim} \text{Ext}_A^n(X, Y)$$

for all  $B$ -modules  $X, Y$  and all  $n \geq 0$ , because  $f$  is a homological epimorphism. This isomorphism implies that the induced functor  $f_*: \mathbf{D}(\text{Mod } B) \rightarrow \mathbf{D}(\text{Mod } A)$  is fully faithful with essential image  $\mathbf{D}_{\mathcal{Y}}(\text{Mod } A)$ .

(1) The functor  $f_*$  admits a left adjoint  $f^* = - \otimes_A^L B$  and we have therefore a localization functor  $L: \mathbf{D}(\text{Mod } A) \rightarrow \mathbf{D}(\text{Mod } A)$  sending a complex  $X$  to  $f_* f^*(X)$ ; see [4, Lemma 3.1]. It remains to note that the essential images of  $L$  and  $f_*$  coincide.

(2) The functor  $f_*$  admits a right adjoint  $f^! = \mathbf{R}\text{Hom}_A(B, -)$  and we have therefore a colocalization functor  $\Gamma: \mathbf{D}(\text{Mod } A) \rightarrow \mathbf{D}(\text{Mod } A)$  sending a complex  $X$  to  $f_* f^!(X)$ . Note that the adjunction morphism  $\Gamma X \rightarrow X$  is an isomorphism if and only if  $X$  belongs to  $\mathbf{D}_{\mathcal{Y}}(\text{Mod } A)$ . Completing  $\Gamma X \rightarrow X$  to a triangle yields a well defined localization functor  $\mathbf{D}(\text{Mod } B) \rightarrow \mathbf{D}(\text{Mod } A)$  with kernel  $\mathbf{D}_{\mathcal{Y}}(\text{Mod } A)$ ; see [4, Lemma 3.3].  $\square$

**Corollary 3.4.** *Let  $A$  be a hereditary ring and  $\mathcal{Y}$  an extension closed abelian subcategory of  $\text{Mod } A$  that is closed under taking products and coproducts. Set  $\mathcal{X} = {}^\perp \mathcal{Y}$  and  $\mathcal{Z} = \mathcal{Y}^\perp$ . Then  $(\mathcal{X}, \mathcal{Y})$  and  $(\mathcal{Y}, \mathcal{Z})$  are both complete Ext-orthogonal pairs.*

*Proof.* The assertion is an immediate consequence of Propositions 3.1 and 3.2. However, we prefer to give an alternative proof because it is more explicit.

There exists a homological epimorphism  $f: A \rightarrow B$  such that restriction identifies  $\text{Mod } B$  with  $\mathcal{Y}$ ; see Proposition 3.2. Then Lemma 3.3 produces two localization functors  $L_1, L_2: \mathbf{D}(\text{Mod } A) \rightarrow \mathbf{D}(\text{Mod } A)$  with  $\text{Im } L_1 = \mathbf{D}_{\mathcal{Y}}(\text{Mod } A) = \text{Ker } L_2$ . Thus

$$\text{Ker } L_1 = {}^\perp(\text{Im } L_1) = \mathbf{D}_{\mathcal{X}}(\text{Mod } A) \quad \text{and} \quad \text{Im } L_2 = (\text{Ker } L_2)^\perp = \mathbf{D}_{\mathcal{Z}}(\text{Mod } A),$$

where in both cases the first equality follows from [4, Lemma 3.3] and the second from Proposition 2.6. It remains to apply Proposition 2.7 which yields in both cases for each  $A$ -module the desired 5-term exact sequence.  $\square$

*Remark 3.5.* The proof of Corollary 3.4 yields for any  $A$ -module  $M$  an explicit description of some terms of the 5-term exact sequence  $\varepsilon_M$ , using the homological epimorphism  $A \rightarrow B$ . In the first case, we have

$$\varepsilon_M: 0 \rightarrow \text{Tor}_1^A(M, B) \rightarrow X_M \rightarrow M \rightarrow M \otimes_A B \rightarrow X^M \rightarrow 0,$$

and in the second case, we have

$$\varepsilon_M: 0 \rightarrow Z_M \rightarrow \mathrm{Hom}_A(B, M) \rightarrow M \rightarrow Z^M \rightarrow \mathrm{Ext}_A^1(B, M) \rightarrow 0.$$

The following result reflects the recollement of the derived category  $\mathbf{D}(\mathrm{Mod} A)$  which arises from a homological epimorphism  $A \rightarrow B$ ; it is an immediate consequence of Corollary 3.4.

**Corollary 3.6.** *Let  $A$  be a hereditary ring and  $(\mathcal{X}, \mathcal{Y})$  an Ext-orthogonal pair for the category of  $A$ -modules.*

- (1) *There is an Ext-orthogonal pair  $(\mathcal{W}, \mathcal{X})$  if and only if  $\mathcal{X}$  is closed under products.*
- (2) *There is an Ext-orthogonal pair  $(\mathcal{Y}, \mathcal{Z})$  if and only if  $\mathcal{Y}$  is closed under coproducts.*

#### 4. EXAMPLES

We present a number of examples of Ext-orthogonal pairs which illustrate the results of this work. The first example is classical and provides one of the motivations for studying perpendicular categories in representation theory of finite dimensional algebras. We refer to Schofield's work [26, 27] which contains some explicit calculations; see also [11, 12].

**Example 4.1.** Let  $A$  be a finite dimensional hereditary algebra over a field  $k$  and  $X$  a finite dimensional  $A$ -module. Then  $X^\perp = \mathcal{Y}$  identifies via a homological epimorphism  $A \rightarrow B$  with the category of modules over a  $k$ -algebra  $B$  and this yields a complete Ext-orthogonal pair  $(\mathcal{X}, \mathcal{Y})$ . If  $X$  is *exceptional*, that is,  $\mathrm{Ext}_A^1(X, X) = 0$ , then  $B$  is finite dimensional and can be constructed explicitly. Note that in this case for each finite dimensional  $A$ -module  $M$  the corresponding 5-term exact sequence  $\varepsilon_M$  consists of finite dimensional modules. Moreover, the category  $\mathcal{X}$  is equivalent to the module category of another finite dimensional algebra. We do not know of a criterion on  $X$  that characterizes the fact that  $B$  is finite dimensional; see however the following proposition.

**Proposition 4.2.** *Let  $A$  be a finite dimensional hereditary algebra over a field and  $(\mathcal{X}, \mathcal{Y})$  a complete Ext-orthogonal pair such that  $\mathcal{Y}$  is closed under coproducts. Fix a homological epimorphism  $A \rightarrow B$  inducing an equivalence  $\mathrm{Mod} B \xrightarrow{\sim} \mathcal{Y}$ . Then the following are equivalent.*

- (1) *There exists an exceptional module  $X \in \mathrm{mod} A$  such that  $\mathcal{Y} = X^\perp$ .*
- (2) *The algebra  $B$  belongs to  $\mathrm{mod} A$  when viewed as an  $A$ -module.*
- (3) *For each  $M \in \mathrm{mod} A$ , the 5-term exact sequence  $\varepsilon_M$  belongs to  $\mathrm{mod} A$ .*

*Proof.* (1)  $\Rightarrow$  (2): This follows, for example, from [11, Proposition 3.2].

(2)  $\Rightarrow$  (3): This follows from Remark 3.5.

(3)  $\Rightarrow$  (1): Let  $\mathcal{X}_{\mathrm{fp}} = \mathcal{X} \cap \mathrm{mod} A$  and  $\mathcal{Y}_{\mathrm{fp}} = \mathcal{Y} \cap \mathrm{mod} A$ . The assumption on  $(\mathcal{X}, \mathcal{Y})$  implies that  $(\mathcal{X}_{\mathrm{fp}}, \mathcal{Y}_{\mathrm{fp}})$  is a complete Ext-orthogonal pair for  $\mathrm{mod} A$  and that every object in  $\mathcal{X}$  is a filtered colimit of objects in  $\mathcal{X}_{\mathrm{fp}}$ . Now choose an injective cogenerator  $Q$  in  $\mathrm{mod} A$  and let  $X = X_Q$  be the module from the 5-term exact sequence  $\varepsilon_Q$ . This module is the image of  $Q$  under a right adjoint of the inclusion  $\mathcal{X}_{\mathrm{fp}} \rightarrow \mathrm{mod} A$ . Note that a right adjoint of an exact functor preserves injectivity. It follows that  $X$  is an exceptional object and that  $\mathcal{X}_{\mathrm{fp}}$  is the smallest extension closed abelian subcategory of  $\mathrm{mod} A$  containing  $X$ . Thus  $X^\perp = \mathcal{X}_{\mathrm{fp}}^\perp = \mathcal{X}^\perp = \mathcal{Y}$ , since every object in  $\mathcal{X}$  is a filtered colimit of objects in  $\mathcal{X}_{\mathrm{fp}}$ .  $\square$

Any finitely generated projective module generates an Ext-orthogonal pair that can be described explicitly; see [11, §5].

**Example 4.3.** Let  $A$  be a hereditary ring and  $e^2 = e \in A$  an idempotent. Let  $\mathcal{X}$  denote the category of  $A$ -modules  $M$  such that the natural map  $Me \otimes_{eAe} eA \rightarrow M$  is an isomorphism, and let  $\mathcal{Y} = eA^\perp = \{M \in \text{Mod } A \mid Me = 0\}$ . Thus  $- \otimes_{eAe} eA$  identifies  $\text{Mod } eAe$  with  $\mathcal{X}$  and restriction via  $A \rightarrow A/AeA$  identifies  $\text{Mod } A/AeA$  with  $\mathcal{Y}$ . Then  $(\mathcal{X}, \mathcal{Y})$  is a complete Ext-orthogonal pair for  $\text{Mod } A$ , and for each  $A$ -module  $M$  the 5-term exact sequence  $\varepsilon_M$  is of the form

$$0 \rightarrow \text{Tor}_1^A(M, A/AeA) \rightarrow Me \otimes_{eAe} eA \rightarrow M \rightarrow M \otimes_A A/AeA \rightarrow 0.$$

The next example<sup>1</sup> arises from the work of Reiten and Ringel on infinite dimensional representations of canonical algebras; see [23] which is our reference for all concepts and results in the following discussion. Note that these algebras are not necessarily hereditary. The example shows the interplay between Ext-orthogonal pairs and (co)torsion pairs.

**Example 4.4.** Let  $A$  be a finite dimensional canonical algebra over a field  $k$ . Take for example a tame hereditary algebra, or, more specifically, the Kronecker algebra  $\begin{bmatrix} k & k^2 \\ 0 & k \end{bmatrix}$ . For such algebras, there is the concept of a *separating tubular family*. We fix such a family and denote by  $\mathcal{T}$  the category of finite dimensional modules belonging to this family. There is also a particular *generic module* over  $A$  which depends in some cases on the choice of  $\mathcal{T}$ ; it is denoted by  $G$ . Then the full subcategory  $\mathcal{X} = \varinjlim \mathcal{T}$  consisting of all filtered colimits of modules in  $\mathcal{T}$  and the full subcategory  $\mathcal{Y} = \text{Add } \overline{G}$  consisting of all coproducts of copies of  $G$  form an Ext-orthogonal pair  $(\mathcal{X}, \mathcal{Y})$  for  $\text{Mod } A$ . Note that the endomorphism ring  $D = \text{End}_A(G)$  of  $G$  is a division ring and that the canonical map  $A \rightarrow B$  with  $B = \text{End}_D(G)$  is a homological epimorphism which induces an equivalence  $\text{Mod } B \xrightarrow{\sim} \mathcal{Y}$ .

The category of  $A$ -modules which are generated by  $\mathcal{T}$  and the category of  $A$ -modules which are cogenerated by  $G$  form a torsion pair  $(\text{Fac } \mathcal{X}, \text{Sub } \mathcal{Y})$  for  $\text{Mod } A$  which equals the torsion pair  $(\mathcal{X}_0, \mathcal{Y}_0)$  generated by  $\mathcal{X}$ . On the other hand, let  $\mathcal{C}$  denote the category of  $A$ -modules which are cogenerated by  $\mathcal{X}$ , and let  $\mathcal{D}$  denote the category of  $A$ -modules  $M$  satisfying  $\text{Hom}_A(M, \mathcal{T}) = 0$ . Then the pair  $(\mathcal{C}, \mathcal{D})$  forms a cotorsion pair for  $\text{Mod } A$  which identifies with the cotorsion pair  $(\mathcal{X}_1, \mathcal{Y}_1)$  generated by  $\mathcal{X}$ .

If  $A$  is hereditary, then the Ext-orthogonal pair  $(\mathcal{X}, \mathcal{Y})$  is complete by Corollary 3.4; see also Remark 3.5 for an explicit description of the 5-term approximation sequence  $\varepsilon_M$  for each  $A$ -module  $M$ . Alternatively, one obtains the sequence  $\varepsilon_M$  by splicing together appropriate approximation sequences which arise from  $(\mathcal{X}_0, \mathcal{Y}_0)$  and  $(\mathcal{X}_1, \mathcal{Y}_1)$ .

The following example of an Ext-orthogonal pair arises from a localizing subcategory; it provides a simple model for the previous example.

**Example 4.5.** Let  $A$  be an integral domain with quotient field  $Q$ . Let  $\mathcal{X}$  denote the category of torsion modules and  $\mathcal{Y}$  the category of torsion free divisible modules. Note that the modules in  $\mathcal{Y}$  are precisely the coproducts of copies of  $Q$ . Then  $(\mathcal{X}, \mathcal{Y})$  is a complete Ext-orthogonal pair for  $\text{Mod } A$ , and for each  $A$ -module  $M$  the 5-term exact sequence  $\varepsilon_M$  is of the form

$$0 \rightarrow tM \rightarrow M \rightarrow M \otimes_A Q \rightarrow \overline{M} \rightarrow 0.$$

<sup>1</sup>The first author is grateful to Lidia Angeleri Hügel for suggesting this example.

There are examples of abelian categories that admit only trivial Ext-orthogonal pairs.

**Example 4.6.** Let  $A$  be a local artinian ring and set  $\mathcal{A} = \text{Mod } A$ . Then  $\text{Hom}_A(X, Y) \neq 0$  for any pair  $X, Y$  of non-zero  $A$ -modules. Thus if  $(\mathcal{X}, \mathcal{Y})$  is an Ext-orthogonal pair for  $\mathcal{A}$ , then  $\mathcal{X} = \mathcal{A}$  or  $\mathcal{Y} = \mathcal{A}$ .

## 5. EXT-ORTHOGONAL PAIRS OF FINITE TYPE

We characterize for a hereditary ring the Ext-orthogonal pairs of finite type.

**Theorem 5.1.** *Let  $A$  be a hereditary ring and  $(\mathcal{X}, \mathcal{Y})$  an Ext-orthogonal pair for the module category of  $A$ . Then the following are equivalent.*

- (1) *The subcategory  $\mathcal{Y}$  is closed under taking coproducts.*
- (2) *Every module in  $\mathcal{X}$  is a filtered colimit of finitely presented modules from  $\mathcal{X}$ .*
- (3) *There exists a category  $\mathcal{C}$  of finitely presented modules such that  $\mathcal{C}^\perp = \mathcal{Y}$ .*

We need some preparations for the proof of this result. The first lemma is a slight modification of [3, Proposition 2.1].

**Lemma 5.2.** *Let  $A$  be a ring and  $\mathcal{Y}$  a subcategory of its module category. Denote by  $\mathcal{X}$  the category  $A$ -modules  $X$  of projective dimension at most 1 satisfying  $\text{Ext}_A^1(X, Y) = 0$  for all  $Y \in \mathcal{Y}$ . Then any module in  $\mathcal{X}$  is a filtered colimit of finitely presented modules from  $\mathcal{X}$ .*

*Proof.* Let  $X \in \mathcal{X}$ . Choose an exact sequence  $0 \rightarrow P \xrightarrow{\phi} Q \rightarrow X \rightarrow 0$  such that  $P$  is free and  $Q$  is projective. The commuting squares of  $A$ -module morphisms

$$\begin{array}{ccc} P_i & \xrightarrow{\phi_i} & Q_i \\ \downarrow & & \downarrow \\ P & \xrightarrow{\phi} & Q \end{array}$$

with  $P_i$  and  $Q_i$  finitely generated projective form a filtered system such that  $\varinjlim \phi_i = \phi$ . We may assume that each morphism  $P_i \rightarrow P$  is a split monomorphism since  $P$  is free. Now set  $X_i = \text{Coker } \phi_i$ . Then  $\varinjlim X_i = X$ , and it is easily checked that  $\text{Ext}_A^1(X_i, \mathcal{Y}) = 0$  for all  $i$ .  $\square$

**Lemma 5.3.** *Let  $A$  be a hereditary ring and  $(\mathcal{X}, \mathcal{Y})$  a complete Ext-orthogonal pair for  $\text{Mod } A$ . Let  $M$  be an  $A$ -module and  $\varepsilon_M$  the corresponding 5-term exact sequence.*

- (1) *If  $\text{Ext}_A^1(M, \mathcal{Y}) = 0$ , then  $Y_M = 0$ .*
- (2) *Suppose that  $\mathcal{Y}$  is closed under coproducts and let  $M = \varinjlim M_i$  be a filtered colimit of  $A$ -modules  $M_i$ . Then  $\varepsilon_M = \varinjlim \varepsilon_{M_i}$ .*

*Proof.* If  $\text{Ext}_A^1(M, \mathcal{Y}) = 0$ , then the image of the morphism  $X_M \rightarrow M$  belongs to  $\mathcal{X}$ . Thus  $X_M \rightarrow M$  is a monomorphism and this yields (1).

To prove (2), one uses that  $\mathcal{X}$  and  $\mathcal{Y}$  are closed under taking colimits and that taking filtered colimits is exact. Thus  $\varinjlim \varepsilon_{M_i}$  is an exact sequence with middle term  $M$  and all other terms in  $\mathcal{X}$  or  $\mathcal{Y}$ . Now the uniqueness of  $\varepsilon_M$  implies that  $\varepsilon_M = \varinjlim \varepsilon_{M_i}$ ; see Lemma 2.8.  $\square$

The following lemma is needed for hereditary rings which are not noetherian.

**Lemma 5.4.** *Let  $M$  be a finitely presented module over a hereditary ring and  $N \subseteq M$  any submodule. Then  $N$  is a direct sum of finitely presented modules.*

*Proof.* We combine two results. Over a hereditary ring, any submodule of a finitely presented module is a direct sum of a finitely presented module and a projective module; see [7, Theorem 5.1.6]. In addition, one uses that any projective module is a direct sum of finitely generated projective modules; see [1].  $\square$

*Proof of Theorem 5.1.* (1)  $\Rightarrow$  (2): Suppose that  $\mathcal{Y}$  is closed under taking coproducts. We apply Corollary 3.4 and obtain for each module  $M$  the natural exact sequence  $\varepsilon_M$ . Now suppose that  $M$  belongs  $\mathcal{X}$ . Then one can write  $M = \varinjlim M_i$  as a filtered colimit of finitely presented modules with  $\text{Ext}_A^1(M_i, \mathcal{Y}) = 0$  for all  $i$ ; see Lemma 5.2. Next we apply Lemma 5.3. Thus

$$\varinjlim X_{M_i} \xrightarrow{\sim} X_M \xrightarrow{\sim} M,$$

and each  $X_{M_i}$  is a submodule of the finitely presented module  $M_i$ . Finally, each  $X_{M_i}$  is a filtered colimit of finitely presented direct summands by Lemma 5.4. Thus  $M$  is a filtered colimit of finitely presented modules from  $\mathcal{X}$ .

(2)  $\Rightarrow$  (3): Let  $\mathcal{X}_{\text{fp}}$  denote the full subcategory that is formed by all finitely presented modules in  $\mathcal{X}$ . Observe that  ${}^\perp Y$  is closed under taking colimits for each module  $Y$ , because  ${}^\perp Y$  is closed under taking coproducts and cokernels. Thus  $\mathcal{X}_{\text{fp}}^\perp = \mathcal{X}^\perp = \mathcal{Y}$  provided that  $\mathcal{X} = \varinjlim \mathcal{X}_{\text{fp}}$ .

(3)  $\Rightarrow$  (1): Use that for each finitely presented  $A$ -module  $X$ , the functor  $\text{Ext}_A^*(X, -)$  preserves all coproducts.  $\square$

Theorem 5.1 gives rise to a bijection between subcategories of finitely presented modules and Ext-orthogonal pairs of finite type. This is a consequence of the following proposition.

**Proposition 5.5.** *Let  $A$  be a hereditary ring and  $\mathcal{C}$  a category of finitely presented  $A$ -modules. Then  ${}^\perp(\mathcal{C}^\perp) \cap \text{mod } A$  equals the smallest extension closed abelian subcategory of  $\text{mod } A$  containing  $\mathcal{C}$ .*

*Proof.* Let  $\mathcal{D}$  denote the smallest extension closed abelian subcategory of  $\text{mod } A$  containing  $\mathcal{C}$ . We claim that the category  $\varinjlim \mathcal{D}$  which is formed by all filtered colimits of modules in  $\mathcal{D}$  is an extension closed abelian subcategory of  $\text{Mod } A$ .

On the other hand, Theorem 2.2 implies that  $\mathcal{X} = {}^\perp(\mathcal{C}^\perp)$  equals the smallest extension closed abelian subcategory of  $\text{Mod } A$  closed under coproducts and containing  $\mathcal{C}$ . Thus the first claim implies  $\mathcal{X} = \varinjlim \mathcal{D}$  and therefore  $\mathcal{X} \cap \text{mod } A = \mathcal{D}$ .

To prove the claim, observe that every morphism in  $\varinjlim \mathcal{D}$  can be written as a filtered colimit of morphisms in  $\mathcal{D}$ . Using that taking filtered colimits is exact, it follows immediately that  $\varinjlim \mathcal{D}$  is closed under kernels and cokernels in  $\text{Mod } A$ .

It remains to show that  $\varinjlim \mathcal{D}$  is closed under extensions. To this end let  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  be an exact sequence with  $L$  and  $N$  in  $\varinjlim \mathcal{D}$ . We can without loss of generality assume that  $N$  belongs to  $\mathcal{D}$ , because the sequence is a filtered colimit of the pull-back exact sequences with the last term in  $\mathcal{D}$ . We choose a morphism  $\phi: M' \rightarrow M$  with  $M'$  finitely presented and need to show that  $\phi$  factors through an object in  $\mathcal{D}$ ; see [16]. We may assume that the composite with  $M \rightarrow N$  is an epimorphism and denote by  $L'$  its kernel which is finitely presented. Thus the induced map  $L' \rightarrow L$  factors through an object  $L''$  in  $\mathcal{D}$  since  $L$  belongs to  $\varinjlim \mathcal{D}$ . Forming the push-out exact sequence

of  $0 \rightarrow L' \rightarrow M' \rightarrow N \rightarrow 0$  along the morphism  $L' \rightarrow L''$  gives an exact sequence  $0 \rightarrow L'' \rightarrow M'' \rightarrow N \rightarrow 0$ . Now  $\phi$  factors through  $M''$  which belongs to  $\mathcal{D}$ .  $\square$

## 6. UNIVERSAL LOCALIZATIONS

A ring homomorphism  $A \rightarrow B$  is called *universal localization* if there exists a set  $\Sigma$  of morphisms between finitely generated projective  $A$ -modules such that

- (1)  $\sigma \otimes_A B$  is an isomorphism of  $B$ -modules for all  $\sigma \in \Sigma$ , and
- (2) every ring homomorphism  $A \rightarrow B'$  such that  $\sigma \otimes_A B'$  is an isomorphism of  $B'$ -modules for all  $\sigma \in \Sigma$  factors uniquely through  $A \rightarrow B$ .

Let  $A$  be a ring and  $\Sigma$  a set of morphisms between finitely generated projective  $A$ -modules. Then there exists a universal localization inverting  $\Sigma$  and this is unique up to a unique isomorphism; see [25] for details. The universal localization is denoted by  $A \rightarrow A_\Sigma$  and restriction identifies  $\text{Mod } A_\Sigma$  with the full subcategory consisting of all  $A$ -modules  $M$  such that  $\text{Hom}_A(\sigma, M)$  is an isomorphism for all  $\sigma \in \Sigma$ . Note that  $\text{Hom}_A(\sigma, M)$  is an isomorphism if and only if  $M$  belongs to  $\{\text{Ker } \sigma, \text{Coker } \sigma\}^\perp$ , provided that  $A$  is hereditary.

**Theorem 6.1.** *Let  $A$  be a hereditary ring. A ring homomorphism  $f: A \rightarrow B$  is a homological epimorphism if and only if  $f$  is a universal localization.*

*Proof.* Suppose first that  $f: A \rightarrow B$  is a homological epimorphism. This gives rise to an Ext-orthogonal pair  $(\mathcal{X}, \mathcal{Y})$  for  $\text{Mod } A$ , if we identify  $\text{Mod } B$  with a full subcategory  $\mathcal{Y}$  of  $\text{Mod } A$ ; see Proposition 3.1. Let  $\mathcal{X}_{\text{fp}}$  denote the subcategory that is formed by all finitely presented modules in  $\mathcal{X}$ . It follows from Theorem 5.1 that  $\mathcal{X}_{\text{fp}}^\perp = \mathcal{Y}$ . Now fix for each  $X \in \mathcal{X}_{\text{fp}}$  an exact sequence

$$0 \rightarrow P_X \xrightarrow{\sigma_X} Q_X \rightarrow X \rightarrow 0$$

such that  $P_X$  and  $Q_X$  are finitely generated projective, and let  $\Sigma = \{\sigma_X \mid X \in \mathcal{X}_{\text{fp}}\}$ . Then

$$\text{Mod } B = \mathcal{X}_{\text{fp}}^\perp = \text{Mod } A_\Sigma,$$

and therefore  $f: A \rightarrow B$  is a universal localization.

Now suppose  $f: A \rightarrow B$  is a universal localization. Then restriction identifies the category of  $B$ -modules with an extension closed subcategory of  $\text{Mod } A$ . Thus we have induced isomorphisms

$$\text{Ext}_B^*(X, Y) \xrightarrow{\sim} \text{Ext}_A^*(X, Y)$$

for all  $B$ -modules  $X, Y$ , since  $A$  is hereditary. It follows that  $f$  is a homological epimorphism.  $\square$

*Remark 6.2.* Neither implication in Theorem 6.1 is true if one drops the assumption on the ring  $A$  to be hereditary. In [13], Keller gives an example of a Bézout domain  $A$  and a non-zero ideal  $I$  such that the canonical map  $A \rightarrow A/I$  is a homological epimorphism, but any map  $\sigma$  between finitely generated projective  $A$ -modules needs to be invertible if  $\sigma \otimes_A A/I$  is invertible. On the other hand, Neeman, Ranicki, and Schofield use finite dimensional algebras to construct in [20] examples of universal localizations that are not homological epimorphisms.

## 7. THE TELESCOPE CONJECTURE

Let  $A$  be a ring. A complex of  $A$ -modules is called *perfect* if it is a bounded complex of finitely generated projective modules. Note that a complex  $X$  is isomorphic to a perfect complex if and only if the functor  $\mathrm{Hom}_{\mathbf{D}(\mathrm{Mod} A)}(X, -)$  preserves coproducts.

A localizing subcategory  $\mathcal{C}$  of  $\mathbf{D}(\mathrm{Mod} A)$  is *generated* by perfect complexes if  $\mathcal{C}$  admits no proper localizing subcategory containing all perfect complexes from  $\mathcal{C}$ .

**Theorem 7.1.** *Let  $A$  be a hereditary ring. For a localizing subcategory  $\mathcal{C}$  of  $\mathbf{D}(\mathrm{Mod} A)$  the following conditions are equivalent:*

- (1) *There exists a localization functor  $L: \mathbf{D}(\mathrm{Mod} A) \rightarrow \mathbf{D}(\mathrm{Mod} A)$  that preserves coproducts such that  $\mathcal{C} = \mathrm{Ker} L$ .*
- (2) *The localizing subcategory  $\mathcal{C}$  is generated by perfect complexes.*
- (3) *There exists a localizing subcategory  $\mathcal{D}$  of  $\mathbf{D}(\mathrm{Mod} A)$  that is closed under products such that  $\mathcal{C} = {}^\perp \mathcal{D}$ .*

*Proof.* (1)  $\Rightarrow$  (2): The kernel  $\mathrm{Ker} L$  and the essential image  $\mathrm{Im} L$  of a localization functor  $L$  form an Ext-orthogonal pair for  $\mathbf{D}(\mathrm{Mod} A)$ ; see [4, Lemma 3.3]. We obtain an Ext-orthogonal pair  $(\mathcal{X}, \mathcal{Y})$  for  $\mathrm{Mod} A$  by taking  $\mathcal{X} = H^0 \mathrm{Ker} L$  and  $\mathcal{Y} = H^0 \mathrm{Im} L$ ; see Proposition 2.6. The fact that  $L$  preserves coproducts implies that  $\mathcal{Y}$  is closed under taking coproducts. It follows from Theorem 5.1 that  $\mathcal{X}$  is generated by finitely presented modules. Each finitely presented module is isomorphic in  $\mathbf{D}(\mathrm{Mod} A)$  to a perfect complex, and therefore  $\mathrm{Ker} L$  is generated by perfect complexes.

(2)  $\Rightarrow$  (3): Suppose that  $\mathcal{C}$  is generated by perfect complexes. Then there exists a localization functor  $L: \mathbf{D}(\mathrm{Mod} A) \rightarrow \mathbf{D}(\mathrm{Mod} A)$  such that  $\mathrm{Ker} L = \mathcal{C}$ . Thus we have an Ext-orthogonal pair  $(\mathcal{C}, \mathcal{D})$  for  $\mathbf{D}(\mathrm{Mod} A)$  with  $\mathcal{D} = \mathrm{Im} L$ ; see [4, Lemma 3.3]. Now observe that  $\mathcal{D} = \mathcal{C}^\perp$  is closed under coproducts, since for any perfect complex  $X$  the functor  $\mathrm{Hom}_{\mathbf{D}(\mathrm{Mod} A)}(X, -)$  preserves coproducts. It follows that  $\mathcal{D}$  is a localizing subcategory.

(3)  $\Rightarrow$  (1): Let  $\mathcal{D}$  be a localizing subcategory that is closed under products such that  $\mathcal{C} = {}^\perp \mathcal{D}$ . Then  $\mathcal{Y} = H^0 \mathcal{D}$  is an extension closed abelian subcategory of  $\mathrm{Mod} A$  that is closed under products and coproducts; see Proposition 2.4. In the proof of Corollary 3.4 we have constructed a localization functor  $L: \mathbf{D}(\mathrm{Mod} A) \rightarrow \mathbf{D}(\mathrm{Mod} A)$  such that  $\mathcal{C} = \mathrm{Ker} L$ . More precisely, there exists a homological epimorphism  $A \rightarrow B$  such that  $L = - \otimes_A^{\mathbf{L}} B$ . It remains to notice that this functor preserves coproducts.  $\square$

*Remark 7.2.* The implication (1)  $\Rightarrow$  (2) is known as “telescope conjecture”. Let us sketch the essential ingredients of the proof of this implication. In fact, the proof is not as involved as one might expect from the references to preceding results of this work.

We need the 5-term exact sequence  $\varepsilon_M$  for each module  $M$  which one gets immediately from the the localization functor  $L$ ; see Proposition 2.7. The perfect complexes generating  $\mathcal{C}$  are constructed in the proof of Theorem 5.1, where the relevant implication is (1)  $\Rightarrow$  (2). For this proof, one uses Lemmas 5.2 – 5.4, but this is all.

## 8. A BIJECTIVE CORRESPONDENCE

In this final section we summarize our findings by stating explicitly the correspondence between various structures arising from Ext-orthogonal pairs for hereditary rings.

**Theorem 8.1.** *For a hereditary ring  $A$  there are bijections between the following sets:*

- (1) *Ext-orthogonal pairs  $(\mathcal{X}, \mathcal{Y})$  for  $\text{Mod } A$  such that  $\mathcal{Y}$  is closed under coproducts.*
- (2) *Ext-orthogonal pairs  $(\mathcal{Y}, \mathcal{Z})$  for  $\text{Mod } A$  such that  $\mathcal{Y}$  is closed under products.*
- (3) *Extension closed abelian subcategories of  $\text{Mod } A$  that are closed under products and coproducts.*
- (4) *Extension closed abelian subcategories of  $\text{mod } A$ .*
- (5) *Homological epimorphisms  $A \rightarrow B$  (up to isomorphism).*
- (6) *Universal localizations  $A \rightarrow B$  (up to isomorphism).*
- (7) *Localizing subcategories of  $\mathbf{D}(\text{Mod } A)$  that are closed under products.*
- (8) *Localization functors  $\mathbf{D}(\text{Mod } A) \rightarrow \mathbf{D}(\text{Mod } A)$  preserving coproducts (up to natural isomorphism).*
- (9) *Thick subcategories of  $\mathbf{D}^b(\text{mod } A)$ .*

*Proof.* We state the bijections explicitly in the following table and give the references to the places where these bijections are established.

Direction	Map	Reference
(1) $\leftrightarrow$ (3)	$(\mathcal{X}, \mathcal{Y}) \mapsto \mathcal{Y}$	Corollary 3.4
(2) $\leftrightarrow$ (3)	$(\mathcal{Y}, \mathcal{Z}) \mapsto \mathcal{Y}$	Corollary 3.4
(3) $\rightarrow$ (4)	$\mathcal{Y} \mapsto (\perp \mathcal{Y}) \cap \text{mod } A$	Thm. 5.1 & Prop. 5.5
(4) $\rightarrow$ (3)	$\mathcal{C} \mapsto \mathcal{C}^\perp$	Thm. 5.1 & Prop. 5.5
(3) $\rightarrow$ (5)	$\mathcal{Y} \mapsto (A \rightarrow \text{End}_A(FA))^\perp$	Proposition 3.2
(5) $\rightarrow$ (3)	$f \mapsto (\text{Ker } f \oplus \text{Coker } f)^\perp$	Proposition 3.1
(5) $\leftrightarrow$ (6)	$f \mapsto f$	Theorem 6.1
(3) $\rightarrow$ (7)	$\mathcal{Y} \mapsto \mathbf{D}_{\mathcal{Y}}(\text{Mod } A)$	Proposition 2.4
(7) $\rightarrow$ (3)	$\mathcal{C} \mapsto H^0 \mathcal{C}$	Proposition 2.4
(7) $\rightarrow$ (8)	$\mathcal{C} \mapsto (X \mapsto GX)^2$	Theorem 7.1
(8) $\rightarrow$ (7)	$L \mapsto \text{Im } L$	Theorem 7.1
(4) $\rightarrow$ (9)	$\mathcal{X} \mapsto \mathbf{D}_{\mathcal{X}}^b(\text{mod } A)$	Remark 2.5
(9) $\rightarrow$ (4)	$\mathcal{C} \mapsto H^0 \mathcal{C}$	Remark 2.5

□

Let us mention that this correspondence is related to recent work of some other authors. In [28], Schofield establishes for any hereditary ring the bijection (4)  $\leftrightarrow$  (6). In [21], Nicolás and Saorín establish for a differential graded algebra  $A$  a correspondence between recollements for the derived category  $\mathbf{D}(A)$  and differential graded homological epimorphisms  $A \rightarrow B$ . This correspondence specializes for a hereditary ring to the bijection (5)  $\leftrightarrow$  (8).

**A finiteness condition.** Given an Ext-orthogonal pair for the category of  $A$ -modules as in Theorem 8.1, it is a natural question to ask when its restriction to the category of finitely presented modules yields a complete Ext-orthogonal pair for  $\text{mod } A$ . This finiteness condition we characterize in terms of finitely presented modules for any finite dimensional algebra; see also Proposition 4.2.

**Proposition 8.2.** *Let  $A$  be a finite dimensional hereditary algebra over a field and  $\mathcal{C}$  an extension closed abelian subcategory of  $\text{mod } A$ . Then the following are equivalent.*

<sup>1</sup>The functor  $F$  denotes a left adjoint of the inclusion  $\mathcal{Y} \rightarrow \text{Mod } A$

<sup>2</sup>The functor  $G$  denotes a left adjoint of the inclusion  $\mathcal{C} \rightarrow \mathbf{D}(\text{Mod } A)$

- (1) *There exists a complete Ext-orthogonal pair  $(\mathcal{C}, \mathcal{D})$  for  $\text{mod } A$ .*
- (2) *The inclusion  $\mathcal{C} \rightarrow \text{mod } A$  admits a right adjoint.*
- (3) *There exists an exceptional object  $X \in \mathcal{C}$  such that  $\mathcal{C}$  is the smallest extension closed abelian subcategory of  $\text{mod } A$  containing  $X$ .*
- (4) *Let  $(\mathcal{X}, \mathcal{Y})$  be the Ext-orthogonal pair for  $\text{Mod } A$  generated by  $\mathcal{C}$ . Then for each  $M \in \text{mod } A$  the 5-term exact sequence  $\varepsilon_M$  belongs to  $\text{mod } A$ .*

*Proof.* (1)  $\Rightarrow$  (2): For  $M \in \text{mod } A$  let  $0 \rightarrow D_M \rightarrow C_M \rightarrow M \rightarrow D^M \rightarrow C^M \rightarrow 0$  be its 5-term exact sequence. Sending a module  $M$  to  $C_M$  induces a right adjoint for the inclusion  $\mathcal{C} \rightarrow \text{mod } A$ ; see Lemma 2.8.

(2)  $\Rightarrow$  (3): Choose an injective cogenerator  $Q$  in  $\text{mod } A$  and let  $X$  denote its image under the right adjoint of the inclusion of  $\mathcal{C}$ . A right adjoint of an exact functor preserves injectivity. It follows that  $X$  is an exceptional object and that  $\mathcal{C}$  is the smallest extension closed abelian subcategory of  $\text{mod } A$  containing  $X$ .

(3)  $\Rightarrow$  (4): See Proposition 4.2.

(4)  $\Rightarrow$  (1): The property of the pair  $(\mathcal{X}, \mathcal{Y})$  implies that  $(\mathcal{X} \cap \text{mod } A, \mathcal{Y} \cap \text{mod } A)$  is a complete Ext-orthogonal pair for  $\text{mod } A$ . An application of Proposition 5.5 yields the equality  $\mathcal{X} \cap \text{mod } A = \mathcal{C}$ . Thus there exists a complete Ext-orthogonal pair  $(\mathcal{C}, \mathcal{D})$  for  $\text{mod } A$ .  $\square$

*Remark 8.3.* There is a dual result which is obtained by applying the duality between modules over the algebra  $A$  and its opposite  $A^{\text{op}}$ . Note that condition (3) is self-dual.

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