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Epimorphisms of additive categories up to direct factors

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Abstract

We characterize additive functors which are epimorphisms up to direct factors.

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Let $F : \mathscr{C} \to \mathscr{D}$ be an additive functor between small additive categories and suppose that F is bijective on objects. Then it is well known that F is an epimorphism in the category of small additive categories if and only if the induced functor $F_* : (\mathscr{D}, \mathrm{Ab}) \to (\mathscr{C}, \mathrm{Ab})$ is fully faithful [5]. Here, $(\mathscr{C}, \mathrm{Ab})$ denotes the category of additive functors from $\mathscr{C}$ to the category Ab of abelian groups, and the natural transformations between such functors form the maps in $(\mathscr{C}, \mathrm{Ab})$. The functor F_* sends a functor $\mathscr{D} \to \mathrm{Ab}$ to its composite with F.

In this note, we generalize this result by removing the assumption on F to be bijective on objects. Our motivation for this work is explained in an Appendix. We need the following simple definition; see also Lemma 1.

Definition. Let $F : \mathscr{C} \to \mathscr{D}$ be any functor. Then F has a factorization $F = F_1 \circ F_0$ such that F_0 is bijective on objects and F_1 is fully faithful. This factorization is unique up to a unique isomorphism and we call it the *canonical factorization* of F.

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Theorem 1. *Let $F : \mathscr{C} \to \mathscr{D}$ be an additive functor between small additive categories and let $F = F_1 \circ F_0$ be its canonical factorization. Then the functor $F_* : (\mathscr{D}, \mathrm{Ab}) \to (\mathscr{C}, \mathrm{Ab})$ is fully faithful if and only if*

(EP1) *F_0 is an epimorphism, and*
(EP2) *for every object Y in $\mathscr{D}$ there is a finite set of objects X_i in $\mathscr{C}$ with maps $\phi_i : Y \to FX_i \to Y$ in $\mathscr{D}$ such that $\mathrm{id}_Y = \sum_i \phi_i$.*

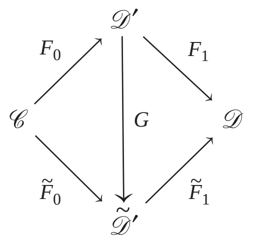
We call an additive functor F an *epimorphism up to direct factors* if its canonical factorization $F = F_1 \circ F_0$ has the above properties (EPl) and (EP2).

Remark. (1) If the category $\mathscr{C}$ has finite products, then (EP2) says that every object in $\mathscr{D}$ is a direct factor of some object in the image of F. Another reformulation of (EP2) says that the induced functor $(F_1)_*$ is an equivalence; see Lemma 4.

(2) There is a non-additive analogue of Theorem 1 for functors between small categories; it is discussed at the end of this note.

The proof of the theorem is based on a sequence of elementary facts. Before we start, let us fix some notation and terminology. Given a category $\mathscr{C}$, we denote for each pair of objects X, Y by $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ the set of maps $X \to Y$. The composition of maps in $\mathscr{C}$ is written from right to left. The category $\mathscr{C}$ is *additive*, if all sets $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ are equipped with an abelian group structure such that all composition maps are bilinear.

Lemma 1. *Let $F : \mathscr{C} \to \mathscr{D}$ be any functor. Then F has a factorization $F = F_1 \circ F_0$ such that F_0 is bijective on objects and F_1 is fully faithful. Given a second factorization $F = \tilde{F}_1 \circ \tilde{F}_0$ such that $\tilde{F}_0$ is bijective on objects and $\tilde{F}_1$ is fully faithful, there is a unique functor $G : \mathscr{D}' \to \tilde{\mathscr{D}}'$ making the following diagram commutative.*



Moreover, the functor G is an isomorphism.

Proof. We define a factorization

$$\mathscr{C} \xrightarrow{F_0} \mathscr{D}' \xrightarrow{F_1} \mathscr{D}$$

as follows. The objects of $\mathscr{D}'$ are those of $\mathscr{C}$ and F_0 is the identity on objects. Let

$$\mathrm{Hom}_{\mathscr{D}'}(X, Y) = \mathrm{Hom}_{\mathscr{D}}(FX, FY)$$

for all X, Y in $\mathscr{C}$, and let $F_0\alpha = F\alpha$ for each map α in $\mathscr{C}$. The functor F_1 equals F on objects and is the identity on maps. The uniqueness of this factorization is clear. $\square$

Lemma 2. *Let $F : \mathscr{C} \to \mathscr{D}$ be an additive functor between additive categories. Suppose the functor $F_* : (\mathscr{D}, \mathrm{Ab}) \to (\mathscr{C}, \mathrm{Ab})$ is full and F is surjective on objects. Then F is an epimorphism.*

Proof. Let $G, G' : \mathscr{D} \to \mathscr{E}$ be a pair of additive functors satisfying $G \circ F = G' \circ F$. Clearly, G and G' coincide on objects since F is surjective on objects. Now choose a map $\alpha : X \to Y$ in $\mathscr{D}$. We need to show that $G\alpha = G'\alpha$. The functor G' induces a natural transformation

$$\gamma : F_* \operatorname{Hom}_{\mathscr{D}}(X, -) \longrightarrow (F_* \circ G_*)\operatorname{Hom}_{\mathscr{E}}(GX, -),$$

which is defined by

$$\gamma_C : \operatorname{Hom}_{\mathscr{D}}(X, FC) \longrightarrow \operatorname{Hom}_{\mathscr{E}}(GX, G(FC)), \quad \phi \mapsto G'\phi$$

for each C in $\mathscr{C}$. The fact that F_* is full implies that $\gamma = F_*\delta$ for some natural transformation $\delta : \operatorname{Hom}_{\mathscr{D}}(X, -) \to G_*\operatorname{Hom}_{\mathscr{E}}(GX, -)$. In particular, $\delta_X = \gamma_C$ for some C in $\mathscr{C}$ satisfying $FC = X$. Thus we obtain the following commutative diagram:

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathscr{D}}(X, X) & \xrightarrow{\delta_Y} & \operatorname{Hom}_{\mathscr{E}}(GX, GX) \\
\downarrow{\scriptstyle \operatorname{Hom}_{\mathscr{D}}(X,\alpha)} & & \downarrow{\scriptstyle \operatorname{Hom}_{\mathscr{E}}(GX,G\alpha)} \\
\operatorname{Hom}_{\mathscr{D}}(X, Y) & \xrightarrow{\delta_x} & \operatorname{Hom}_{\mathscr{E}}(GX, GY),
\end{array}$$

which shows $G\alpha = G'\alpha$ if we apply it to id_X. We conclude that $G = G'$. □

Lemma 3. *Let $F : \mathscr{C} \to \mathscr{D}$ be an additive functor between small additive categories. Then the functor $F_* : (\mathscr{D}, \mathrm{Ab}) \to (\mathscr{C}, \mathrm{Ab})$ is faithful if and only if for every object Y in $\mathscr{D}$ there is a finite set of objects X_i in $\mathscr{C}$ with maps $\phi_i : Y \to FX_i \to Y$ in $\mathscr{D}$ such that $\mathrm{id}_Y = \sum_i \phi_i$.*

Proof. Suppose first that F_* is faithful. Note that F_* has a left adjoint $F^* : (\mathscr{C}, \mathrm{Ab}) \to (\mathscr{D}, \mathrm{Ab})$. The assumption on F_* implies that for each object M in $(\mathscr{D}, \mathrm{Ab})$, the natural map $(F^* \circ F_*)M \to M$ is an epimorphism. Now fix an object Y in $\mathscr{D}$. Every functor $\mathscr{C} \to \mathrm{Ab}$ is a quotient of a coproduct of representable functors. Thus we have an epimorphism

$$\coprod_{i \in \Lambda} \operatorname{Hom}_{\mathscr{C}}(X_i, -) \longrightarrow F_*\operatorname{Hom}_{\mathscr{D}}(Y, -)$$

and applying F^* induces an epimorphism

$$\coprod_{i \in \Lambda} \operatorname{Hom}_{\mathscr{D}}(FX_i, -) \longrightarrow (F^* \circ F_*)\operatorname{Hom}_{\mathscr{D}}(Y, -) \longrightarrow \operatorname{Hom}_{\mathscr{D}}(Y, -).$$

Using Yoneda's lemma, we find an element $\sum_i \psi_i$ in $\coprod_{i \in \Lambda} \operatorname{Hom}_{\mathscr{D}}(FX_i, Y)$ which is sent to id_Y. Now compose for each i the map ψ_i with the map $Y \to FX_i$ corresponding to $\operatorname{Hom}_{\mathscr{D}}(FX_i, -) \to \operatorname{Hom}_{\mathscr{D}}(Y, -)$. This yields the maps $\phi_i : Y \to Y$.

To prove the converse, let $\alpha, \beta : M \to N$ be a pair of natural transformations and suppose $\alpha \neq \beta$. Thus $\alpha_Y \neq \beta_Y$ for some Y in $\mathscr{D}$. Let $\mathrm{id}_Y = \sum_i \phi_i$ for some finite set of maps $\phi_i : Y \to FX_i \to Y$ in $\mathscr{D}$. It follows that $F_*\alpha \neq F_*\beta$ since $(F_*\alpha)_{X_i} \neq (F_*\beta)_{X_i}$ for some i. Thus F_* is faithful. $\square$

Next recall that a functor is an equivalence if and only if it is fully faithful and surjective on isomorphism classes of objects.

Lemma 4. *Let $F : \mathscr{C} \to \mathscr{D}$ be an additive functor between small additive categories. Then the functor $F_* : (\mathscr{D}, \mathrm{Ab}) \to (\mathscr{C}, \mathrm{Ab})$ is an equivalence if and only if F is fully faithful and for every object Y in $\mathscr{D}$ there is a finite set of objects X_i in $\mathscr{C}$ with maps $\phi_i : Y \to FX_i \to Y$ in $\mathscr{D}$ such that* $\mathrm{id}_Y = \sum_i \phi_i$.

Proof. The functor F_* is an equivalence if and only if its left adjoint F^* is an equivalence. Denote by $\mathrm{proj}\,\mathscr{C}$ the full subcategory of $(\mathscr{C}, \mathrm{Ab})$ formed by all direct factors of finite coproducts $\coprod_i \mathrm{Hom}_{\mathscr{C}}(X_i, -)$ of representable functors. The functor F^* sends $\mathrm{Hom}_{\mathscr{C}}(X, -)$ to $\mathrm{Hom}_{\mathscr{D}}(FX, -)$ and induces therefore a functor $\mathrm{proj}\,F : \mathrm{proj}\,\mathscr{C} \to \mathrm{proj}\,\mathscr{D}$ which is an equivalence if and only if F^* is an equivalence. It follows from Yoneda's lemma that the induced functor $\mathrm{proj}\,F$ is fully faithful if and only if F is fully faithful. Moreover, $\mathrm{proj}\,F$ is surjective on objects up to isomorphism if and only if for every object Y in $\mathscr{D}$ there is a finite set of objects X_i in $\mathscr{C}$ with maps $\phi_i : Y \to FX_i \to Y$ in $\mathscr{D}$ such that $\mathrm{id}_Y = \sum_i \phi_i$. Thus we have characterized in terms of F the fact that F_* is fully faithful. $\square$

Lemma 5. *Let $F : \mathscr{C} \to \mathscr{D}$ be an additive functor between additive categories. Suppose F is an epimorphism and bijective on objects. Then the functor $F_* : (\mathscr{D}, \mathrm{Ab}) \to (\mathscr{C}, \mathrm{Ab})$ is fully faithful.*

Proof. Let M, N be a pair of objects in $(\mathscr{D}, \mathrm{Ab})$. We need to show that the canonical map

$$(F_*)_{M,N} : \mathrm{Hom}_{(\mathscr{D},\mathrm{Ab})}(M, N) \to \mathrm{Hom}_{(\mathscr{C},\mathrm{Ab})}(F_*M, F_*N)$$

is bijective. Given a family $\phi = (\phi_X)_{X\in\mathscr{D}}$ of maps $\phi_X : MX \to NX$, we define a functor $H_\phi : \mathscr{D} \to \mathrm{Ab}$ by

$$H_\phi X = MX \amalg NX \quad \text{and} \quad H_\phi\alpha = \begin{bmatrix} M\alpha & 0 \\ N\alpha \circ \phi_X - \phi_Y \circ M\alpha & N\alpha \end{bmatrix}$$

for each object X and each map $\alpha : X \to Y$ in $\mathscr{D}$. Note that

$$(\phi_X)_{X\in\mathscr{D}} : M \longrightarrow N$$

is a natural transformation if and only if $H_\phi = M \amalg N$.

To prove that $(F_*)_{M,N}$ is surjective, fix a natural transformation

$$\psi = (\psi_X)_{X\in\mathscr{C}} : F_*M \longrightarrow F_*N.$$

For each X in $\mathscr{D}$ put $\phi_X = \psi_{F^{-1}X}$. We have $H_\phi \circ F = (M \amalg N) \circ F$ since ψ is a natural transformation. Thus (ϕ) is a natural transformation because $H_\phi \circ F = (M \amalg N) \circ F$ implies $H_\phi = M \amalg N$. We have $F_*\phi = \psi$ and conclude that the map $(F_*)_{M,N}$ is surjective.

To prove that $(F_*)_{M,N}$ is injective, apply Lemma 3. $\square$

Proof of Theorem 1. We fix the canonical factorization $F = F_1 \circ F_0$ of F. Suppose first that F_* is fully faithful. It follows from Lemma 3 that for each object Y in $\mathscr{D}$, there is a family of maps $\phi_i : Y \to FX_i \to Y$ such that $\sum_i \phi_i = \mathrm{id}_Y$. Now Lemma 4 implies that $(F_1)_*$ is fully faithful. Thus $(F_0)_*$ is fully faithful, since $F_* = (F_1)_* \circ (F_0)_*$. We conclude from Lemma 2 that F_0 is an epimorphism. Thus (EP1) and (EP2) hold.

Now suppose that the canonical factorization $F = F_1 \circ F_0$ satisfies (EP1) and (EP2). Then $(F_0)_*$ is fully faithful by Lemma 5, and $(F_1)_*$ is fully faithful by Lemma 4. Thus F_* is fully faithful. □

We end this note by pointing out the analogue of Theorem 1 for functors between small categories. Thus we provide a characterization of lax epimorphisms in the category of small categories. Recall that a functor $F : \mathscr{C} \to \mathscr{D}$ between small categories is a *lax epimorphism* if the induced functor

$$(F, \mathscr{E}) : (\mathscr{D}, \mathscr{E}) \longrightarrow (\mathscr{C}, \mathscr{E}), \quad X \mapsto F \circ X$$

is fully faithful for every small category $\mathscr{E}$. In [1], it is shown that F is a lax epimorphism if and only if (F, Set) is fully faithful, where Set denotes the category of sets.

Theorem 2. *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between small categories and let $F = F_1 \circ F_0$ be its canonical factorization. Then the functor $(F, \mathrm{Set}) : (\mathscr{D}, \mathrm{Set}) \to (\mathscr{C}, \mathrm{Set})$ is fully faithful if and only if*

(1) *F_0 is an epimorphism, and*
(2) *every object in $\mathscr{D}$ is a retract of an object in the image of F.*

Proof. Adapt the proof of Theorem 1. □

Appendix. The telescope conjecture

In this appendix we explain the relevance of epimorphism up to direct factors for the telescope conjecture from stable homotopy theory. This is based on the following definition. Let us recall from [3] that for any category $\mathscr{C}$ and any class of maps Σ in $\mathscr{C}$, there exists a functor $Q : \mathscr{C} \to \mathscr{C}[\Sigma^{-1}]$ such that

(1) $Q\phi$ is invertible for all $\phi \in \Sigma$, and
(2) given any functor $F : \mathscr{C} \to \mathscr{D}$ such that $F\phi$ is invertible for all $\phi \in \Sigma$, there is a unique functor $\bar{F} : \mathscr{C}[\Sigma^{-1}] \to \mathscr{D}$ such that $F = \bar{F} \circ Q$.

Definition. Let $F : \mathscr{C} \to \mathscr{D}$ be an additive functor between additive categories and denote by Σ the class of maps ϕ in $\mathscr{C}$ such that $F\phi$ is invertible. We call F a *localization up to direct factors*, if

(1) F induces a fully faithful functor $\mathscr{C}[\Sigma^{-1}] \to \mathscr{D}$, and
(2) for every object Y in $\mathscr{D}$ there is a finite set of objects X_i in $\mathscr{C}$ with maps $\phi_i : Y \to FX_i \to Y$ in $\mathscr{D}$ such that $\mathrm{id}_Y = \sum_i \phi_i$.

Let us give an example of a localization up to direct factors. Consider a separated noetherian scheme X and denote by $\mathbf{D}^{\mathrm{perf}}(X)$ the category of perfect complexes on X.

Theorem (*Thomason and Trobaugh*). *Let U be an open subscheme of a noetherian scheme X. Then restriction onto U induces a functor* $\mathbf{D}^{\mathrm{perf}}(X) \to \mathbf{D}^{\mathrm{perf}}(U)$ *which is a localization up to direct factors.*

This is a key result in the work of Thomason and Trobaugh on the K-theory of schemes [8]. Next, observe that every localization up to direct factors is an epimorphism up to direct factors. Without serious assumptions, one cannot expect the converse to be true. However, work in [4] shows that for certain triangulated categories there is a bijective correspondence between smashing localizations and epimorphism up to direct factors. This leads to the following reformulation of a result due to Neeman [6].

Theorem (*Neeman*). *Let R be a commutative noetherian ring and let* $F : \mathbf{D}^{\mathrm{perf}}(R) \to \mathscr{T}$ *be an exact functor from the category of perfect complexes over R to a triangulated category* $\mathscr{T}$*. If F is an epimorphism up to direct factors, then F is a localization up to direct factors.*

There is a similar reformulation of the telescope conjecture due to Bousfield and Ravenel [2,7].

Telescope Conjecture (*Bousfield and Ravenel*). *Let* $\mathscr{S}$ *be the stable homotopy category of finite CW-spectra and let* $F : \mathscr{S} \to \mathscr{T}$ *be an exact functor to a triangulated category* $\mathscr{T}$*. If F is an epimorphism up to direct factors, then F is a localization up to direct factors.*

We refer to [4] for details and the proof that the above statement is equivalent to the original conjecture of Bousfield and Ravenel.

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