

# GENERIC IDEMPOTENT MODULES FOR A FINITE GROUP

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ABSTRACT. Let  $G$  be a finite group and  $k$  an algebraically closed field of characteristic  $p$ . Let  $F_{\mathcal{U}}$  be the Rickard idempotent  $kG$ -module corresponding to the set  $\mathcal{U}$  of subvarieties of the cohomology variety  $V_G$  which are not irreducible components. We show that  $F_{\mathcal{U}}$  is a finite sum of *generic modules* corresponding to the irreducible components of  $V_G$ . In this context, a generic module is an indecomposable module of infinite length over  $kG$  but finite length as a module over its endomorphism ring.

*This article is dedicated to Klaus Roggenkamp on the occasion of his sixtieth birthday.*

## 1. INTRODUCTION

The purpose of this paper is to draw attention to a particular infinite dimensional module for a finite group algebra, which plays the role of a generic module in a certain precise sense which will be described. The module in question is a particular case of Rickard's construction [14] of idempotent modules in the stable category. These idempotent modules have played a pivotal role in some recent developments in modular representation theory, see for example [1, 3, 4, 5].

**Theorem 1.1.** *Let  $\mathcal{U}$  denote the set of closed homogeneous irreducible subvarieties of the cohomology variety  $V_G$  which are not irreducible components, and let  $F_{\mathcal{U}}$  be the corresponding idempotent module. Then  $F_{\mathcal{U}}$  is endofinite.*

## 2. ENDOFINITE MODULES IN THE STABLE CATEGORY

For any ring  $\Lambda$ , a  $\Lambda$ -module  $M$  is said to be *endofinite* if it has finite length as a module over  $\text{End}_{\Lambda}(M)$ . This concept is investigated by Crawley–Boevey in [9]. The following are some properties of endofinite modules over a finite dimensional algebra; their proofs can be found in Section 4 of [9].

- (1) If  $M$  has finite length then  $M$  is endofinite.
- (2) An arbitrary direct sum or direct product of (possibly infinitely many) copies of an endofinite module is endofinite.
- (3) Every finite direct sum of endofinite modules is endofinite.
- (4) Every direct summand of an endofinite module is endofinite.
- (5) Every projective module and every injective module is endofinite. This follows from (2) and (3), since there are finitely many isomorphism classes of indecomposable projectives, and every projective is a direct sum of indecomposable projectives; and similarly for injectives.

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(6) Every endofinite module is pure injective. It is even  $\Sigma$ -pure injective, since this is equivalent to d.c.c. on additive subgroups of finite definition (in the first order language of  $\Lambda$ -modules). Endofinite is equivalent to a.c.c. and d.c.c. on additive subgroups of finite definition.

A *generic* module is defined to be an indecomposable endofinite module of infinite length. As an example, let  $G = \langle g_1, g_2 \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$  and let  $k$  have characteristic two. Let  $t$  be an indeterminate, and let  $M$  be the  $kG$ -module with underlying vector space  $k(t) \oplus k(t)$ , with  $G$  acting via

$$g_1 \mapsto \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \quad g_2 \mapsto \begin{pmatrix} I & T \\ 0 & I \end{pmatrix}.$$

Here,  $I$  denotes the identity map on  $k(t)$  and  $T$  denotes multiplication by  $t$  on  $k(t)$ . It is easy to see that  $M$  has infinite length over  $kG$  but length two over  $\text{End}_{kG}(M) \cong k(t)[u]/(u^2)$ . Since  $M$  is also indecomposable, it is a generic module. In fact, it is the module  $F_{\mathcal{U}}$  for this group, and there are no other generic modules.

More generally, Crawley–Boevey has shown that for finite dimensional algebras over an algebraically closed field there is a strong connection between generic modules and the representation type; see [9]. Of course, for an algebra of finite representation type, there are no generic modules. For a finite dimensional algebra of tame representation type, there is a natural correspondence between generic modules and one parameter families of finite length modules. A finite dimensional algebra has wild representation type if and only if there exists a generic module such that the endomorphism ring modulo its radical is the full division ring of fractions of the free ring in two variables  $k\langle x, y \rangle$ . This division ring has infinite dimension over its center.

Let  $\Lambda$  be a finite dimensional self-injective algebra. We write  $\text{Mod}(\Lambda)$  for the category of all  $\Lambda$ -modules and homomorphisms, and  $\text{mod}(\Lambda)$  for the full subcategory of finitely generated modules. The stable category  $\text{StMod}(\Lambda)$  has the same objects as  $\text{Mod}(\Lambda)$ , but the arrows are  $\text{Hom}_{\Lambda}(M, N)$  modulo the homomorphisms which factor through a projective (= injective) module. The stable category of finitely generated modules is written  $\text{stmod}(\Lambda)$ . The categories  $\text{StMod}(\Lambda)$  and  $\text{stmod}(\Lambda)$  are triangulated categories, in which the distinguished triangles correspond to short exact sequences in  $\text{Mod}(\Lambda)$ . Recall from [11] that an object  $M$  in  $\text{StMod}(\Lambda)$  is said to be endofinite if  $\underline{\text{Hom}}_{\Lambda}(X, M)$  has finite length over  $\underline{\text{End}}_{\Lambda}(M)$  for all  $X$  in  $\text{stmod}(\Lambda)$ . This does not conflict with the previous notion of endofiniteness, because of the following proposition.

**Proposition 2.1.** *The following are equivalent for a  $\Lambda$ -module  $M$ :*

- (1)  $M$  is an endofinite  $\Lambda$ -module;
- (2)  $M$  is an endofinite object in  $\text{StMod}\Lambda$ ;
- (3)  $\underline{\text{Hom}}_{\Lambda}(S, M)$  is of finite length over  $\underline{\text{End}}_{\Lambda}(M)$  for every simple  $\Lambda$ -module  $S$ .

*Proof.* (1)  $\Rightarrow$  (2) Suppose that  $M$  is endofinite and let  $X \in \text{mod}\Lambda$ . Clearly,  $\text{Hom}_{\Lambda}(X, M)$  has finite length over  $\text{End}_{\Lambda}(M)$ . Therefore the quotient  $\underline{\text{Hom}}_{\Lambda}(X, M)$  has finite length over  $\underline{\text{End}}_{\Lambda}(M)$  since it is of finite length over  $\text{End}_{\Lambda}(M)$ .

(2)  $\Rightarrow$  (3) Clear.

(3)  $\Rightarrow$  (1) Direct limits of injective modules are injective. Therefore we can apply Zorn's lemma and find a maximal injective submodule  $I$  of  $M$ . Let  $M = I \oplus M'$ . Given any simple  $\Lambda$ -module  $S$ , we have

$$\underline{\text{Hom}}_{\Lambda}(S, M) \cong \underline{\text{Hom}}_{\Lambda}(S, M') = \text{Hom}_{\Lambda}(S, M').$$

Assuming (3), this module is of finite length over  $\text{End}_\Lambda(M')$  since it is of finite length over  $\underline{\text{End}}_\Lambda(M') \cong \underline{\text{End}}_\Lambda(M)$ . Thus  $\text{Hom}_\Lambda(\Lambda, M')$  has finite length over  $\text{End}_\Lambda(M')$  since  $\Lambda$  is of finite length over  $\Lambda$ . We conclude that  $M'$  is endofinite. Every injective  $\Lambda$ -module is endofinite, so we conclude that  $M = I \oplus M'$  is endofinite and therefore the proof is complete.  $\square$

For a  $\Lambda$ -module  $M$ , we denote by  $\text{endol}(M)$  the length of  $M$  as a module over  $\text{End}_\Lambda(M)$ .

**Lemma 2.2.** *Let  $H$  be a subgroup of a group  $G$ , and let  $k$  be a field.*

- (1) *If  $M$  is a  $kG$ -module then  $\text{endol}(M \downarrow_H) \leq \text{endol}(M)$ .*
- (2) *If  $M$  is a  $kH$ -module then  $\text{endol}(M \uparrow^G) \leq \text{endol}(M) \cdot |G : H|$ .*

*Proof.* (1) The action of  $\text{End}_{kH}(M \downarrow_H)$  on  $M$  extends the action of  $\text{End}_{kG}(M)$ . Therefore  $\text{endol}(M)$  is a bound for  $\text{endol}(M \downarrow_H)$ .

(2) The endomorphism ring  $\text{End}_{kH}(M)$  acts on  $M \uparrow^G = kG \otimes_{kH} M$  via the ring homomorphism  $\text{End}_{kH}(M) \rightarrow \text{End}_{kG}(M \uparrow^G)$  which sends an endomorphism  $\alpha$  to the endomorphism  $\text{id} \otimes \alpha$ . The  $kH$ -module  $kG$  is free of rank  $|G : H|$ , and therefore the length of  $M \uparrow^G$  over  $\text{End}_{kH}(M)$  is equal to  $\text{endol}(M) \cdot |G : H|$ . We conclude that the length of  $M \uparrow^G$  over  $\text{End}_{kG}(M \uparrow^G)$  is bounded by  $\text{endol}(M) \cdot |G : H|$ .  $\square$

**Proposition 2.3.** *Let  $G$  be a finite group and  $k$  be a field of characteristic  $p$ . Then a  $kG$ -module  $M$  is endofinite if and only if its restriction to a Sylow  $p$ -subgroup of  $G$  is endofinite.*

*Proof.* Let  $P$  be a Sylow  $p$ -subgroup of  $G$ . If  $M$  is endofinite then Lemma 2.2 (1) shows that  $M \downarrow_P$  is endofinite. Conversely, if  $M \downarrow_P$  is endofinite then by Lemma 2.2 (2),  $M \downarrow_P \uparrow^G$  is endofinite. But since  $|G : P|$  is coprime to  $p$ ,  $M$  is a direct summand of  $M \downarrow_P \uparrow^G$ .  $\square$

### 3. IDEMPOTENT MODULES

**Definition 3.1.** Let  $G$  be a finite group and  $k$  be a field.

- (1) A  $kG$ -module  $E$  is called *counital idempotent* if there exists a map  $\varepsilon: E \rightarrow k$  such that  $\varepsilon \otimes \text{id}_E$  induces an isomorphism  $E \otimes_k E \rightarrow E$  in  $\text{StMod}kG$ .
- (2) A  $kG$ -module  $F$  is called *unital idempotent* if there exists a map  $\phi: k \rightarrow F$  such that  $\phi \otimes \text{id}_F$  induces an isomorphism  $F \rightarrow F \otimes_k F$  in  $\text{StMod}kG$ .

Suppose that  $E$  is a counital idempotent  $kG$ -module. Then the  $kG$ -modules  $M$  such that  $\varepsilon \otimes \text{id}_M$  induces an isomorphism  $E \otimes_k M \rightarrow M$  in  $\text{StMod}kG$  form a localizing subcategory  $\mathcal{L}_E$  of  $\text{StMod}kG$ . Recall that a full subcategory of  $\text{StMod}kG$  is *localizing* if it is a triangulated subcategory which is closed under arbitrary direct sums. Moreover, the inclusion functor  $\mathcal{L}_E \rightarrow \text{StMod}kG$  has a right adjoint which preserves arbitrary direct sums; it sends a  $kG$ -module  $M$  to  $E \otimes_k M$ . Note that  $\mathcal{L}_E$  does not depend on the map  $\varepsilon: E \rightarrow k$ . In fact, if  $\varepsilon': E \rightarrow k$  is a second map inducing an isomorphism  $E \otimes_k E \rightarrow E$  in  $\text{StMod}kG$  then there exists an isomorphism  $\alpha: E \rightarrow E$  such that  $\varepsilon' = \varepsilon\alpha$  in  $\text{StMod}kG$ .

**Lemma 3.2.** *Let  $\phi: k \rightarrow F$  be a map in  $\text{StMod}kG$  such that  $\phi \otimes \text{id}_F$  is an isomorphism. Then  $\phi$  induces an isomorphism  $\underline{\text{Hom}}_{kG}(F, F) \rightarrow \underline{\text{Hom}}_{kG}(k, F)$  of  $\underline{\text{End}}_{kG}(F)$ -modules.*

*Proof.* Complete the map  $\phi$  to a triangle  $E \rightarrow k \rightarrow F \rightarrow \Omega^{-1}(E)$ . Tensoring this triangle with  $F$  gives a new triangle which shows that  $F \otimes_k E = 0$  (in the stable category). It follows that

$\underline{\mathrm{Hom}}_{kG}(E, F) = 0$  since any map  $E \rightarrow F$  induces a commutative square

$$\begin{array}{ccc} k \otimes_k E & \longrightarrow & k \otimes_k F \\ \downarrow & & \downarrow \cong \\ F \otimes_k E & \longrightarrow & F \otimes_k F \end{array}$$

The same argument shows that  $\underline{\mathrm{Hom}}_{kG}(\Omega^{-1}(E), F) = 0$ . Applying  $\underline{\mathrm{Hom}}_{kG}(-, F)$  to the above triangle then shows that  $\underline{\mathrm{Hom}}_{kG}(F, F) \cong \underline{\mathrm{Hom}}_{kG}(k, F)$ .  $\square$

**Theorem 3.3.** *Let  $G$  be a  $p$ -group. Then a unital idempotent  $kG$ -module  $F$  is endofinite if and only if  $\underline{\mathrm{End}}_{kG}(F)$  is an Artinian ring.*

*Proof.* We may assume that the field  $k$  is of characteristic  $p$ , and therefore the trivial  $kG$ -module  $k$  is, up to isomorphism, the unique simple  $kG$ -module since  $G$  is a  $p$ -group. Lemma 3.2 shows that  $\underline{\mathrm{Hom}}_{kG}(k, F)$  has finite length over  $\underline{\mathrm{End}}_{kG}(F)$  if and only if  $\underline{\mathrm{End}}_{kG}(F)$  is Artinian. Therefore the assertion follows from the above criterion for endofiniteness.  $\square$

**Remark 3.4.** A unital idempotent  $kG$ -module  $F$  with Artinian endomorphism ring  $\underline{\mathrm{End}}_{kG}(F)$  has a unique decomposition  $F = F_1 \oplus \cdots \oplus F_n$  into finitely many indecomposable objects in  $\mathrm{StMod}kG$ . Moreover, each  $F_i$  is a unital idempotent module and  $F_i \otimes_k F_j = 0$  for all  $i \neq j$ ; see [11].

#### 4. THE MODULE $F_{\mathcal{U}}$

Let  $G$  be a finite group and  $k$  an algebraically closed field of characteristic  $p$ . Let  $V_G$  be the maximal ideal spectrum of the graded cohomology ring  $H^*(G, k) = \mathrm{Ext}_{kG}^*(k, k)$  (Note that this ring is only graded commutative; if this bothers you, notice that for  $p = 2$  this means it is commutative; for  $p$  odd, elements of odd degree square to zero, so modulo the nil radical this ring is again commutative). Let  $V_1, \dots, V_t$  be the irreducible components of  $V_G$ , and let  $\mathcal{U}$  be the collection of closed homogeneous irreducible subvarieties of  $V_G$  which are not irreducible components. This collection  $\mathcal{U}$  is clearly closed under specialization, in the sense that if  $V \in \mathcal{U}$  and  $W \subseteq V$  then  $W \in \mathcal{U}$ . Denote by  $\mathcal{C}(\mathcal{U})$  the thick subcategory of  $\mathrm{stmod}(kG)$  consisting of finitely generated  $kG$ -modules  $M$  such that  $V_G(M) \in \mathcal{U}$ . Let

$$E_{\mathcal{U}} \rightarrow k \rightarrow F_{\mathcal{U}} \rightarrow \Omega^{-1}E_{\mathcal{U}}$$

be the corresponding distinguished triangle of idempotent modules, as constructed by Rickard ([14], Section 5). Let  $\mathcal{C}(\mathcal{U})^{\oplus}$  denote the triangulated subcategory of  $\mathrm{StMod}(kG)$  generated by (possibly infinite) direct sums of objects in  $\mathcal{C}(\mathcal{U})$ . Then passage to the quotient category  $\mathrm{StMod}(kG) \rightarrow \mathrm{StMod}(kG)/\mathcal{C}(\mathcal{U})^{\oplus}$  has a left adjoint, which takes a module  $M$  to  $F(\mathcal{U}) \otimes_k M$ . Furthermore, the natural functor  $\mathrm{stmod}(kG)/\mathcal{C}(\mathcal{U}) \rightarrow \mathrm{StMod}(kG)/\mathcal{C}(\mathcal{U})^{\oplus}$  is fully faithful.

**Proposition 4.1.** *The stable endomorphism ring  $\underline{\mathrm{End}}_{kG}(F_{\mathcal{U}})$  is Artinian. It is a direct product of local rings,  $E_1 \times \cdots \times E_t$ , corresponding to the irreducible components  $V_1, \dots, V_t$ . The quotient  $E_i/J(E_i)$  of  $E_i$  by its radical is isomorphic to the function field  $k(V_i)$ .*

*Proof.* By the above remarks, the ring  $\underline{\mathrm{End}}_{kG}(F_{\mathcal{U}})$  is isomorphic to the endomorphism ring of  $k$  in the quotient category  $\mathrm{StMod}(kG)/\mathcal{C}(\mathcal{U})^{\oplus}$ , or equivalently in  $\mathrm{stmod}(kG)/\mathcal{C}(\mathcal{U})$ . So the method of Carlson, Donovan and Wheeler [7] can be used to give the following explicit description of  $\underline{\mathrm{End}}_{kG}(F_{\mathcal{U}})$ . Let  $S_{\mathcal{U}}$  be the multiplicative subset of  $H^*(G, k)$  consisting of elements which are not identically zero on any irreducible component of  $V_G$ , or equivalently, elements which do not

belong to any minimal prime ideal in  $H^*(G, k)$ . Then  $\underline{\text{End}}_{kG}(F_U)$  is isomorphic to the ring of degree zero elements in the homogeneous localization  $S_U^{-1}H^*(G, k)$ . The effect of this localization is that the homogeneous prime ideals in  $S_U^{-1}H^*(G, k)$  correspond precisely to the minimal prime ideals in  $H^*(G, k)$ , namely to the irreducible components of  $V_G$ . The prime ideals in  $\underline{\text{End}}_{kG}(F_U)$  are then obtained by intersecting these homogeneous primes with the degree zero part. It follows that in  $\underline{\text{End}}_{kG}(F_U)$ , every prime ideal is maximal. So this is a zero dimensional Noetherian graded commutative ring, and is hence Artinian.

A commutative Artinian ring is necessarily a direct product of local rings with nilpotent Jacobson radicals. For  $\underline{\text{End}}_{kG}(F_U)$ , these local rings correspond to the homogeneous maximal ideals in  $S_U^{-1}H^*(G, k)$ . Furthermore, by definition, the function field of  $V_i$  is equal to the field of fractions of the integral domain of degree zero elements in the quotient of  $H^*(G, k)$  by the corresponding minimal prime ideal. This is the same as the degree zero elements in the quotient of  $S_U^{-1}H^*(G, k)$  by the corresponding maximal ideal. This in turn is the same as the quotient of  $E_i$  by its Jacobson radical.  $\square$

**Remark 4.2.** Greenlees and Lyubeznik [10] (Corollary 7.4) have shown that the ring  $S_U^{-1}H^*(G, k)$  is a finite product of Gorenstein local rings. It is tempting to suppose that it follows from this that the degree zero part  $\underline{\text{End}}_{kG}(F_U)$  is a symmetric algebra. That this is not the case may be seen by examining the case where  $p$  is odd and  $G$  is an elementary abelian  $p$ -group of rank three. In this case,  $\underline{\text{End}}_{kG}(F_U)$  is an Artinian local ring of length four and square zero radical, so it cannot be a symmetric algebra. The theorem of Greenlees and Lyubeznik should therefore be interpreted as applying to the *graded* endomorphism ring

$$\underline{\text{End}}_{kG}^*(F_U) = \bigoplus_{n \in \mathbb{Z}} \underline{\text{Hom}}_{kG}(\Omega^n F_U, F_U).$$

**Remark 4.3.** For other collections  $\mathcal{V}$  of closed homogeneous irreducible subvarieties of  $V_G$ , closed under specialization, the structure of the stable endomorphism ring of  $F_{\mathcal{V}}$  can be deduced from the work of Carlson and Wheeler [8].

**Theorem 4.4.** *The module  $F_U$  is endofinite.*

*Proof.* By Proposition 2.3, it suffices to show that the restriction of  $F_U$  to a Sylow  $p$ -subgroup  $P$  of  $G$  is endofinite. But the restriction of  $F_U$  to  $P$  is stably isomorphic to the corresponding module  $F_U$  there. By Proposition 4.1 this module has Artinian stable endomorphism ring, so by Theorem 3.3 it is endofinite.  $\square$

## 5. LOCAL CONTROL

In this section, we use the techniques of [2] to reduce the description of the summands of  $F_U$  down to local subgroups, and there we give an explicit construction for this module.

By Proposition 4.1,  $F_U$  decomposes into indecomposable pieces  $F_1 \oplus \cdots \oplus F_t$  corresponding to the irreducible components  $V_1, \dots, V_t$ . The correspondence has the property that

$$\underline{\text{End}}_{kG}(F_i) / J \underline{\text{End}}_{kG}(F_i) \cong k(V_i).$$

By a theorem of Quillen [12, 13], the irreducible components of  $V_G$  are in one-one correspondence with the conjugacy classes of elementary abelian  $p$ -subgroups of  $G$ . Let  $E_i$  be a representative of the conjugacy class corresponding to  $V_i$ , and set  $N_i = N_G(E_i)$ .

**Lemma 5.1.** *The module  $F_i$  is isomorphic to the module induced from the corresponding module  $F_{\mathcal{U}_i}$  for  $N_i$ .*

*Proof.* We apply the methods of [2], but with  $\mathcal{M}_{r-1}$  replaced by  $\mathcal{C}(\mathcal{U})$ , and  $\mathcal{M}_{r,r-1}$  replaced by  $\mathbf{stmod}(kG)/\mathcal{C}(\mathcal{U})$ . The arguments work equally well in this context, and show the following. The category  $\mathbf{stmod}(kG)/\mathcal{C}(\mathcal{U})$  decomposes as a direct sum of triangulated categories  $\mathcal{M}_{G,E_i}$ , with the  $E_i$  as above. Induction and restriction between  $N_i$  and  $G$  establishes an equivalence of categories  $\mathbf{StMod}(kN_i)/\mathcal{C}(\mathcal{U}_i) \cong \mathcal{M}_{G,E_i}$ , under which the trivial module corresponds to the module called  $\pi_{G,E_i}(k)$  in [2].

Now we apply the adjunction mentioned before the proof of Proposition 4.1. This takes the trivial module in  $\mathbf{stmod}(kG)/\mathcal{C}(\mathcal{U})$  to  $F_{\mathcal{U}}$ , and the module  $\pi_{G,E_i}(k)$  to  $F_i$ . The corresponding adjunction at the level of  $N_i$  takes the trivial module in  $\mathbf{stmod}(kN_i)/\mathcal{C}(\mathcal{U}_i)$  to  $F_{\mathcal{U}_i}$ . Since passage to the quotient category commutes with induction, it follows that  $F_{\mathcal{U}_i} \uparrow^G \cong F_i$ .  $\square$

Now  $N_i$  has a unique maximal elementary abelian  $p$ -subgroup, namely  $E_i$ . The map from the cohomology variety of  $N_i$  to the cohomology variety of  $G$  induces a birational equivalence  $V_{N_i} \rightarrow V_i$ . In this situation, there is an explicit construction for  $F_{\mathcal{U}_i}$  which we now describe. This generalizes the construction given in Theorems 5.1.2 and 5.4.6 of [6].

So from now on, let  $G$  be a group with a unique maximal elementary abelian  $p$ -subgroup  $E$ , and assume that  $E$  is normal in  $G$ . Let  $r$  be the  $p$ -rank of  $E$ , and let  $X_1, \dots, X_{r-1}$  be indeterminates. Choose elements  $\zeta_1, \dots, \zeta_r$  forming a homogeneous system of parameters for  $H^*(G, k)$  and having the same degree  $n$ . Set  $K = k(X_1, \dots, X_{r-1})$ , and for  $1 \leq j \leq r-1$ , set

$$\xi_j = X_j \zeta_r - \zeta_j \in H^n(G, K).$$

Represent  $\xi_j$  by a map  $\hat{\xi}_j : K \rightarrow \Omega^{-n}K$ , and write  $L_j$  for the cokernel. Write  $\phi_j : k \rightarrow L_j$  for the composite  $k \hookrightarrow K \xrightarrow{\hat{\xi}_j} \Omega^{-n}K \rightarrow L_j$ .

**Theorem 5.2.** *The  $kG$ -module  $F_{\mathcal{U}}$  is isomorphic to the tensor product  $L_1 \otimes_K \cdots \otimes_K L_{r-1}$ , with the coefficients restricted to  $k$ , and with the map  $\phi_1 \otimes \cdots \otimes \phi_{r-1}$  as  $\phi : k \rightarrow F_{\mathcal{U}}$ .*

*Proof.* Since  $E$  is the unique maximal elementary abelian subgroup of  $G$ , it suffices to restrict to  $E = \langle g_1, \dots, g_r \rangle$  and use the theory developed in [4]. Namely, let  $V$  be a closed homogeneous irreducible subvariety of the rank variety  $V_E^r \cong \mathbb{A}^r(k)$ , and write  $L$  for the tensor product  $L_1 \otimes_K \cdots \otimes_K L_{r-1}$ . We need to complete the map  $\phi : k \rightarrow L$  to a triangle  $X \rightarrow k \rightarrow L \rightarrow \Omega^{-1}X$ , and then tensor with the function field  $k(V)$ . We then need to check that the triangle restricts to  $k(V) \rightarrow k(V) \rightarrow 0 \rightarrow \Omega^{-1}k(V)$  if  $V$  is a proper subvariety of  $V_E^r$ , and to  $0 \rightarrow k(V) \rightarrow k(V) \rightarrow 0$  otherwise. In other words, we need to show that the rank variety of  $L$  is  $\{V_E^r\}$ , while that of  $X$  is the complement of this.

Let  $H^*(E, k)/(\text{nilpotents}) = k[x_1, \dots, x_r]$  with the  $x_j$  in degree one if  $p = 2$  and two if  $p$  is odd. Let  $m$  denote  $n$  if  $p = 2$  and  $n/2$  if  $p$  is odd. Then  $\text{res}_{G,E}(\zeta_j) \pmod{\text{nilpotents}}$  is equal to some homogeneous polynomial  $f_j(x_1, \dots, x_r)$  of degree  $m$ . The polynomials  $f_1, \dots, f_r$  form a regular sequence in  $k[x_1, \dots, x_r]$ .

Denote by  $t_1, \dots, t_r$  the coordinate functions on  $V_E^r$ , regarded as elements of  $k(V)$ . Thus if  $V = V_E^r$  then the  $t_j$  are algebraically independent, while if  $V$  is a proper subvariety of  $V_E^r$  then there exists a homogeneous irreducible polynomial in the elements  $t_1, \dots, t_r$  which equals zero in  $k(V)$ .

The generic cyclic shifted subgroup of  $E$  corresponding to  $V$  is generated by

$$u_V = 1 + t_1(g_1 - 1) + \cdots + t_r(g_r - 1) \in k(V)E.$$

We must restrict the map  $k \rightarrow L$  to  $u_V$ , being careful to observe that tensor product does not commute with restriction. To get around this difficulty, we form the tensor product of the complexes

$$0 \rightarrow K \xrightarrow{\hat{\xi}_j} \Omega^{-n} K \rightarrow 0$$

to form a Koszul-like complex whose homology is  $L$ . The point of this is that Yoneda composition of cohomology elements does commute with restriction. So we are restricting a tensor product of complexes of the form

$$0 \rightarrow k(V) \otimes_k K \xrightarrow{1 \otimes \hat{\xi}_j} k(V) \otimes_k \Omega^{-n} K \rightarrow 0$$

and then taking the homology. We can act as though tensor product commutes with restriction.

Let  $\zeta$  be a nonzero element of  $H^2(\langle u_V \rangle, k(V))$  if  $p$  is odd, and of  $H^1(\langle u_V \rangle, k(V))$  if  $p = 2$ , chosen so that  $\text{res}_{E, \langle u_V \rangle}(x_j) = t_j \zeta$ . Then

$$\text{res}_{G, \langle u_V \rangle}(\zeta_j) = \text{res}_{E, \langle u_V \rangle}(f_j(x_1, \dots, x_r)) = f_j(t_1, \dots, t_r) \zeta^m,$$

and

$$\text{res}_{G, \langle u_V \rangle}(\xi_j) = (X_j f_r(t_1, \dots, t_r) - f_j(t_1, \dots, t_r)) \zeta^m.$$

These are the elements involved in the restriction to  $\langle u_V \rangle$  of the above Koszul-like complex.

If  $f_r(t_1, \dots, t_r) = 0$ , so that  $V$  is contained in the corresponding hypersurface, then

$$\text{res}_{G, \langle u_V \rangle}(\xi_j) = -f_j(t_1, \dots, t_r) \zeta^m$$

does not involve any of the  $X_j$ . So we may perform the construction of the triangle over  $k(V)$  and restrict, and then tensor with  $K$  at the end. Since some  $f_j(t_1, \dots, t_r)$  is nonzero, the corresponding  $\hat{\xi}_j$  restricts to a map with projective cokernel, and so  $k(V) \otimes_k L \downarrow_{\langle u_V \rangle}$  is projective. So from now on, assume that  $f_r(t_1, \dots, t_r) \neq 0$ .

Using  $\hat{\zeta}^m$  to identify  $k(V)$  with  $\Omega^{-m} k(V)$ , it follows that up to irrelevant projective summands, the restriction is equal to the Koszul complex obtained using the elements  $X_j f_r - f_j$  on the ring  $k(V) \otimes_k K$ . Since  $f_r(t_1, \dots, t_r) \neq 0$ , these elements form a regular sequence, in this ring, because each uses an  $X_j$  not used in any of the other terms. So the homology is equal to the quotient

$$k(V) \otimes_k K / (X_1 f_r - f_1, \dots, X_{r-1} f_r - f_{r-1}).$$

If  $V = V_E$ , so that the  $t_j$  are algebraically independent, the map (given by restricting  $\phi_1 \otimes \dots \otimes \phi_{r-1}$ ) from  $k(V)$  to this quotient is an isomorphism, since each occurrence of  $X_j$  can be replaced by  $f_j/f_r$ . On the other hand, if  $V$  is a proper subvariety, so that there is a nontrivial homogeneous relation between the elements  $t_1, \dots, t_r$ , then there is also a nontrivial homogeneous relation between the elements  $f_1, \dots, f_r$ , say  $g(f_1, \dots, f_r) = 0$ . But then

$$\begin{aligned} 1 \otimes 1 &= 1 \otimes g(X_1, \dots, X_{r-1}, 1) / g(X_1, \dots, X_{r-1}, 1) \\ &= g(f_1/f_r, \dots, f_{r-1}/f_r, 1) \otimes 1 / g(X_1, \dots, X_{r-1}, 1) = 0, \end{aligned}$$

and so the quotient is zero, and the restriction of  $L$  to  $\langle u_V \rangle$  is projective.

The conclusion of this calculation is that the rank variety of  $L$  is equal to  $\{V_E^r\}$ , while the rank variety of  $X$  is the complement of this. Rickard's characterization of  $F_{\mathcal{U}}$  (see the remark after Proposition 5.7 of [14]) now completes the proof of the theorem.  $\square$

**Remark 5.3.** This theorem can be used as the basis for an alternative proof that  $F_{\mathcal{U}}$  is endofinite.

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