

# SUPPORT OF NOETHERIAN MODULES OVER COMMUTATIVE RINGS

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Let  $A$  be a commutative ring. We consider the category of  $A$ -modules and its full subcategory  $\text{noeth } A$  which is formed by all noetherian  $A$ -modules. An  $A$ -module is *locally noetherian* if all its finitely generated submodules are noetherian. We denote by  $\text{Noeth } A$  the category of locally noetherian  $A$ -modules. Note that the ring  $A$  is noetherian precisely when every  $A$ -module is locally noetherian.

The *spectrum*  $\text{Spec } A$  of  $A$  is the set of prime ideals of  $A$ . A subset of  $\text{Spec } A$  is *Zariski closed* if it is of the form

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \text{Spec } A \mid \mathfrak{a} \subseteq \mathfrak{p}\}$$

for some ideal  $\mathfrak{a}$  of  $A$ . The *support* of an  $A$ -module  $M$  is by definition

$$\text{supp } M = \{\mathfrak{p} \in \text{Spec } A \mid M_{\mathfrak{p}} \neq 0\}.$$

**Lemma 1.** *Let  $\mathfrak{a}$  be an ideal of  $A$ . Then  $\text{supp } A/\mathfrak{a} = V(\mathfrak{a})$ .*

*Proof.* Fix  $\mathfrak{p} \in \text{Spec } A$  and let  $S = A \setminus \mathfrak{p}$ . Recall that for any  $A$ -module  $M$ , an element  $x/s$  in  $S^{-1}M = M_{\mathfrak{p}}$  is zero iff there exists  $t \in S$  such that  $tx = 0$ . Thus we have  $(A/\mathfrak{a})_{\mathfrak{p}} = 0$  iff there exists  $t \in S$  with  $t(1 + \mathfrak{a}) = t + \mathfrak{a} = 0$  iff  $\mathfrak{a} \not\subseteq \mathfrak{p}$ .  $\square$

**Lemma 2.** *Let  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  be an exact sequence of  $A$ -modules. Then  $\text{supp } M = \text{supp } M' \cup \text{supp } M''$ .*

*Proof.* For each prime ideal  $\mathfrak{p}$ , the sequence  $0 \rightarrow M'_{\mathfrak{p}} \rightarrow M_{\mathfrak{p}} \rightarrow M''_{\mathfrak{p}} \rightarrow 0$  is exact.  $\square$

**Lemma 3.** *Let  $M = \sum_i M_i$  be an  $A$ -module, written as a sum of submodules  $M_i$ . Then  $\text{supp } M = \bigcup_i \text{supp } M_i$ .*

*Proof.* The assertion is clear if the sum  $\sum_i M_i$  is direct, since

$$\bigoplus_i (M_i)_{\mathfrak{p}} = (\bigoplus_i M_i)_{\mathfrak{p}}.$$

Now apply Lemma 2. More precisely,  $M_i \subseteq M$  for all  $i$  implies  $\bigcup_i \text{supp } M_i \subseteq \text{supp } M$ . On the other hand,  $M = \sum_i M_i$  is a factor of  $\bigoplus_i M_i$ , and therefore  $\text{supp } M \subseteq \bigcup_i \text{supp } M_i$ .  $\square$

**Lemma 4.** *Let  $M$  be a finitely generated  $A$ -module and  $\mathfrak{a}$  its annihilator. Then  $\text{supp } M = V(\mathfrak{a})$ .*

*Proof.* Let  $M = \sum_{i=1}^n M_i$  and  $M_i \cong A/\mathfrak{a}_i$  for each  $i$ . Then

$$\text{supp } M = \bigcup_{i=1}^n \text{supp } M_i = \bigcup_{i=1}^n V(\mathfrak{a}_i) = V\left(\bigcap_{i=1}^n \mathfrak{a}_i\right) = V(\mathfrak{a}).$$

$\square$

**Lemma 5.** *Let  $M \neq 0$  be an  $A$ -module. If  $\mathfrak{p}$  is maximal in the set of ideals which annihilate a non-zero element of  $M$ , then  $\mathfrak{p}$  is prime.*

*Proof.* Suppose  $0 \neq x \in M$  and  $\mathfrak{p}x = 0$ . Let  $a, b \in A$  with  $ab \in \mathfrak{p}$  and  $a \notin \mathfrak{p}$ . Then  $(\mathfrak{p}, b)$  annihilates  $ax \neq 0$ , and the maximality of  $\mathfrak{p}$  implies  $b \in \mathfrak{p}$ . Thus  $\mathfrak{p}$  is prime.  $\square$

**Lemma 6.** *Let  $M \neq 0$  be a noetherian  $A$ -module. There exists a submodule of  $M$  which is isomorphic to  $A/\mathfrak{p}$  for some prime ideal  $\mathfrak{p}$ .*

*Proof.* The set of ideals annihilating a non-zero element has a maximal element, because  $M$  is noetherian. Now apply Lemma 5.  $\square$

**Lemma 7.** *Let  $M$  be a noetherian  $A$ -module. Then there exists a chain*

$$0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = M$$

*such that each factor  $M_i/M_{i-1}$  is isomorphic to  $A/\mathfrak{p}_i$  for some prime ideal  $\mathfrak{p}_i$ .*

*Proof.* Let  $M'$  be a maximal submodule of  $M$  which admits a filtration with factors of the form  $A/\mathfrak{p}$ . If  $M' \neq M$ , then there exists a submodule of  $M/M'$  which is isomorphic to  $A/\mathfrak{q}$  for some prime ideal  $\mathfrak{q}$ , by Lemma 6. We obtain a submodule  $M'' \subseteq M$  containing  $M'$  with  $M''/M' \cong A/\mathfrak{q}$ , and this contradicts the maximality of  $M'$ . Thus  $M' = M$ .  $\square$

A full subcategory  $\mathcal{C}$  of  $A$ -modules is called a *Serre subcategory* if for every exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  of  $A$ -modules,  $M$  belongs to  $\mathcal{C}$  if and only if  $M'$  and  $M''$  belong to  $\mathcal{C}$ . A subset  $V$  of  $\text{Spec } A$  is *specialization closed* if for any pair  $\mathfrak{p} \subseteq \mathfrak{q}$  of prime ideals,  $\mathfrak{p} \in V$  implies  $\mathfrak{q} \in V$ .

**Proposition 8.** *There exists a bijection between*

- (1) *the set of Serre subcategories of  $\text{noeth } A$ , and*
- (2) *the set of specialization closed subsets of  $\{\mathfrak{p} \in \text{Spec } A \mid A/\mathfrak{p} \in \text{noeth } A\}$ .*

*This bijection takes a Serre subcategory  $\mathcal{C}$  to  $\bigcup_{M \in \mathcal{C}} \text{supp } M$ ; its inverse takes  $V \subseteq \text{Spec } A$  to  $\{M \in \text{noeth } A \mid \text{supp } M \subseteq V\}$ .*

*Proof.* Both maps are well defined by Lemmas 2 and 4. Let us show that the map sending a Serre subcategory  $\mathcal{C}$  to

$$\text{supp } \mathcal{C} = \bigcup_{M \in \mathcal{C}} \text{supp } M$$

is bijective. If  $V \subseteq \text{Spec } A$  is a specialization closed subset, let  $\mathcal{C}_V$  denote the smallest Serre subcategory containing  $\{A/\mathfrak{p} \mid \mathfrak{p} \in V\}$ . Then we have  $\text{supp } \mathcal{C}_V = V$ , by Lemmas 1 and 2. Now let  $\mathcal{C}$  be a Serre subcategory of  $\text{noeth } A$ . Then

$$\text{supp } \mathcal{C} = \{\mathfrak{p} \in \text{Spec } A \mid A/\mathfrak{p} \in \mathcal{C}\}.$$

This follows from Lemma 7. In fact, each  $M \in \mathcal{C}$  admits a filtration

$$0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = M$$

such that each  $M_i/M_{i-1}$  is isomorphic to  $A/\mathfrak{p}_i$  for some  $\mathfrak{p}_i \in \text{Spec } A$ , and we have therefore

$$\text{supp } M = \bigcup_{i=1}^n V(\mathfrak{p}_i).$$

It follows that  $\mathcal{C} = \mathcal{C}_V$  for each Serre subcategory  $\mathcal{C}$ , where  $V = \text{supp } \mathcal{C}$ . Thus  $\text{supp } \mathcal{C}_1 = \text{supp } \mathcal{C}_2$  implies  $\mathcal{C}_1 = \mathcal{C}_2$  for each pair  $\mathcal{C}_1, \mathcal{C}_2$  of Serre subcategories.  $\square$

**Corollary 9.** *Let  $M$  and  $N$  be noetherian  $A$ -modules. Then  $\text{supp } N \subseteq \text{supp } M$  if and only if  $N$  belongs to the smallest Serre subcategory containing  $M$ .*

*Proof.* Let  $\mathcal{C}$  denote the smallest Serre subcategory containing  $M$ . Then  $\text{supp } \mathcal{C} = \text{supp } M$  by Lemma 2. Now apply Proposition 8.  $\square$

A full subcategory  $\mathcal{C}$  of  $A$ -modules is called a *localizing subcategory* if  $\mathcal{C}$  is a Serre subcategory and if for any family of  $A$ -modules  $M_i \in \mathcal{C}$  the direct sum  $\bigoplus_i M_i$  belongs to  $\mathcal{C}$ .

**Corollary 10** ([1, p. 425]). *There exists a bijection between*

- (1) *the set of localizing subcategories of  $\text{Noeth } A$ , and*
- (2) *the set of specialization closed subsets of  $\{\mathfrak{p} \in \text{Spec } A \mid A/\mathfrak{p} \in \text{noeth } A\}$ .*

*This bijection takes a localizing subcategory  $\mathcal{C}$  to  $\bigcup_{M \in \mathcal{C}} \text{supp } M$ ; its inverse takes  $V \subseteq \text{Spec } A$  to  $\{M \in \text{Noeth } A \mid \text{supp } M \subseteq V\}$ .*

*Proof.* The proof is essentially the same as the one of Proposition 8 if we observe that any  $A$ -module  $M$  is the sum  $M = \sum_i M_i$  of its finitely generated submodules. Note that  $M$  belongs to a localizing subcategory  $\mathcal{C}$  if and only if all  $M_i$  belong to  $\mathcal{C}$ . In addition, we use that  $\text{supp } M = \bigcup_i \text{supp } M_i$ ; see Lemma 3.  $\square$

#### REFERENCES

- [1] P. GABRIEL: Des catégories abéliennes. Bull. Soc. Math. France **90** (1962), 323–448.

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