REPRESENTATIONS OF QUIVERS VIA REFLECTION FUNCTORS

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Contents

These are the notes for a course on representations of quivers for second year students in Paderborn in summer 2007. My aim was to provide a basic introduction without using any advanced methods. It turns out that a good knowledge of linear algebra is sufficient for proving Gabriel's theorem. Thus we classify the quivers of finite representation type and study their representations using reflection functors. The course was complemented by problem sessions run by Andrew Hubery; see his homepage for interesting exercises and further material. I wish to thank him for many useful discussions on the subject and the students of this course for their enthusiasm.

1. Representations of quivers

In this section we introduce our basic concepts: quivers and their representations. Throughout we fix a field k.

1.1. **Quivers.** A *quiver* is a directed graph, which is assumed to be finite. More precisely, a quiver is a quadruple $Q = (Q_0, Q_1, s, t)$ consisting of a finite set Q_0 of *vertices*, a finite set Q_1 of *arrows*, and two maps $s, t\colon Q_1 \to Q_0$. An arrow $\alpha \in Q_1$ *starts* at $s(\alpha)$ and *terminates* at $t(\alpha)$. We sometimes write $\alpha\colon s(\alpha) \to t(\alpha)$.

A *non-trivial path* of length $r \geq 1$ in Q is a sequence $\xi = \xi_r \dots \xi_1$ of arrows satisfying $t(\xi_p) = s(\xi_{p+1})$ for $1 \leq p < r$. We write

$$i_1 \xrightarrow{\xi_1} i_2 \xrightarrow{\xi_2} \cdots \xrightarrow{\xi_r} i_{r+1}.$$

The path ξ starts at $s(\xi) = s(\xi_1)$ and terminates at $t(\xi) = t(\xi_r)$. For each vertex i we have in addition the *trivial path* ε_i of length zero with $s(\varepsilon_i) = i = t(\varepsilon_i)$.

For a pair i, j of vertices let $Q(i,j)$ denote the set of paths ξ with $s(\xi) = i$ and $t(\xi) = j$. The obvious composition of paths induces maps

$$Q(i,\xi)\colon Q(i,s(\xi)) \longrightarrow Q(i,t(\xi)) \quad \text{and} \quad Q(\xi,j)\colon Q(t(\xi),j) \longrightarrow Q(s(\xi),j)$$

with $Q(i,\xi)(\mu) = \xi\mu$ and $Q(\xi,j)(\nu) = \nu\xi$.

1.2. **Representations.** Let Q be a quiver. A *representation* of Q is a collection

$$X = (X_i, X_\alpha)_{i\in Q_0, \alpha\in Q_1}$$

consisting of a vector space X_i for each vertex i and a linear map $X_\alpha\colon X_{s(\alpha)} \to X_{t(\alpha)}$ for each arrow α. A *morphism* of representations $\phi\colon X \to Y$ is a collection $\phi = (\phi_i)_{i\in Q_0}$ of linear maps $\phi_i\colon X_i \to Y_i$ for each vertex i such that $Y_\alpha\phi_{s(\alpha)} = \phi_{t(\alpha)}X_\alpha$ for each arrow α. In other words, for each arrow α we have a commutative diagram

$$\begin{tikzcd}
X_{s(\alpha)} \arrow[r, "\phi_{s(\alpha)}"] \arrow[d, "X_\alpha"] & Y_{s(\alpha)} \arrow[d, "Y_\alpha"] \\
X_{t(\alpha)} \arrow[r, "\phi_{t(\alpha)}"] & Y_{t(\alpha)}
\end{tikzcd}$$

The *composition* of ϕ with $\psi\colon Y \to Z$ is given by $(\psi\phi)_i = \psi_i\phi_i$ for each vertex i. For each representation X we have the *identity morphism* $\mathrm{id}_X\colon X \to X$ with $(\mathrm{id}_X)_i = \mathrm{id}_{X_i}$ for all i. The set of morphisms $X \to Y$ we denote by $\mathrm{Hom}(X,Y)$ and we write $\mathrm{End}(X)$ for the set of endomorphisms $X \to X$.

This defines a category $\mathrm{Rep}(Q,k)$. We denote by $\mathrm{rep}(Q,k)$ the full subcategory with objects the finite dimensional representations. Here, a representation X is *finite dimensional* if each vector space X_i is finite dimensional.

1.3. **The category of representations.** Various concepts that are defined for vector spaces carry over to representations of Q by applying the vector space definition *point-wise*, that is, for each vertex i.

Fix a pair X, Y of representations of Q. We call X a *subrepresentation* of Y and write $X \subseteq Y$, if X_i is a subspace of Y_i for each vertex i and $X_\alpha(x) = Y_\alpha(x)$ for each arrow α and $x \in X_{s(\alpha)}$.

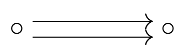
Given a morphism $\phi\colon X \to Y$, its *kernel* $\operatorname{Ker}\phi$ is by definition the subrepresentation of X with $(\operatorname{Ker}\phi)_i = \operatorname{Ker}\phi_i$ for each vertex i. The *cokernel* $\operatorname{Coker}\phi$ and the *image* $\operatorname{Im}\phi$ are defined analogously. The morphism ϕ is an *isomorphism* if each ϕ_i is an isomorphism. One defines addition and scalar multiplication for morphisms $X \to Y$ point-wise and that makes $\mathrm{Hom}(X,Y)$ into a vector space.

The *dimension vector* of a finite dimensional representation X is the vector $\dim X$ in $\mathbb{Z}^{Q_0}$ with

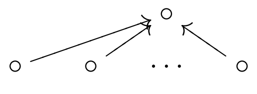
$$(\dim X)_i = \dim X_i \quad (i \in Q_0).$$

1.4. **Some special quivers.** We consider three quivers in more detail. The repre-

(a) Jordan quiver



(b) Kronecker quiver



(c) n-subspace quiver

sentations of the *Jordan quiver* are endomorphisms of a single vector space and their classification can be formulated in terms of Jordan blocks. The representations of the *Kronecker quiver* are pairs of linear maps up to simultaneous conjugation. The *n-subspace quiver* has $n+1$ vertices and its representations are basically configurations of n subspaces of a fixed vector space.

1.5. **Duality.** Let Q^{op} denote the *opposite quiver* which is obtained from Q by reversing all arrows. The vector space duality $D = \operatorname{Hom}_k(-,k)$ induces a *duality*

$$D\colon \operatorname{Rep}(Q,k) \longrightarrow \operatorname{Rep}(Q^{\mathrm{op}},k).$$

Given a representation X of Q we let $(DX)_i = D(X_i)$ and $(DX)_\alpha = D(X_\alpha)$ for $i \in Q_0$ and $\alpha \in Q_1$. For a morphism $\phi\colon X \to Y$ we let $(D\phi)_i = D(\phi_i)$.

Let V, W be a pair of vector spaces. Recall that there is a canonical monomorphism $\varepsilon_V\colon V \to D^2V$ defined by $\varepsilon_V(x)(\phi) = \phi(x)$ for $x \in V$ and $\phi \in DV$. This induces an isomorphism

$$\operatorname{Hom}(W, DV) \xrightarrow{\sim} \operatorname{Hom}(V, DW)$$

by sending $\phi\colon W \to DV$ to $(D\phi)\varepsilon_V$.

Lemma 1.5.1. *Let $X \in \operatorname{Rep}(Q,k)$ and $Y \in \operatorname{Rep}(Q^{\mathrm{op}},k)$. There is a canonical monomorphism $\varepsilon_X\colon X \to D^2X$ which induces a natural isomorphism*

$$\operatorname{Hom}(Y, DX) \xrightarrow{\sim} \operatorname{Hom}(X, DY)$$

by sending $\phi\colon Y \to DX$ to $(D\phi)\varepsilon_X$.

Proof. Use the linear maps ε_{X_i} and $\operatorname{Hom}(Y_i, DX_i) \xrightarrow{\sim} \operatorname{Hom}(X_i, DY_i)$. ☐

1.6. **Simple representations.** A *simple* representation is defined to be a non-zero representation with no proper subrepresentations.

Given a vertex i, let $S(i)$ be the representation with

$$S(i)_j = \begin{cases} k & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases} \quad \text{and} \quad S(i)_\alpha = 0$$

for $j \in Q_0$ and $\alpha \in Q_1$. This representation is simple.

Now assume that Q has no *oriented cycles* (that is, non-trivial paths from a vertex to itself). Then for any simple representation S of Q, there exists a unique vertex i such that $S \cong S(i)$.

Example 1.6.1. The finite dimensional simple representations of the Jordan quiver are parametrised by the monic irreducible polynomials over k. More precisely, the representation corresponding to such a polynomial $\sum_{i=0}^d \lambda_i t^i$ of degree d is the pair (X, ϕ) consisting of the vector space $X = k^d$ and the endomorphism $\phi\colon X \to X$ with $\phi(e_i) = e_{i+1}$ for $1 \le i < d$ and $\phi(e_d) = \sum_{i=1}^d -\lambda_{i-1} e_i$.

1.7. **Projective and injective representations.** For each vertex i we define a *projective representation* $P(i)$ and an *injective representation* $I(i)$.

Given any set X we denote by $k[X]$ the vector space with basis X, that is, $k[X]$ is the set of linear combinations $\sum_p \lambda_p x_p$ with $x_p \in X$, $\lambda_p \in k$, and almost all $\lambda_p = 0$. For a map $\phi\colon X \to Y$ let $k[\phi]\colon k[X] \to k[Y]$ be the linear map sending $\sum_p \lambda_p x_p$ to $\sum_p \lambda_p \phi(x_p)$.

Now define

$$P(i)_j = k[Q(i,j)] \quad \text{and} \quad P(i)_\alpha = k[Q(i,\alpha)]$$

for $j \in Q_0$ and $\alpha \in Q_1$. Dually, we define

$$I(i)_j = Dk[Q(j,i)] \quad \text{and} \quad I(i)_\alpha = Dk[Q(\alpha,i)].$$

Note that $I(i) = D\bar{P}(i)$ where $\bar{P}(i)$ refers to the projective representation of Q^{op}.

Lemma 1.7.1. *Let X be a representation of Q. Then there are natural isomorphisms*

$$\operatorname{Hom}(P(i),X) \cong X_i \quad \textit{and} \quad \operatorname{Hom}(X,I(i)) \cong DX_i.$$

Proof. The isomorphism $\operatorname{Hom}(P(i),X) \to X_i$ sends ϕ to $\phi_i(\varepsilon_i)$. Its inverse map sends $x \in X_i$ to the morphism $\phi\colon P(i) \to X$ with

$$\phi_j(\xi) = X_{\xi_r} \dots X_{\xi_1}(x)$$

for a basis element $\xi = \xi_r \dots \xi_1$ of $P(i)_j$.

The second isomorphism $\operatorname{Hom}(X,I(i)) \cong DX_i$ follows from the first using Lemma 1.5.1 since

$$\operatorname{Hom}(X,I(i)) = \operatorname{Hom}(X,D\bar{P}(i)) \cong \operatorname{Hom}(\bar{P}(i),DX) \cong DX_i. \qquad \square$$

Lemma 1.7.2. (1) *The representations $P(i)$ ($i \in Q_0$) are pairwise non-isomorphic.*
(2) *The representations $I(i)$ ($i \in Q_0$) are pairwise non-isomorphic.*

Proof. Fix two vertices i,j and suppose $P(j) \cong P(i)$. Then

$$k = S(i)_i \cong \operatorname{Hom}(P(i),S(i)) \cong \operatorname{Hom}(P(j),S(i)) \cong S(i)_j = \begin{cases} k & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases}$$

by Lemma 1.7.1. Thus $j = i$. The proof for the $I(i)$ is analogous. $\square$

Lemma 1.7.3. *Suppose Q has no oriented cycles. Then $P(i)$ and $I(i)$ are finite dimensional with* $\operatorname{End}(P(i)) \cong k \cong \operatorname{End}(I(i))$.

Proof. If Q has no oriented cycle then $Q(i,j)$ is finite for all pairs of vertices i,j because the quiver is finite. We have

$$\operatorname{End}(P(i)) \cong P(i)_i = k[Q(i,i)] = k$$

by Lemma 1.7.1, and a similar argument works for $I(i)$. $\square$

Remark 1.7.4. The representation $P(i)$ is a *projective object* in the sense that for every epimorphism $X \to Y$ the induced map $\operatorname{Hom}(P(i),X) \to \operatorname{Hom}(P(i),Y)$ is surjective. Dually, $I(i)$ is an *injective object* in the sense that for every monomorphism $X \to Y$ the induced map $\operatorname{Hom}(Y,I(i)) \to \operatorname{Hom}(X,I(i))$ is surjective. This follows from Lemma 1.7.1.

2. Direct sum decompositions

In this section we consider finite dimensional representations. We show that each representation decomposes essentially uniquely into indecomposable representations.

2.1. **Direct sums.** Let $X_1, \dots, X_r$ be a finite number of representations. A *direct sum*

$$X = X_1 \oplus \dots \oplus X_r$$

is a representation X together with morphisms $\iota_i \colon X_i \to X$ and $\pi_i \colon X \to X_i$ for $1 \leqslant i \leqslant r$ such that $\sum_{i=1}^{r} \iota_i \pi_i = \mathrm{id}_X$ and $\pi_i \iota_i = \mathrm{id}_{X_i}$ for all i. Note that we can identify each X_i via ι_i with a subrepresentation of X. Then we obtain

$$X = \sum_{i=1}^{r} X_i \quad \text{and} \quad X_i \cap \sum_{i' \neq i} X_{i'} = 0 \quad \text{for} \quad 1 \leq i \leq r. \tag{2.1.1}$$

Here $\sum_{i \in I} X_i$ refers to the smallest subrepresentation of X containing X_i for all $i \in I$. Conversely, if $X_1, \dots, X_r$ is a family of subrepresentations of a representation X satisfying (2.1.1) then $X = X_1 \oplus \dots \oplus X_r$. In that case we take for $\iota_i \colon X_i \to X$ the inclusion morphism and let $\pi_i = (\rho_i \iota_i)^{-1} \rho_i$, where $\rho_i \colon X \to X / \sum_{i' \neq i} X_{i'}$ denotes the canonical morphism.

A family of subrepresentations $X_1, \dots, X_r$ of X satisfying (2.1.1) is called a *direct sum decomposition* of X. Note that one can check (2.1.1) point-wise for each vertex. Thus we have a decomposition $X = X_1 \oplus \dots \oplus X_r$ if and only if we have a vector space decomposition $X_j = (X_1)_j \oplus \dots \oplus (X_r)_j$ for each vertex j.

Lemma 2.1.1. *Let $X = X_1 \oplus \dots \oplus X_r$ and $Y = Y_1 \oplus \dots \oplus Y_s$. Then we have induced vector space decompositions*

$$\bigoplus_{i=1}^{r} \mathrm{Hom}(X_i, Y) = \mathrm{Hom}(X, Y) = \bigoplus_{j=1}^{s} \mathrm{Hom}(X, Y_j).$$

Proof. Let $\iota_i^* = \mathrm{Hom}(\iota_i, Y)$ and $\pi_i^* = \mathrm{Hom}(\pi_i, Y)$ for $1 \leqslant i \leqslant r$. Then we have $\sum_{i=1}^{r} \pi_i^* \iota_i^* = \mathrm{id}_{\mathrm{Hom}(X,Y)}$ and $\iota_i^* \pi_i^* = \mathrm{id}_{\mathrm{Hom}(X_i,Y)}$ for all i. This proves the first equality. The argument for the second equality is analogous. □

It follows from Lemma 2.1.1 that a direct sum of $X_1, \dots, X_r$ is unique up to an isomorphism. Thus we may speak of *the* direct sum and the notation $X_1 \oplus \dots \oplus X_r$ is well-defined. We write $X^r = X \oplus \dots \oplus X$ for the direct sum of r copies of a representation X.

A representation X is called *indecomposable* if $X \neq 0$ and $X = X_1 \oplus X_2$ implies $X_1 = 0$ or $X_2 = 0$.

2.2. **Fitting's lemma.** We fix a representation X and study the ring of endomorphisms $\mathrm{End}(X)$.

Lemma 2.2.1. *Let X be a representation and ϕ an endomorphism.*

(1) *For large enough r, we have $X = \mathrm{Im}\,\phi^r \oplus \mathrm{Ker}\,\phi^r$.*
(2) *If X is indecomposable, then ϕ is either an automorphism or nilpotent.*

Proof. Because X is finite dimensional, we may choose r large enough so that $\mathrm{Im}\,\phi^r = \mathrm{Im}\,\phi^{r+1}$. Thus $\phi^r \colon \mathrm{Im}\,\phi^r \to \mathrm{Im}\,\phi^{2r}$ is an isomorphism and we denote by ψ its inverse. Furthermore, let $\iota_1 \colon \mathrm{Im}\,\phi^r \to X$ and $\iota_2 \colon \mathrm{Ker}\,\phi^r \to X$ denote the inclusions. We put $\pi_1 = \psi\phi^r \colon X \to \mathrm{Im}\,\phi^r$ and $\pi_2 = \mathrm{id}_X - \psi\phi^r \colon X \to \mathrm{Ker}\,\phi^r$. Then $\iota_1\pi_1 + \iota_2\pi_2 = \mathrm{id}_X$ and $\pi_i\iota_i = \mathrm{id}_{X_i}$ for $i = 1, 2$. Thus $X = \mathrm{Im}\,\phi^r \oplus \mathrm{Ker}\,\phi^r$. Part (2) is an immediate consequence of (1). □

A ring is called *local* if the sum of two non-units is again a non-unit.

Proposition 2.2.2. *A representation X is indecomposable if and only if* $\mathrm{End}(X)$ *is local.*

Proof. Let X be indecomposable and $\phi, \phi' \in \mathrm{End}(X)$. Suppose $\phi + \phi'$ is invertible, say $\rho(\phi + \phi') = \mathrm{id}_X$. If ϕ is non-invertible then $\rho\phi$ is non-invertible. Thus $\rho\phi$ is nilpotent, say $(\rho\phi)^r = 0$, by Lemma 2.2.1. We obtain

$$(\mathrm{id}_X - \rho\phi)(\mathrm{id}_X + \rho\phi + \ldots + (\rho\phi)^{r-1}) = \mathrm{id}_X .$$

Therefore $\rho\phi' = \mathrm{id}_X - \rho\phi$ is invertible whence ϕ' is invertible.

If $X = X_1 \oplus X_2$ with $X_i \neq 0$ for $i = 1, 2$, then we have idempotent endomorphisms ε_i of X with $\mathrm{Im}\, \varepsilon_i = X_i$. Clearly, each ε_i is non-invertible but $\mathrm{id}_X = \varepsilon_1 + \varepsilon_2$. □

The assumption on X to be finite dimensional is necessary.

Example 2.2.3. Let $k[t]$ denote the polynomial ring in one variable and consider the following representation of the Kronecker quiver.

$$\begin{tikzcd} X : & k[t] \arrow[r, shift left, "\cdot t"] \arrow[r, shift right, "\mathrm{id}"'] & k[t] \end{tikzcd}$$

The endomorphism ring of X is isomorphic to $k[t]$. Thus X is indecomposable but $\mathrm{End}(X)$ is not local.

2.3. The radical.

Given a pair X, Y of representations, we define the *radical*

$$\mathrm{Rad}(X,Y) = \left\{ \phi \in \mathrm{Hom}(X,Y) \,\middle|\, \begin{array}{c} \tau\phi\sigma \text{ is non-invertible for every pair } Z \xrightarrow{\sigma} X \\ \text{and } Y \xrightarrow{\tau} Z \text{ with } Z \text{ indecomposable} \end{array} \right\} .$$

Lemma 2.3.1. *Let X, Y be a pair of representations.*

1. $\mathrm{Rad}(X,Y)$ *is a subspace of* $\mathrm{Hom}(X,Y)$.
2. $\mathrm{Rad}(X, Y_1 \oplus Y_2) = \mathrm{Rad}(X, Y_1) \oplus \mathrm{Rad}(X, Y_2)$.
3. $\mathrm{Rad}(X_1 \oplus X_2, Y) = \mathrm{Rad}(X_1, Y) \oplus \mathrm{Rad}(X_2, Y)$.
4. *If X and Y are indecomposable, then* $\mathrm{Hom}(X,Y) \setminus \mathrm{Rad}(X,Y)$ *equals the set of isomorphisms $X \to Y$.*

Proof. (1) Let $\phi_1, \phi_2 \in \mathrm{Rad}(X,Y)$. Choose $\sigma \in \mathrm{Hom}(Z,X)$ and $\tau \in \mathrm{Hom}(Y,Z)$ with Z indecomposable. Then $\tau\phi_1\sigma$ and $\tau\phi_2\sigma$ are non-invertible, and therefore $\tau(\phi_1 + \phi_2)\sigma = \tau\phi_1\sigma + \tau\phi_2\sigma$ is non-invertible, since $\mathrm{End}(Z)$ is local by Proposition 2.2.2. Thus $\phi_1 + \phi_2$ belongs to $\mathrm{Rad}(X,Y)$.

(2) Let $Y = Y_1 \oplus Y_2$ and $\phi = (\phi_i) \in \mathrm{Hom}(X,Y)$ with $\phi_i \in \mathrm{Hom}(X, Y_i)$ for $i = 1, 2$. Choose $\sigma \in \mathrm{Hom}(Z,X)$ and $\tau = (\tau_i) \in \mathrm{Hom}(Y,Z)$ with Z indecomposable and $\tau_i \in \mathrm{Hom}(Y_i, Z)$ for $i = 1, 2$. Then $\tau\phi\sigma = \tau_1\phi_1\sigma + \tau_2\phi_2\sigma$.

If $\phi_i \in \mathrm{Rad}(X, Y_i)$ for $i = 1, 2$, then $\tau_i\phi_i\sigma$ is non-invertible for $i = 1, 2$, and therefore $\tau\phi\sigma$ is non-invertible, since $\mathrm{End}(Z)$ is local by Proposition 2.2.2. Thus ϕ belongs to $\mathrm{Rad}(X,Y)$. Conversely, let $\phi \in \mathrm{Rad}(X,Y)$ and fix $i \in \{1,2\}$. Then $\phi_i \in \mathrm{Rad}(X, Y_i)$ because we can put $\tau_j = 0$ for $j \neq i$ and have that $\tau_i\phi_i\sigma = \tau\phi\sigma$ is non-invertible.

(3) Analogous to part (2).

(4) Let $\phi \in \mathrm{Hom}(X,Y) \setminus \mathrm{Rad}(X,Y)$. Choose $\sigma \in \mathrm{Hom}(Z,X)$ and $\tau \in \mathrm{Hom}(Y,Z)$ with Z indecomposable such that $\tau\phi\sigma$ is invertible. Then σ is invertible because X is indecomposable, and τ is invertible because Y is indecomposable. Thus ϕ is invertible. It is clear that an isomorphism $X \to Y$ does not belong to $\mathrm{Rad}(X,Y)$. □

2.4. **The Krull-Remak-Schmidt theorem.**

Theorem 2.4.1. *Let X be a finite dimensional representation. Then there exists a decomposition $X = X_1^{a_1} \oplus \ldots \oplus X_r^{a_r}$ with the X_i pairwise non-isomorphic indecomposable representations and each $a_i \geqslant 1$. If $X = Y_1^{b_1} \oplus \ldots \oplus Y_s^{b_s}$ is another such decomposition, then $r = s$ and, after reordering, $X_i \cong Y_i$ and $a_i = b_i$ for $i \leq i \leq r$.*

Proof. Induction on dimension shows that X decomposes into a finite direct sum of indecomposable representations. Suppose that $X = X_1^{a_1} \oplus \ldots \oplus X_r^{a_r}$ is such a direct sum decomposition with the X_i pairwise non-isomorphic indecomposable representations and each $a_i \geqslant 1$. Let Y be indecomposable and consider

$$\frac{\dim \operatorname{Hom}(X,Y) - \dim \operatorname{Rad}(X,Y)}{\dim \operatorname{Hom}(Y,Y) - \dim \operatorname{Rad}(Y,Y)}.$$

We see by Lemma 2.3.1 that this number equals a_i if $Y \cong X_i$ (there is at most one such i) and 0 otherwise. In particular, this number is independent of the decomposition. □

The Krull-Remak-Schmidt theorem says that the classification of finite dimensional representations reduces to the classification of indecomposable representations. There is a similar statement about morphisms between representations. Let $X = X_1 \oplus \ldots \oplus X_r$ and $Y = Y_1 \oplus \ldots \oplus Y_s$ be two representations with their decompositions into indecomposable representations. Then we have

$$\operatorname{Hom}(X,Y) = \bigoplus_{i,j} \operatorname{Hom}(X_i, Y_j)$$

by Lemma 2.1.1. Thus each morphism $\phi\colon X \to Y$ can be written uniquely as a matrix $\phi = (\phi_{ij})$ where each entry $\phi_{ij}\colon X_i \to Y_j$ is a morphism between indecomposable representations.

3. Reflection functors

In this section we introduce reflection functors. These functors form our basic tool for classifying representations in terms of their dimension vectors.

3.1. **Orientations.**

A vertex i of Q is called a *sink* (resp. *source*) if there is no arrow in Q starting (resp. ending) at i.

Given any vertex i, the quiver $\sigma_i Q$ is obtained from Q by reversing all arrows which start or end at i.

An ordering $i_1, \ldots, i_n$ of the vertices of Q is called *admissable* if for each p the vertex i_p is a sink for $\sigma_{i_{p-1}} \ldots \sigma_{i_1} Q$. In that case we have

$$\sigma_{i_n} \ldots \sigma_{i_1} Q = Q.$$

Lemma 3.1.1. *There exists an admissable ordering of the vertices of Q if and only if there is no oriented cycle in Q.*

Proof. We show one implication by induction on the number of vertices. So suppose Q has no oriented cycle and let i_n be the starting vertex of a path of maximal length. Then i_n is a source and we remove it from Q. There is an admissable ordering $i_1, \ldots, i_{n-1}$ of the remaining vertices and we get an admissable ordering $i_1, \ldots, i_n$ of the vertices of Q. □

3.2. **The Euler form.** Let $n = \operatorname{card} Q_0$. The *Euler form* is the bilinear form

$$\langle -,-\rangle \colon \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z} \quad \text{with} \quad \langle x,y\rangle = \sum_{i\in Q_0} x_i y_i - \sum_{\alpha \in Q_1} x_{s(\alpha)} y_{t(\alpha)}.$$

We obtain on $\mathbb{Z}^n$ a *symmetric bilinear form* by defining

$$(x,y) = \langle x,y\rangle + \langle y,x\rangle. \tag{3.2.1}$$

Suppose that Q has no *loops* (that is, arrows from a vertex to itself). The *reflection* with respect to a vertex i is by definition the map

$$\sigma_i \colon \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \quad \text{with} \quad \sigma_i(x) = x - \frac{2(x,e_i)}{(e_i,e_i)} e_i$$

where e_i is the ith coordinate vector. It is easily checked that the σ_i are automorphisms of order two preserving the bilinear form $(-,-)$.

For the set $\mathbb{Z}^n$ we use the *partial order* which is defined as follows:

$$x \le y \quad \Longleftrightarrow \quad x_i \le y_i \quad \text{for all} \quad i.$$

3.3. **Reflection functors.** Let i be a vertex of Q. We define a pair of *reflection functors* S_i^+ and S_i^-. To this end fix representations X, X' of Q and a morphism $\phi \colon X \to X'$.

(1) If the vertex i is a sink of Q, then we construct

$$S_i^+ \colon \operatorname{Rep}(Q,k) \longrightarrow \operatorname{Rep}(\sigma_i Q,k)$$

as follows. We define $S_i^+ X = Y$ by letting $Y_j = X_j$ for a vertex $j \ne i$, and letting Y_i be the kernel of the map $\xi = (X_\alpha)$ in the following sequence

$$Y_i \xrightarrow{\check{\xi}} \bigoplus_{\substack{\alpha\in Q_1\\ t(\alpha)=i}} X_{s(\alpha)} \xrightarrow{\xi} X_i$$

where $\check{\xi}$ denotes the inclusion map of the kernel. For an arrow α in Q, let $Y_\alpha = X_\alpha$ if $t(\alpha) \ne i$, and $Y_\alpha \colon Y_i \to X_{s(\alpha)} = Y_{s(\alpha)}$ be the map $\check{\xi}$ followed by the canonical projection onto $X_{s(\alpha)}$ if $t(\alpha) = i$. For the morphism $S_i^+\phi = \psi$ let $\psi_j = \phi_j$ if $j \ne i$ and let $\psi_i \colon Y_i \to Y_i'$ be the restriction of the map

$$(\phi_{s(\alpha)}) \colon \bigoplus_{\substack{\alpha\in Q_1\\ t(\alpha)=i}} X_{s(\alpha)} \longrightarrow \bigoplus_{\substack{\alpha\in Q_1\\ t(\alpha)=i}} X'_{s(\alpha)}.$$

(2) If the vertex i is a source of Q, then we construct dually

$$S_i^- \colon \operatorname{Rep}(Q,k) \longrightarrow \operatorname{Rep}(\sigma_i Q,k)$$

as follows. We define $S_i^- X = Y$ by letting $Y_j = X_j$ for a vertex $j \ne i$, and letting Y_i be the cokernel of the map $\xi = (X_\alpha)$ in the following sequence

$$X_i \xrightarrow{\xi} \bigoplus_{\substack{\alpha\in Q_1\\ s(\alpha)=i}} X_{t(\alpha)} \xrightarrow{\hat{\xi}} Y_i$$

where $\hat{\xi}$ denotes the canonical map onto the cokernel. For an arrow α in Q, let $Y_\alpha = X_\alpha$ if $s(\alpha) \ne i$, and $Y_\alpha \colon Y_{t(\alpha)} = X_{t(\alpha)} \to Y_i$ be the restriction of $\hat{\xi}$ to $X_{t(\alpha)}$ if $s(\alpha) = i$. For the morphism $S_i^- \phi = \psi$ let $\psi_j = \phi_j$ if $j \neq i$ and let $\psi_i \colon Y_i \to Y_i'$ be the map which is induced by

$$(\phi_{t(\alpha)}) \colon \bigoplus_{\substack{\alpha\in Q_1\\ s(\alpha)=i}} X_{t(\alpha)} \longrightarrow \bigoplus_{\substack{\alpha\in Q_1\\ s(\alpha)=i}} X'_{t(\alpha)}.$$

(3) Let i be a sink of Q. Then we define a natural monomorphism

$$\iota_i X \colon S_i^- S_i^+ X \longrightarrow X \tag{3.3.1}$$

by letting $(\iota_i X)_j = \mathrm{id}_{X_j}$ for a vertex $j \neq i$, and letting $(\iota_i X)_i$ be the canonical map

$$(S_i^- S_i^+ X)_i = \operatorname{Coker} \check{\xi} \cong \operatorname{Im} \xi \longrightarrow X_i.$$

(4) Let i be a source of Q. Then we define a natural epimorphism

$$\pi_i X \colon X \longrightarrow S_i^+ S_i^- X \tag{3.3.2}$$

by letting $(\pi_i X)_j = \mathrm{id}_{X_j}$ for a vertex $j \neq i$, and letting $(\pi_i X)_i$ be the canonical map

$$X_i \longrightarrow \operatorname{Im} \xi \cong \operatorname{Ker} \hat{\xi} = (S_i^+ S_i^- X)_i.$$

Lemma 3.3.1. *S_i^+ and S_i^- are functors, that is, $S_i^\pm \mathrm{id}_X = \mathrm{id}_{S_i^\pm X}$ for every representation X and $S_i^\pm(\psi\phi) = (S_i^\pm \psi)(S_i^\pm \phi)$ for every pair $\phi\colon X \to Y$ and $\psi \colon Y \to Z$ of morphisms.*

Proof. Clear. □

Lemma 3.3.2. *Let X, X' be representations of Q and i be a vertex.*

1. *$S_i^\pm(X \oplus X') = S_i^\pm X \oplus S_i^\pm X'$.*
2. *$X = (S_i^- S_i^+ X) \oplus \operatorname{Coker} \iota_i X$ and $X = (S_i^+ S_i^- X) \oplus \operatorname{Ker} \pi_i X$.*
3. *If $\operatorname{Coker} \iota_i X = 0$, then $\dim S_i^+ X = \sigma_i(\dim X)$.*
4. *If $\operatorname{Ker} \pi_i X = 0$, then $\dim S_i^- X = \sigma_i(\dim X)$.*

Proof. (1) Use that $S_i^\pm$ is a functor satisfying $S_i^\pm(\phi + \psi) = S_i^\pm \phi + S_i^\pm \psi$ for any pair of parallel morphisms ϕ, ψ.

(2) The canonical map $\rho_i' \colon X_i \to \operatorname{Coker} \xi$ has a section $\rho_i \colon \operatorname{Coker} \xi \to X_i$, that is, $\rho_i' \rho_i = \mathrm{id}_{\operatorname{Coker} \xi}$. This gives a morphism $\rho\colon \operatorname{Coker} \iota_i X \to X$ if we put $\rho_j = 0$ for $j \neq i$. It is clear that $\iota_i X \colon S_i^- S_i^+ X \to X$ and $\rho \colon \operatorname{Coker} \iota_i X \to X$ give a direct sum decomposition of X. The proof for $X = (S_i^+ S_i^- X) \oplus \operatorname{Ker} \pi_i X$ is similar.

(3) If $\operatorname{Coker} \iota_i X = 0$, then we have

$$\dim Y_i = \sum_{\substack{\alpha\in Q_1\\ t(\alpha)=i}} \dim X_{s(\alpha)} - \dim X_i$$

and $\dim Y_j = \dim X_j$ for $j \neq i$. Thus $\dim Y = \sigma_i(\dim X)$. The proof of (4) is similar. □

Note that the representations $\operatorname{Coker} \iota_i X$ and $\operatorname{Ker} \pi_i X$ are concentrated at the vertex i. Thus they are direct sums of copies of the simple representation $S(i)$.

Lemma 3.3.3. *Let i be a sink and X an indecomposable representation of Q. Then the following are equivalent:*

1. *$X \not\cong S(i)$.*
2. *$S_i^+ X$ is indecomposable.*
3. *$S_i^+ X \neq 0$.*

(4) $S_i^- S_i^+ X \cong X$.
(5) *The map* $(X_\alpha)\colon \bigoplus_{\substack{\alpha \in Q_1\\ t(\alpha)=i}} X_{s(\alpha)} \to X_i$ *is an epimorphism.*
(6) $\sigma_i(\dim X) > 0$.
(7) $\dim S_i^+ X = \sigma_i(\dim X)$.

Proof. Apply Lemma 3.3.2. □

Remark 3.3.4. There is an analogue of Lemma 3.3.3 for a source of Q and the corresponding functor S_i^-.

The following theorem is a consequence and summarises the basic properties of the reflection functors.

Theorem 3.3.5. *The functors S_i^+ and S_i^- induce mutually inverse bijections between the indecomposable representations of Q and the indecomposable representations of $\sigma_i Q$, with the exception of the simple representation $S(i)$ corresponding to i, which is annihilated by these functors. Moreover,* $\dim S_i^{\pm} X = \sigma_i(\dim X)$ *for every indecomposable representation X not isomorphic to $S(i)$.*

3.4. **Coxeter functors.** Let Q be a quiver without oriented cycles and let $i_1, \dots, i_n$ be an admissable ordering of the vertices of Q. The *Coxeter functor* with respect to this ordering is the functor

$$C^+ = S_{i_n}^+ \dots S_{i_1}^+ \colon \operatorname{Rep}(Q,k) \longrightarrow \operatorname{Rep}(Q,k).$$

We also define

$$C^- = S_{i_1}^- \dots S_{i_n}^- \colon \operatorname{Rep}(Q,k) \longrightarrow \operatorname{Rep}(Q,k).$$

For $r \in \mathbb{Z}$ we write

$$C^r = \begin{cases} (C^+)^r & \text{if } r > 0, \\ \operatorname{Id} & \text{if } r = 0, \\ (C^-)^{-r} & \text{if } r < 0. \end{cases}$$

Lemma 3.4.1. *The functors C^+ and C^- do not depend on the choice of the ordering of the vertices of Q.*

Proof. First observe that $S_i^+ S_j^+ = S_j^+ S_i^+$ if i and j are sinks with respect to some orientation and if both are not joined by an arrow. Now fix two admissable orderings $i_1, \dots, i_n$ and $i'_1, \dots, i'_n$ of the vertices of Q. Let $i_1 = i'_m$. Then $i'_1, \dots, i'_{m-1}$ are not joined to i_1 by an arrow. Therefore

$$S_{i'_m}^+ \dots S_{i'_1}^+ = S_{i'_{m-1}}^+ \dots S_{i'_1}^+ S_{i_1}^+.$$

Applying a similar argument for i_2, then i_3, and so on, we obtain

$$S_{i'_n}^+ \dots S_{i'_1}^+ = S_{i_n}^+ \dots S_{i_1}^+.$$ □

For simplicity we assume in the following that $Q_0 = \{1, \dots, n\}$ with $1, \dots, n$ an admissable ordering.

Lemma 3.4.2. *Let i be a vertex.*

(1) $\dim P(i) = \sigma_1 \dots \sigma_{i-1}(e_i)$ *and* $\dim I(i) = \sigma_n \dots \sigma_{i+1}(e_i)$.
(2) $P(i) \cong S_1^- \dots S_{i-1}^- S(i)$ *and* $I(i) \cong S_n^+ \dots S_{i+1}^+ S(i)$

Proof. We provide the proof for $P(i)$; the proof for $I(i)$ is dual.

(1) For $0 \leq l < i$ one shows by induction that

$$\sigma_{i-l} \dots \sigma_{i-1}(e_i) = \sum_{j=0}^{l} \operatorname{card} Q(i, i-j) e_{i-j} \tag{3.4.1}$$

in $\mathbb{Z}^n$. For $l = i-1$ we then obtain $\sigma_1 \dots \sigma_{i-1}(e_i) = \dim P(i)$ because there are no paths from i to j in Q for $j > i$.

(2) We use the first part, in particular (3.4.1), and apply Lemma 3.3.3. An induction yields for $0 \leq l < i$ that

$$\dim S_l^+ \dots S_1^+ P(i) = \sigma_{l+1} \dots \sigma_{i-1}(e_i).$$

Thus $S_{i-1}^+ \dots S_1^+ P(i) \cong S(i)$ and therefore $P(i) \cong S_1^- \dots S_{i-1}^- S(i)$. □

Proposition 3.4.3. *Let X be an indecomposable representation of Q.*

1. $C^+ X = 0$ *if and only if* $X \cong P(i)$ *for some vertex* i.
2. $C^- X = 0$ *if and only if* $X \cong I(i)$ *for some vertex* i.

Proof. (1) We have $P(i) \cong S_1^- \dots S_{i-1}^- S(i)$ by Lemma 3.4.2. Now apply reflection functors and use Lemma 3.3.3 to obtain

$$C^+ P(i) = S_n^+ \dots S_i^+ S(i) = 0.$$

Conversely, $C^+ X = 0$ implies $X \cong S_1^- \dots S_{i-1}^- S(i)$ for some vertex i. Thus $X \cong P(i)$. The proof of (2) is analogous. □

3.5. **Preprojective and preinjective representations.** Let Q be a quiver without oriented cycles. We introduce three classes of representations.

Definition 3.5.1. Let X be an indecomposable representation of Q.

1. X is *preprojective* if $X \cong C^r P(i)$ for some vertex i and some $r \leq 0$.
2. X is *preinjective* if $X \cong C^r I(i)$ for some vertex i and some $r \geq 0$.
3. X is *regular* if $C^r X \neq 0$ for all $r \in \mathbb{Z}$.

Note that X is preprojective if and only $C^r X = 0$ for some $r > 0$, and X is preinjective if and only $C^r X = 0$ for some $r < 0$. This is an immediate consequence of Proposition 3.4.3.

Proposition 3.5.2. *An indecomposable representation is preprojective, preinjective or regular. Given indecomposable representations X, Y with X preprojective or preinjective, we have $X \cong Y$ if and only if* $\dim X = \dim Y$. *Moreover,*

1. $C^r P(i) \cong C^s P(j) \neq 0$ *implies* $i = j$ *and* $r = s$;
2. $C^r I(i) \cong C^s I(j) \neq 0$ *implies* $i = j$ *and* $r = s$.

Proof. The first assertion is an immediate consequence of Proposition 3.4.3. For the rest we use reflection functors, in particular Lemma 3.3.3. Suppose $\dim X = \dim Y$ and let X be preprojective, say $X \cong C^r P(i)$. We know $\dim P(i)$ from Lemma 3.4.2 and have therefore

$$\dim Y = (\sigma_n \dots \sigma_1)^r \sigma_1 \dots \sigma_{i-1}(e_i).$$

Using reflection functors we obtain $S_{i-1}^+ \dots S_1^+ C^{-r}(Y) \cong S(i)$ and this gives

$$Y \cong C^r S_1^- \dots S_{i-1}^- S(i) \cong C^r P(i) \cong X.$$

The proof for preinjective X is analogous.

(1) If $C^rP(i) \cong C^sP(j) \neq 0$ then $P(i) \cong C^{s-r}P(j)$ and therefore $s - r \leq 0$ by Propopsition 3.4.3. The same argument gives $r - s \leq 0$ whence $r = s$. We obtain that $P(i) \cong P(j)$ and this implies $i = j$ by Lemma 1.7.2. The proof of (2) is analogous. $\square$

4. Dynkin and Euclidean diagrams

A finite graph arises from a quiver when one forgets the orientation of its arrows. In this section we classify finite graphs using properties of quadratic forms. For graphs of Dynkin or Euclidean type we study the corresponding root systems.

4.1. Finite graphs.

Let Γ be a finite graph with set of vertices $\{1, \ldots, n\}$. The finite number of edges joining two vertices i and j is denoted by $d_{ij} = d_{ji}$. The graph Γ induces a *symmetric bilinear form*

$$(-,-)\colon \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z} \quad \text{with} \quad (e_i, e_j) = \begin{cases} -d_{ij} & \text{if } i \neq j, \\ 2 - 2d_{ii} & \text{if } i = j, \end{cases}$$

where e_i is the ith coordinate vector, and a *quadratic form*

$$q\colon \mathbb{Z}^n \longrightarrow \mathbb{Z} \quad \text{with} \quad q(x) = \sum_{i=1}^{n} x_i^2 - \sum_{i \leq j} d_{ij} x_i x_j.$$

Note that Γ, $(-,-)$ and q determine each other, since $q(x) = \frac{1}{2}(x,x)$ and $(x,y) = q(x+y) - q(x) - q(y)$. The *radical* of the form q is by definition the set

$$\operatorname{rad} q = \{x \in \mathbb{Z}^n \mid (x,-) = 0\}.$$

A vector $x \in \mathbb{Z}^n$ is *sincere* if $x_i \neq 0$ for all i.

Remark 4.1.1. Let Q be a quiver whose underlying graph is Γ. Then the symmetric bilinear form (3.2.1) for Q coincides with the one defined for Γ.

Definition 4.1.2. Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a quadratic form.

(1) q is *positive definite* if $q(x) > 0$ for all non-zero $x \in \mathbb{Z}^n$.
(2) q is *positive semi-definite* if $q(x) \geq 0$ for all $x \in \mathbb{Z}^n$.

Lemma 4.1.3. *Let Γ be connected and $y \in \mathbb{Z}^n$ be a positive radical vector. Then y is sincere and q is positive semi-definite. For $x \in \mathbb{Z}^n$ we have*

$$q(x) = 0 \iff x \in \mathbb{Q}y \iff x \in \operatorname{rad} q.$$

Proof. The assumption on y yields

$$0 = (e_i, y) = (2 - 2d_{ii})y_i - \sum_{j \neq i} d_{ij} y_j \quad \text{for} \quad 1 \leq i \leq n. \tag{4.1.1}$$

If $y_i = 0$ then $\sum_{j \neq i} d_{ij} y_j = 0$, and since each term is non-negative we have $y_j = 0$ whenever i and j are joined by an edge. It follows that $y = 0$ since Γ is connected.

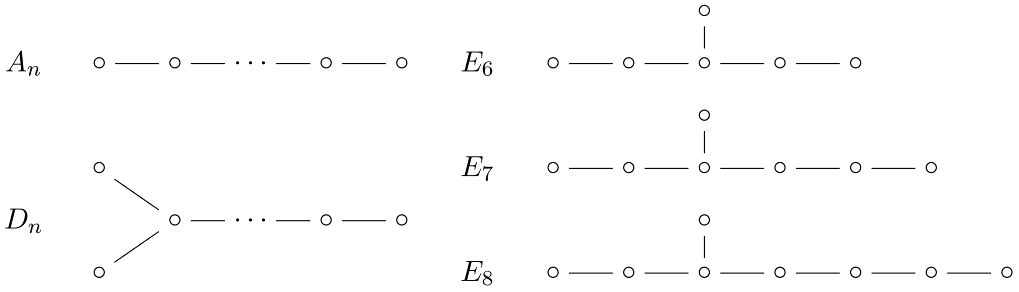
This is a contradiction and therefore y is sincere. The next calculation shows that q is positive semi-definite. For $x \in \mathbb{Z}^n$ we have

$$\begin{aligned} q(x) &= \sum_i (2-2d_{ii}) y_i \frac{1}{2y_i} x_i^2 - \sum_{i<j} d_{ij} x_i x_j \\ &= \sum_{i\neq j} d_{ij} \frac{y_j}{2y_i} x_i^2 - \sum_{i<j} d_{ij} x_i x_j \\ &= \sum_{i<j} d_{ij} \frac{y_j}{2y_i} x_i^2 - \sum_{i<j} d_{ij} x_i x_j + \sum_{i<j} d_{ij} \frac{y_i}{2y_j} x_j^2 \\ &= \sum_{i<j} d_{ij} \frac{y_i y_j}{2} \Big(\frac{x_i}{y_i} - \frac{x_j}{y_j} \Big)^2 \geq 0, \end{aligned}$$

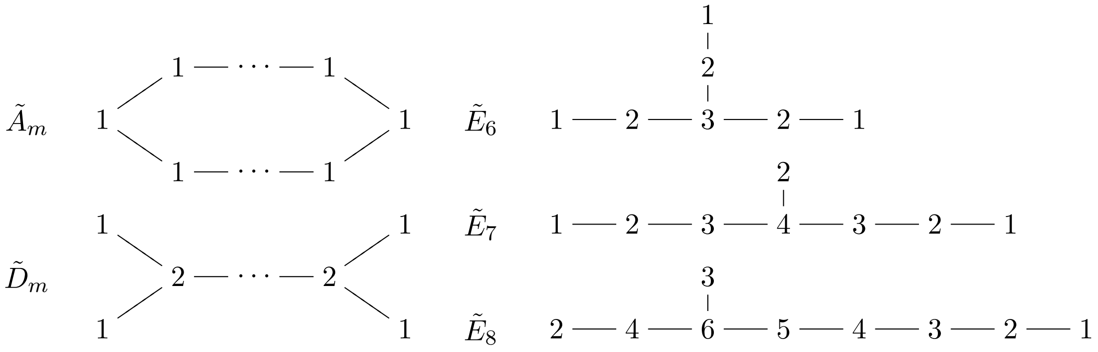
where the second equality follows from (4.1.1). If $q(x) = 0$ then $\frac{x_i}{y_i} = \frac{x_j}{y_j}$ whenever there is an edge joining i and j. Thus $x \in \mathbb{Q}y$ since Γ is connected. If $x \in \mathbb{Q}y$ then $x \in \operatorname{rad} q$ since $y \in \operatorname{rad} q$ by assumption. Finally, $x \in \operatorname{rad} q$ implies $q(x) = 0$. $\square$

4.2. **The classification.** We list the Dynkin and Euclidean diagrams.

Dynkin diagrams (with n vertices).



Euclidean diagrams (with $n = m + 1$ vertices). Each vertex i is marked with the value δ_i of a vector $\delta \in \mathbb{Z}^n$. Let $m \geq 0$ for $\tilde{A}_m$ and $m \geq 4$ for $\tilde{D}_m$.



Theorem 4.2.1. *Let Γ be a connected graph and q the corresponding quadratic form.*

1. *Γ is a Dynkin diagram if and only if q is positive definite.*
2. *Γ is a Euclidean diagram if and only if q is positive semi-definite but not positive definite. In that case there is a unique positive vector $\delta \in \mathbb{Z}^n$ with $\operatorname{rad} q = \mathbb{Z}\delta$.*

Proof. We proceed in three steps.

Step 1. *If* Γ *is Euclidean then* q *is positive semi-definite and* $\operatorname{rad} q = \mathbb{Z}\delta$. The assertion follows from Lemma 4.1.3 once we have shown that δ is a radical vector. This is done by inspection. If Γ has no loops or multiple edges we need to check that

$$0 = (e_i, \delta) = 2\delta_i - \sum_{\substack{j=1 \\ d_{ij} \neq 0}}^{n} \delta_j \quad \text{for} \quad 1 \leq i \leq n.$$

Finally, since some $\delta_i = 1$ we have

$$\operatorname{rad} q = \mathbb{Q}\delta \cap \mathbb{Z}^n = \mathbb{Z}\delta.$$

Step 2. *If* Γ *is Dynkin then* q *is positive definite.* This follows from the first part because there exists a Euclidean diagram $\tilde{\Gamma}$ such that Γ is obtained by deleting some vertex e. For $\tilde{\Gamma}$ we have $q(x) > 0$ for every non-zero vector x with $x_e = 0$.

Step 3. *If* Γ *is not Dynkin or Euclidean then* $q(x) < 0$ *for some* $x \in \mathbb{Z}^n$. It is not difficult to find a Euclidean subgraph Γ' with radical vector δ. Put $x = \delta$ if the vertices of Γ' and Γ coincide. Otherwise let i be a vertex of $\Gamma \setminus \Gamma'$, but connected with Γ' by an edge, and take $x = 2\delta + e_i$.

□

4.3. **Roots.** We define

$$\Delta = \{x \in \mathbb{Z}^n \mid q(x) \leq 1\}$$

and a non-zero element of Δ is called *root.*

Proposition 4.3.1. *Let* Γ *be Dynkin or Euclidean.*

1) *Each* e_i *is a root.*
2) *If* $x \in \Delta$ *and* $y \in \operatorname{rad} q$, *then* $-x, x+y \in \Delta$.
3) *Every root is positive or negative.*
4) *If* Γ *is Euclidean then* $\Delta/\operatorname{rad} q$ *is finite.*
5) *If* Γ *is Dynkin then* Δ *is finite.*

Proof. (1) Clear.

(2) We have $q(y \pm x) = q(y) + q(x) \pm (y, x) = q(x)$.

(3) Let x be a root and write $x = x^+ - x^-$ where $x^+, x^- \geq 0$ and both have disjoint support. Then $(x^+, x^-) \leq 0$ and

$$1 \geq q(x) = q(x^+) + q(x^-) - (x^+, x^-) \geq q(x^+) + q(x^-) \geq 0.$$

This implies $q(x^+) = 0$ or $q(x^-) = 0$. Thus one of x^+ and x^- is sincere if we assume that both vectors are non-zero. This is a contradiction and therefore x is positive or negative.

(4) Fix a vertex e. If x is a root with $x_e = 0$, then $\delta - x$ and $\delta + x$ are positive at e. Both vectors are roots by (2) and therefore positive by (3). Thus

$$\{x \in \Delta \mid x_e = 0\} \subseteq \{x \in \mathbb{Z}^n \mid -\delta \leq x \leq \delta\}$$

which is a finite set. If $x \in \Delta$ then $x - x_e\delta$ belongs to the finite set $\{x \in \Delta \mid x_e = 0\}$.

(5) There exists a Euclidean diagram $\tilde{\Gamma}$ such that Γ is obtained by deleting some vertex e. A root x of Γ can be viewed as a root for $\tilde{\Gamma}$ with $x_e = 0$. Thus the result follows from (4). □

Lemma 4.3.2. *Let Q be a quiver whose underlying graph is Dynkin or Euclidean. If x is a positive root and $\sigma_i(x)$ is not positive, then $x = e_i$.*

Proof. The root $\sigma_i(x)$ is non-positive by assumption and therefore negative by Proposition 4.3.1. For each vertex $j \neq i$ we have $\sigma_i(x)_j = x_j$ and therefore $x_j = 0$. Thus $x = e_i$. ☐

4.4. **The Coxeter transformation.** Let Q be a quiver without oriented cycles and fix an admissable ordering $i_1, \dots, i_n$ of its vertices. The automorphism

$$c \colon \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \quad \text{with} \quad c(x) = \sigma_{i_n} \dots \sigma_{i_1}(x)$$

is called *Coxeter transformation.* The next lemma shows that c does not depend on the numbering of the vertices.

Lemma 4.4.1. (1) *$c(\dim P(i)) = -\dim I(i)$ for every vertex i.*
(2) *$\{\dim P(i) \mid i \in Q_0\}$ and $\{\dim I(i) \mid i \in Q_0\}$ form two bases of $\mathbb{Z}^n$.*

Proof. (1) We simplify the labeling of the vertices and let $i_j = j$ for $1 \leq j \leq n$. Then $\dim P(i) = \sigma_1 \dots \sigma_{i-1}(e_i)$ by Lemma 3.4.2 and we get

$$c(\dim P(i)) = c\sigma_1 \dots \sigma_{i-1}(e_i) = \sigma_n \dots \sigma_i(e_i) = -\sigma_n \dots \sigma_{i+1}(e_i) = -\dim I(i).$$

(2) We have

$$e_i = \dim P(i) - \sum_{\substack{\alpha \in Q_1 \\ s(\alpha) = i}} \dim P(t(\alpha)) = \dim I(i) - \sum_{\substack{\alpha \in Q_1 \\ t(\alpha) = i}} \dim I(s(\alpha)).$$ ☐

Lemma 4.4.2. *Let $x, y \in \mathbb{Z}^n$.*
(1) *$\langle \dim P(i), x \rangle = x_i = \langle x, \dim I(i) \rangle$ for each vertex i.*
(2) *$\langle x, y \rangle = -\langle y, c(x) \rangle = \langle c(x), c(y) \rangle$.*

Proof. (1) It is sufficient to check this for each standard basis vector $x = e_j$.

(2) It is sufficient to check this for $x = \dim P(j)$ where j runs through all vertices; see Lemma 4.4.1. Using (1) and Lemma 4.4.1 we get

$$\langle \dim P(j), y \rangle = \langle y, \dim I(j) \rangle = \langle y, -c(\dim P(j)) \rangle.$$ ☐

Lemma 4.4.3. *Let $x \in \mathbb{Z}^n$. Then $c(x) = x$ if and only if $x \in \operatorname{rad} q$.*

Proof. We have $c(x) = x$ iff $x_i = c(x)_i = \sigma_i(x)_i$ for all i iff $(x, e_i) = 0$ for all i iff $(x, -) = 0$. ☐

From now on we assume that the underlying graph of Q is Dynkin or Euclidean. The map c induces a permutation of the finite set $\Delta/\operatorname{rad} q$. Thus c^h is the identity on $\Delta/\operatorname{rad} q$ for some $h > 0$. In fact, c^h is the identity on $\mathbb{Z}^n/\operatorname{rad} q$ since $e_i \in \Delta$ for all i.

Lemma 4.4.4. *Let Q be of Dynkin type and $x \in \mathbb{Z}^n$. Then there exists $r \geq 0$ such that $c^r(x)$ is not positive.*

Proof. The vector $y = \sum_{r=0}^{h-1} c^r(x)$ is fixed by c, and hence $y = 0$ by Lemma 4.4.3. Thus $c^r(x)$ is not positive for some $r \geq 0$. ☐

Lemma 4.4.5. *Let Q be of Euclidean type and $x \in \mathbb{Z}^n$.*
(1) *If $c^r(x) > 0$ for all $r \in \mathbb{Z}$ then $c^h(x) = x$.*
(2) *If $c^h(x) = x$ then $\langle \delta, x \rangle = 0$.*

Proof. (1) Suppose $c^h(x) = x + v$ for some non-zero $v \in \operatorname{rad} q$. An induction shows that $c^{lh}(x) = x + lv$ for all $l \in \mathbb{Z}$. Thus one finds r such that $c^r(x)$ is not positive since the vector v is sincere and positive or negative.

(2) The vector $y = \sum_{r=0}^{h-1} c^r(x)$ is fixed by c, and hence $y \in \mathbb{Z}\delta$ by Lemma 4.4.3. Now

$$0 = \langle \delta, y \rangle = \sum_{r=0}^{h-1} \langle \delta, c^r(x) \rangle = h\langle \delta, x \rangle$$

since c preserves the Euler form by Lemma 4.4.2. Thus $\langle \delta, x \rangle = 0$. □

5. Finite representation type

In this section we prove Gabriel's theorem. For quivers of Dynkin or Euclidean type we classify indecomposable representations in terms of their dimension vectors.

5.1. **The Dynkin case.**

Theorem 5.1.1 (Gabriel)**.** *Let Q be a quiver whose underlying graph is a Dynkin diagram. Then the assignment $X \mapsto \dim X$ induces a bijection between the isomorphism classes of indecomposable representations of Q and the positive roots corresponding to the diagram of Q. In particular, there are only finitely many isomorphism classes of indecomposable representations.*

Proof. Choose an admissable ordering $i_1, \dots, i_n$ for the vertices of Q. Suppose X is an indecomposable representation of Q with dimension vector $x = \dim X$. Let

$$\tau = \sigma_{i_s} \dots \sigma_{i_1} (\sigma_{i_n} \dots \sigma_{i_1})^r$$

be the shortest expression such that $\tau(x)$ is not positive, which exists by Lemma 4.4.4. Now we apply the reflection functors and use Lemma 3.3.3 to obtain

$$S^+_{i_{s-1}} \dots S^+_{i_1} (S^+_{i_n} \dots S^+_{i_1})^r X \cong S(i_s)$$

and therefore

$$X \cong (S^-_{i_1} \dots S^-_{i_n})^r S^-_{i_1} \dots S^-_{i_{s-1}} S(i_s).$$

Thus

$$\dim X = (\sigma_{i_1} \dots \sigma_{i_n})^r \sigma_{i_1} \dots \sigma_{i_{s-1}} (e_{i_s})$$

is a positive root. The same argument shows for another indecomposable representation X' with $\dim X' = \dim X$ that $X' \cong X$.

Now suppose x is a positive root. Let

$$\tau = \sigma_{i_s} \dots \sigma_{i_1} (\sigma_{i_n} \dots \sigma_{i_1})^r$$

be the shortest expression such that $\tau(x)$ is not positive, which exists by Lemma 4.4.4. We infer from Lemma 4.3.2 that

$$\sigma_{i_{s-1}} \dots \sigma_{i_1} (\sigma_{i_n} \dots \sigma_{i_1})^r (x) = e_{i_s}.$$

Let

$$X = (S^-_{i_1} \dots S^-_{i_n})^r S^-_{i_1} \dots S^-_{i_{s-1}} S(i_s).$$

Another iterated application of reflection functors shows that X is indecomposable with

$$\dim X = (\sigma_{i_1} \dots \sigma_{i_n})^r \sigma_{i_1} \dots \sigma_{i_{s-1}} (e_{i_s}) = x.$$

There are only finitely many isomorphism classes of indecomposable representations because the number of roots is finite by Proposion 4.3.1. □

Remark 5.1.2. The proof of Theorem 5.1.1 shows that for a quiver of Dynkin type all indecomposable representations are preprojective and preinjective. In fact, the proof simplifies a bit if one uses Coxeter functors.

Remark 5.1.3. Given a graph Γ without loops, the corresponding *Weyl group* $W(\Gamma)$ is the group of automorphisms of $\mathbb{Z}^n$ which is generated by the reflections σ_i. If Γ is a Dynkin diagram then the roots are precisely the vectors of the form $\tau(e_i)$ with $\tau \in W(\Gamma)$ and $1 \leq i \leq n$. Clearly, each $\tau(e_i)$ is a root since the σ_i preserve the quadratic form q. Conversely, let $q(x) = 1$. Then the argument used in the proof of Theorem 5.1.1 shows that x is of the form $\tau(e_i)$.

5.2. **The defect.** Let Q be a quiver of Euclidean type. The *defect* of a vector $x \in \mathbb{Z}^n$ is

$$\partial x = \langle \delta, x \rangle = -\langle x, \delta \rangle.$$

The *defect* of a representation X is $\partial X = \partial \dim X$.

Proposition 5.2.1. *Let X be an indecomposable representation.*

(1) *X is preprojective if and only if $\partial X < 0$.*
(2) *X is preinjective if and only if $\partial X > 0$.*
(3) *X is regular if and only if $\partial X = 0$.*

Proof. First observe that for every representation X with $C^r X \neq 0$ we have

$$\dim C^r X = c^r(\dim X)$$

by Lemma 3.3.3. Now let $X = C^r P(i)$ be preprojective. Then

$$\partial X = -\langle c^r(\dim P(i)), \delta \rangle = -\langle \dim P(i), \delta \rangle = -\delta_i < 0$$

by Lemmas 4.4.2 and 4.4.3. Similarly preinjectives have positive defect. If X is regular then $\partial X = 0$ by Lemma 4.4.5. □

5.3. **The Euclidean case.**

Theorem 5.3.1. *Let Q be a quiver without oriented cycles and suppose the underlying graph is a Euclidean diagram with n vertices. Then the assignment $X \mapsto \dim X$ induces a bijection between the isomorphism classes of indecomposable preprojective or preinjective representations of Q and the positive roots with non-zero defect corresponding to the diagram of Q. The preprojective and preinjective indecomposables form $2n$ countably infinite series $C^{-r}P(i)$ and $C^r I(i)$ ($r \in \mathbb{N}_0, i \in Q_0$) of pairwise non-isomorphic representations.*

Proof. Choose an admissable ordering $i_1, \dots, i_n$ for the vertices of Q. Suppose X is an indecomposable representation of Q with dimension vector $x = \dim X$. Let $X \cong C^r P(i_s)$ be preprojective. We have $\dim P(i_s) = \sigma_{i_1} \dots \sigma_{i_{s-1}}(e_{i_s})$ by Lemma 3.4.2 and therefore $x = c^r \sigma_{i_1} \dots \sigma_{i_{s-1}}(e_{i_s})$ by Lemma 3.3.3. This is a positive root, and $\partial x < 0$ by Proposition 5.2.1. A similar argument works if X is preinjective. The map $X \mapsto \dim X$ is injective by Proposition 3.5.2.

Now suppose x is a positive root with $\partial x \neq 0$. We know from Lemma 4.4.5 that $c^t(x)$ is not positive for some $t \in \mathbb{Z}$. Suppose first $t > 0$. Then there are $1 \leq s \leq n$ and $r \geq 0$ such that

$$\tau = \sigma_{i_s} \dots \sigma_{i_1} (\sigma_{i_n} \dots \sigma_{i_1})^r$$

is the shortest expression with $\tau(x)$ not positive. We infer from Lemma 4.3.2 that

$$\sigma_{i_{s-1}} \dots \sigma_{i_1} (\sigma_{i_n} \dots \sigma_{i_1})^r (x) = e_{i_s}.$$

Let

$$X = (S^-_{i_1} \dots S^-_{i_n})^r S^-_{i_1} \dots S^-_{i_{s-1}} S(i_s).$$

An iterated application of reflection functors and Lemma 3.3.3 shows that X is indecomposable with

$$\dim X = (\sigma_{i_1} \dots \sigma_{i_n})^r \sigma_{i_1} \dots \sigma_{i_{s-1}} (e_{i_s}) = x.$$

Moreover, X is preprojective since $X \cong C^{-r} P(i_s)$ by Lemma 3.4.2. In case $t < 0$ a similar argument gives a preinjective representation X with $\dim X = x$.

Finally we show that the preprojectives and preinjectives form $2n$ countably infinite series of pairwise non-isomorphic representations. This follows essentially from Proposition 3.5.2. It remains to show that $C^{-r}P(i) \neq 0$ and $C^r I(i) \neq 0$ for all $r \geq 0$. But this follows from Poposition 3.4.3 because $C^{-r}P(i) = 0$ would imply that $P(i)$ is preinjective, contradicting the fact that preprojectives and preinjectives have different defect by Proposition 5.2.1. A similar argument works for $C^r I(i)$. ☐

Proposition 5.3.2. *Let Q be a quiver of Euclidean type $\tilde{A}_n$ with $n \geq 0$. Then there are infinitely many isomorphism classes of indecomposable representations.*

Proof. We allow any orientation, in particular an oriented cycle, and fix an arrow α_0. Now we define for each $p \geq 1$ a representation $X = X(p)$ as follows. Let $X_i = k^p$ for each vertex i, let $X_{\alpha_0} = J(p, 0)$ be the Jordan block of size p with eigenvalue 0 and let $X_\alpha = \mathrm{id}_{k^p}$ for every arrow $\alpha \neq \alpha_0$. Then $\mathrm{End}(X(p)) \cong k[t]/(t^p)$ and therefore $X(p)$ is indecomposable. ☐

Remark 5.3.3. Let Q be a quiver of Euclidean type. There is an explict description of all regular indecomposable representation. For example, if Q is the Kronecker quiver and the field k is algebraically closed, then the regular indecomposable representations $R(n, \lambda)$ are indexed by integers $n \geq 1$ and points $\lambda = (\lambda_0 : \lambda_1)$ of the projective line $\mathbb{P}^1(k)$. For instance, we have the representation

$$R(n,\lambda): \begin{tikzcd} k^n \arrow[r, shift left, "J(n{,}\lambda_0)"] \arrow[r, shift right, "\mathrm{id}"'] & k^n \end{tikzcd} \quad \text{for} \quad \lambda = (\lambda_0 : 1),$$

where $J(n, \lambda_0)$ denotes the Jordan block of size n with eigenvalue λ_0.

Corollary 5.3.4 (Gabriel)**.** *Let Q be a connected quiver. Then there are only finitely many isomorphism classes of indecomposable representations if and only if the underlying graph is a Dynkin diagram.*

Proof. If Q is of Dynkin type then the classification of the indecomposable representations in Theorem 5.1.1 shows that there are only finitely many. If Q is not of Dynkin type then Q has a Euclidean subquiver Q' which has infinitely many indecomposable representations by Theorem 5.3.1 and Proposition 5.3.2. Each representation X of Q' can be extended to a representation of Q by letting $X_i = 0$ and $X_\alpha = 0$ for all $i \in Q_0$ and $\alpha \in Q_1$ not in Q'. Thus Q has infinitely many pairwise non-isomorphic indecomposable representations. ☐

6. Irreducible morphisms

In this section we investigate the morphisms between two representations using the concept of an irreducible morphism. The irreducible morphisms play the role of generators and we obtain a combinatorial description of all morphisms between preprojective representations.

Throughout this section Q denotes a quiver without oriented cycles.

6.1. **The radical.** Let X, Y be a pair of representations. Recall from Section 2.3 that the radical $\operatorname{Rad}(X, Y)$ is a subspace of $\operatorname{Hom}(X, Y)$. We extend this definition recursively for each $n \geq 0$ as follows. Let $\operatorname{Rad}^0(X, Y) = \operatorname{Hom}(X, Y)$ and for $n > 0$ let $\operatorname{Rad}^n(X, Y)$ be the set of morphisms $\phi \in \operatorname{Hom}(X, Y)$ which admit a factorisation $\phi = \phi''\phi'$ with $\phi' \in \operatorname{Rad}(X, Z)$ and $\phi'' \in \operatorname{Rad}^{n-1}(Z, Y)$ for some representation Z. Note that $\operatorname{Rad}^1(X, Y) = \operatorname{Rad}(X, Y)$.

Lemma 6.1.1. *Let X, Y, Z be representations and $m, n \geq 0$.*

1. *$\operatorname{Rad}^{n+1}(X, Y)$ is a subspace of $\operatorname{Rad}^n(X, Y)$.*
2. *For each finite set of representations X_i and Y_j we have*
$$\operatorname{Rad}^n(\bigoplus_i X_i, \bigoplus_j Y_j) = \bigoplus_{i,j} \operatorname{Rad}^n(X_i, Y_j).$$
3. *If $\phi \in \operatorname{Rad}^n(X, Y)$ and $\psi \in \operatorname{Rad}^m(Y, Z)$, then $\psi\phi \in \operatorname{Rad}^{n+m}(X, Z)$.*

Proof. Use Lemma 2.3.1. □

6.2. **Irreducible morphisms.** Fix a morphism $\phi\colon X \to Y$ between two representations.

The morphism ϕ is called *split monomorphism* if there exists $\phi'\colon Y \to X$ with $\phi'\phi = \mathrm{id}_X$, and ϕ is called *split epimorphism* if there exists $\phi''\colon Y \to X$ with $\phi\phi'' = \mathrm{id}_Y$.

The morphism ϕ is called *irreducible* if ϕ is neither a split monomorphism nor a split epimorphism and if for any factorisation $\phi = \phi''\phi'$ the morphism ϕ' is a split monomorphism or the morphism ϕ'' is a split epimorphism.

Lemma 6.2.1. *An irreducible morphism is a monomorphism or an epimorphism.*

Proof. Let $\phi\colon X \to Y$ be irreducible. Consider the canonical factorisation $X \to \operatorname{Im}\phi \to Y$. If $X \to \operatorname{Im}\phi$ is a split monomorphism then ϕ is a monomorphism. If $\operatorname{Im}\phi \to Y$ is a split epimorphism then ϕ is an epimorphism. □

Lemma 6.2.2. *Let $\phi\colon X \to Y$ be a morphism between two representations.*

1. *If X is indecomposable, then $\phi \in \operatorname{Rad}(X, Y)$ if and only if ϕ is not a split mono.*
2. *If Y is indecomposable, then $\phi \in \operatorname{Rad}(X, Y)$ if and only if ϕ is not a split epi.*
3. *If X and Y are indecomposable, then $\phi \in \operatorname{Rad}^1(X, Y) \setminus \operatorname{Rad}^2(X, Y)$ if and only if ϕ is irreducible.*

Proof. (1) We apply Lemma 2.3.1. Let $Y = \bigoplus_{i=1}^r Y_i$ be a decomposition into indecomposable represenations. Then $\operatorname{Rad}(X, Y) = \bigoplus_{i=1}^r \operatorname{Rad}(X, Y_i)$ and we have $\phi \in \operatorname{Rad}(X, Y)$ if and only if each component ϕ_i belongs to $\operatorname{Rad}(X, Y_i)$. Thus $\phi \notin \operatorname{Rad}(X, Y)$ if and only if ϕ_i is an isomorphism for some i if and only if ϕ is a split monomorphism.

(2) This is the dual statement of (1).

(3) Combine (1) and (2). □

We define for indecomposable representations X, Y

$$\operatorname{Irr}(X,Y) = \operatorname{Rad}^1(X,Y)/\operatorname{Rad}^2(X,Y).$$

Remark 6.2.3. Let $\phi\colon X \to Y$ be an irreducible morphism between indecomposable representation and let $Y' = Y \oplus Y$. Then the morphism $(\phi,\phi)\colon X \to Y'$ is not irreducible but belongs to $\operatorname{Rad}^1(X,Y') \setminus \operatorname{Rad}^2(X,Y')$.

Proposition 6.2.4. *Let X, Y be indecomposable representations and $\operatorname{Rad}^n(X,Y) = 0$ for some n. Then every non-isomorphism $X \to Y$ is a sum of compositions of irreducible morphisms between indecomposable representations.*

Proof. Suppose ϕ is not irreducible. Because ϕ is not an isomorphism, there exists a factorisation $\phi = \phi''\phi'$ with $\phi' \in \operatorname{Rad}(X,Z)$ and $\phi'' \in \operatorname{Rad}(Z,Y)$ for some representation Z by Lemma 6.2.2. Let $Z = \bigoplus_{i=1}^r Z_i$ be a decomposition into indecomposable representations. Then $\phi = \sum_{i=1}^r \phi_i''\phi_i'$ with $\phi_i' \in \operatorname{Rad}(X,Z_i)$ and $\phi_i'' \in \operatorname{Rad}(Z_i,Y)$ for all i. Next we factorise each non-irreducible ϕ_i' and each non-irreducible ϕ_i'' into a composition of two radical morphisms. We continue and obtain in each step a finite sum $\phi = \sum_i \phi_{in_i}\dots\phi_{i1}$ of compositions of radical morphisms between indecomposable representations. This process stops and all ϕ_{ij} are irreducible because a composition of n radical morphisms is zero by our assumption on X and Y. □

6.3. **The Harada-Sai lemma.** The *length* $\ell(X)$ of a representation X is the maximal number n such that there exists a chain of subrepresentations

$$0 = X_n \subsetneq \dots \subsetneq X_1 \subsetneq X_0 = X.$$

Note that we have $\ell(X) = \sum_{i \in Q_0} \dim X_i$ and $\ell(X) = \ell(U) + \ell(X/U)$ for every subrepresentation $U \subseteq X$.

Lemma 6.3.1 (Harada-Sai)**.** *Let $\phi_i\colon X_i \to X_{i+1}$ with $1 \le i \le 2^n - 1$ be a family of non-isomorphisms between indecomposable representations satisfying $\ell(X_i) \le n$ for all i. Then $\phi_{2^n-1}\dots\phi_1 = 0$.*

Proof. We show by induction that the length of the image of $\phi_{2^m-1}\dots\phi_1$ is at most $n-m$. This is clear for $m=1$ since ϕ_1 is not an isomorphism. Let $\phi' = \phi_{2^{m-1}-1}\dots\phi_1$, $\phi = \phi_{2^{m-1}}$, and $\phi'' = \phi_{2^m-1}\dots\phi_{2^{m-1}+1}$. By the inductive hypothesis, the length of $\operatorname{Im}\phi'$ and $\operatorname{Im}\phi''$ is at most $n-m+1$. The assertion follows if either is strictly less. So suppose the images of ϕ',ϕ'' and $\phi''\phi\phi'$ each has length $n-m+1$. Then $\operatorname{Ker}(\phi''\phi) \cap \operatorname{Im}\phi' = 0$ and $\operatorname{Ker}\phi'' \cap \operatorname{Im}(\phi\phi') = 0$. On the other hand, we have

$$\ell(X_{2^{m-1}}) = \ell(\operatorname{Ker}(\phi''\phi)) + \ell(\operatorname{Im}(\phi''\phi)) = \ell(\operatorname{Ker}(\phi''\phi)) + \ell(\operatorname{Im}\phi'),$$
$$\ell(X_{2^{m-1}+1}) = \ell(\operatorname{Ker}\phi'') + \ell(\operatorname{Im}\phi'') = \ell(\operatorname{Ker}\phi'') + \ell(\operatorname{Im}(\phi\phi')).$$

Therefore $X_{2^{m-1}} = \operatorname{Ker}(\phi''\phi) \oplus \operatorname{Im}\phi'$ and $X_{2^{m-1}+1} = \operatorname{Ker}\phi'' \oplus \operatorname{Im}(\phi\phi')$. Since each is indecomposable, $\phi''\phi$ is a monomorphism and $\phi\phi'$ is an epimorphism. Thus ϕ is an isomorphism, contrary to hypothesis. □

Proposition 6.3.2. *Suppose the length of every indecomposable representation is bounded by n. Then $\operatorname{Rad}^{2^n-1}(X,Y) = 0$ for every pair X, Y of representations.*

Proof. We may assume that X and Y are indecomposable, by Lemma 6.1.1. An induction shows that any morphism in $\operatorname{Rad}^r(X,Y)$ can be written as a finite sum $\sum_i \phi_{ir}\dots\phi_{i1}$ of compositions of radical morphisms between indecomposable representations. Thus $\operatorname{Rad}^{2^n-1}(X,Y) = 0$ by Lemma 6.3.1. □

6.4. **Irreducible morphisms between indecomposable projectives.** Given a pair i, j of vertices, each arrow $\alpha\colon i \to j$ induces a morphism

$$\alpha^*\colon P(j) \longrightarrow P(i) \quad \text{with} \quad \alpha^*_l = k[Q(\alpha, l)] \quad (l \in Q_0).$$

Note that α^* is a monomorphism since each map $Q(\alpha, l)$ is injective. Taking all arrows starting at i (resp. ending at j) we obtain two morphisms

$$\sigma(i)\colon \bigoplus_{\alpha\colon i \to i'} P(i') \xrightarrow{(\alpha^*)} P(i) \quad \text{and} \quad \tau(j)\colon P(j) \xrightarrow{(\alpha^*)} \bigoplus_{\alpha\colon j' \to j} P(j').$$

Lemma 6.4.1. *The morphism $\sigma(i)$ is a monomorphism and its image is the unique maximal subrepresentation of $P(i)$.*

Proof. This is a direct consequence of the definition of $P(i)$. Note that a subrepresentation $X \subseteq P(i)$ equals $P(i)$ if and only if the trivial path ε_i belongs to X_i. ☐

Lemma 6.4.2. *For a morphism $\phi\colon X \to P(i)$ the following are equivalent:*

(1) $\phi \in \operatorname{Rad}(X, P(i))$.
(2) *ϕ is not an epimorphism.*
(3) *ϕ admits a factorisation $\phi = \sigma(i)\phi'$.*

Proof. (1) $\Leftrightarrow$ (2): We have $\phi \in \operatorname{Rad}(X, P(i))$ if and only if ϕ is not a split epimorphism, by Lemma 6.2.2. Thus we need to show that every epimorphism is a split epimorphism. So let ϕ be an epimorphism. Then the map ϕ_i is surjective and we find $x \in X_i$ with $\phi_i(x) = \varepsilon_i$. It follows from Lemma 1.7.1 that there is a morphism $\phi'\colon P(i) \to X$ corresponding to x. Then we have $\phi\phi' = \mathrm{id}_{P(i)}$ and therefore ϕ is a split epimorphism.

(2) $\Leftrightarrow$ (3): Apply Lemma 6.4.1. ☐

Lemma 6.4.3. *Let i be a vertex and X an indecomposable representation.*

(1) *If $X \to P(i)$ is an irreducible morphism then there is an arrow $i \to j$ such that $X \cong P(j)$.*
(2) *Suppose $P(i)$ is simple. If $P(i) \to X$ is an irreducible morphism then there is an arrow $j \to i$ such that $X \cong P(j)$.*

Proof. (1) Let $\phi\colon X \to P(i)$ be irreducible. Then $\phi \in \operatorname{Rad}(X, P(i))$ by Lemma 6.2.2 and we obtain a factorisation $\phi = \sigma(i)\phi'$ by Lemma 6.4.2. The morphism $\sigma(i)$ is not a split epimorphism and therefore ϕ' is a split monomorphism. Thus $X \cong P(j)$ for some arrow $i \to j$.

(2) Let $\phi\colon P(i) \to X$ be irreducible. We claim that ϕ factors through $\tau(i)$. First observe that $X \ncong P(i) = S(i)$. This implies

$$\xi\colon \bigoplus_{\alpha\colon i' \to i} X_{i'} \xrightarrow{(X_\alpha)} X_i$$

is an epimorphism, by Lemma 3.3.3. We obtain the following commutative diagram

$$\begin{tikzcd}
\bigoplus_{\alpha\colon i' \to i} \operatorname{Hom}(P(i'), X) \arrow[r, equal] \arrow[d, "\wr"] & \operatorname{Hom}(\bigoplus_{\alpha\colon i' \to i} P(i'), X) \arrow[r, "\operatorname{Hom}(\tau(i), X)"] & \operatorname{Hom}(P(i), X) \arrow[d, "\wr"] \\
\bigoplus_{\alpha\colon i' \to i} X_{i'} \arrow[rr, "\xi"] & & X_i
\end{tikzcd}$$

where the vertical maps are taken from Lemma 1.7.1. Thus $\operatorname{Hom}(\tau(i), X)$ is surjective and therefore ϕ factors through $\tau(i)$. Let $\phi = \phi'\tau(i)$ be a factorisation. The morphism $\tau(i)$ is not a split monomorphism and therefore ϕ' is a split epimorphism. Thus $X \cong P(j)$ for some arrow $j \to i$. ☐

We denote by $Q_1(i, j)$ the set of arrows $i \to j$ in Q.

Lemma 6.4.4. *The map sending an arrow $\alpha\colon i \to j$ to α^* induces an isomorphism*

$$f\colon k[Q_1(i,j)] \xrightarrow{\sim} \operatorname{Irr}(P(j), P(i)).$$

Proof. The map f is well-defined because for each arrow $\alpha\colon i \to j$ the morphism α^* belongs to $\operatorname{Rad}(P(j), P(i))$ and induces therefore an element of $\operatorname{Irr}(P(j), P(i))$.

To show that f is an epimorphism choose a radical morphism $\phi\colon P(j) \to P(i)$. Then ϕ admits a factorisation $\phi = \sigma(i)\phi'$ by Lemma 6.4.2. Let $\phi' = (\phi'_\alpha)$ with $\phi'_\alpha \in \operatorname{Hom}(P(j), P(t(\alpha)))$. If $t(\alpha) = j$ then we have $\phi'_\alpha = \lambda_\alpha \operatorname{id}_{P(j)}$ for some $\lambda_\alpha \in k$ because $\operatorname{End}(P(j)) \cong k$ by Lemma 1.7.3. If $t(\alpha) \neq j$ then $\phi'_\alpha \in \operatorname{Rad}(P(j), P(t(\alpha)))$. It follows that $f(\sum_{\alpha \in Q_1(i,j)} \lambda_\alpha \alpha)$ and ϕ represent the same element in $\operatorname{Irr}(P(j), P(i))$ since $\phi - \sum_{\alpha \in Q_1(i,j)} \lambda_\alpha \alpha^* \in \operatorname{Rad}^2(P(j), P(i))$.

To show that f is a monomorphism choose $x = \sum_{\alpha \in Q_1(i,j)} \lambda_\alpha \alpha$ with $f(x) = 0$. Thus $\theta = \sum_{\alpha \in Q_1(i,j)} \lambda_\alpha \alpha^*$ belongs to $\operatorname{Rad}^2(P(j), P(i))$ and there is a factorization $\theta = \phi\psi$ such that ϕ and ψ are radical morphisms. We obtain a second factorisation $\phi = \sigma(i)\phi'$ by Lemma 6.4.2 and consider $\phi'\psi\colon P(j) \to \bigoplus_{\alpha\colon i \to i'} P(i')$. Then $(\phi'\psi)_\alpha = \lambda_\alpha \operatorname{id}_{P(j)}$ if $t(\alpha) = j$, and $(\phi'\psi)_\alpha = 0$ if $t(\alpha) \neq j$, since

$$\sum_{\alpha \in Q_1(i,j)} \lambda_\alpha \alpha^* = \sigma(i)\phi'\psi = \sum_{\alpha\colon i \to i'} \alpha^*(\phi'\psi)_\alpha$$

and $\sigma(i)$ is a monomorphism. This implies $\lambda_\alpha = 0$ for all $\alpha\colon i \to j$ since $(\phi'\psi)_\alpha$ is a radical morphism. Thus f is a monomorphism. ☐

Lemma 6.4.4 shows that for each arrow α the morphism α^* is irreducible.

6.5. **More irreducible morphisms.** We fix an arrow $\alpha\colon i \to j$ in Q and construct another irreducible morphism

$$\alpha_*\colon P(i) \longrightarrow C^- P(j)$$

as follows. We may assume that $1, \dots, n$ is an admissable numbering of the vertices of Q. Let $\tilde{\alpha}\colon j \to i$ be the arrow in $\tilde{Q} = \sigma_{i-1} \dots \sigma_1 Q$ corresponding to α. This induces the morphism $\tilde{\alpha}^*\colon \tilde{P}(i) \to \tilde{P}(j)$ between representations of $\tilde{Q}$ and we define $\alpha_* = S_1^- \dots S_{i-1}^- \tilde{\alpha}^*$. Note that we can identify $P(i) = S_1^- \dots S_{i-1}^- \tilde{P}(i)$ and $C^- P(j) = S_1^- \dots S_{i-1}^- \tilde{P}(j)$ by Lemma 3.4.2, since $\tilde{P}(i) = S(i)$ and $\tilde{P}(j) = S_i^- \dots S_{j-1}^- S(j)$.

6.6. **Reflection functors and morphisms.**

Lemma 6.6.1. *Let i be a sink and X, Y indecomposable representations not isomorphic to $S(i)$. Then S_i^+ induces isomorphisms*

$$\operatorname{Rad}^n(X, Y) \xrightarrow{\sim} \operatorname{Rad}^n(S_i^+ X, S_i^+ Y) \quad \textit{for} \quad n \geq 0.$$

In particular, S_i^+ induces an isomorphism $\operatorname{Irr}(X, Y) \xrightarrow{\sim} \operatorname{Irr}(S_i^+ X, S_i^+ Y)$.

Proof. We use the natural morphism $\iota_i Z\colon S_i^- S_i^+ Z \to Z$ (3.3.1) which is defined for any representation Z; it is a split monomorphism by Lemma 3.3.2. Thus we can identify $S_i^- S_i^+ X = X$ and $S_i^- S_i^+ Y = Y$. Using this identification the inverse for $\operatorname{Hom}(X,Y) \to \operatorname{Hom}(S_i^+ X, S_i^+ Y)$ sends $\psi \in \operatorname{Hom}(S_i^+ X, S_i^+ Y)$ to $S_i^- \psi$. Now fix $\phi \in \operatorname{Hom}(X,Y)$. Clearly, ϕ is an isomorphism if and only if $S_i^+\phi$ is an isomorphism. Thus S_i^+ induces a bijection

$$\operatorname{Rad}^1(X,Y) \xrightarrow{\sim} \operatorname{Rad}^1(S_i^+ X, S_i^+ Y).$$

Next we suppose $\phi \in \operatorname{Rad}^n(X,Y)$ and $n > 1$. Then ϕ admits a factorisation $\phi = \phi''\phi'$ with $\phi' \in \operatorname{Rad}^1(X,Z)$ and $\phi'' \in \operatorname{Rad}^{n-1}(Z,Y)$ for some representation Z. We know by induction that $S_i^+\phi' \in \operatorname{Rad}^1(S_i^+ X, S_i^+ Z)$ and $\phi'' \in \operatorname{Rad}^{n-1}(S_i^+ Z, S_i^+ Y)$. Thus $S_i^+\phi \in \operatorname{Rad}^n(S_i^+ X, S_i^+ Y)$. The same argument shows that S_i^- maps $\operatorname{Rad}^n(S_i^+ X, S_i^+ Y)$ to $\operatorname{Rad}^n(X,Y)$. This establishes for all $n > 1$ the isomorphism

$$\operatorname{Rad}^n(X,Y) \xrightarrow{\sim} \operatorname{Rad}^n(S_i^+ X, S_i^+ Y). \qquad \square$$

Proposition 6.6.2. *If X is an indecomposable preprojective or preinjective representation then* $\operatorname{End}(X) \cong k$.

Proof. Combine Lemmas 1.7.3 and 6.6.1. ☐

Proposition 6.6.3. *Let $X = C^r P(i)$ and $Y = C^s P(j)$ be two indecomposable preprojective representations. Then we have*

$$\operatorname{Irr}(X,Y) \cong \begin{cases} k[Q_1(j,i)] & \text{if } r = s,\\ k[Q_1(i,j)] & \text{if } r = s+1,\\ 0 & \text{otherwise.}\end{cases}$$

The isomorphism sends an arrow $\alpha\colon j \to i$ to $C^r\alpha^*$ and $\beta\colon i \to j$ to $C^r\beta_*$.

Proof. Suppose there exists an irreducible morphisms $X \to Y$.

Let $r \le s$. Then $C^{r-s}P(i) \neq 0$ and we have

$$\operatorname{Irr}(X,Y) \cong \operatorname{Irr}(C^{r-s}P(i), P(j)) \cong k[Q_1(j,i)],$$

where the first isomorphism follows from Lemma 6.6.1, and the second follows from Lemmas 6.4.3 and 6.4.4. In particular, $r = s$.

A similar argument works if $r > s$. Then $C^{s-r}P(j)$ is an indecomposable and non-projective representation. We may assume that $1, \dots, n$ is an admissable numbering of the vertices of Q and we have

$$\begin{aligned}\operatorname{Irr}(X,Y) &\cong \operatorname{Irr}(P(i), C^{s-r}P(j)\\ &\cong \operatorname{Irr}(S_1^- \dots S_{i-1}^- S(i), S_1^- \dots S_{i-1}^- S_i^- \dots S_n^- C^{s-r+1} S_1^- \dots S_{j-1}^- S(j))\\ &\cong \operatorname{Irr}(\tilde{P}(i), \tilde{P}(j)) \cong k[Q_1(i,j)],\end{aligned}$$

where $\tilde{P}(i) = S(i)$ and $\tilde{P}(j) = S_i^- \dots S_{j-1}^- S(j)$ denote the indecomposable projective representations of $\tilde{Q} = \sigma_{i-1} \dots \sigma_1 Q$ corresponding to i and j respectively. In particular, $r = s+1$.

It is clear that this argument can be reversed. Thus we have a necessary and sufficient criterion for the existence of irreducible morphisms $X \to Y$. ☐

6.7. **Morphisms between preprojective representations.** We define a new quiver Γ as follows. Let

$$\Gamma_0 = \{i[r] \mid i \in Q_0, r \in \mathbb{Z}\}.$$

For each arrow $\alpha\colon i \to j$ in Q and $r \in \mathbb{Z}$ we have in Γ a pair of arrows

$$\alpha^*[r]\colon j[r] \longrightarrow i[r] \quad \text{and} \quad \alpha_*[r]\colon i[r] \longrightarrow j[r-1].$$

Thus

$$\Gamma_1 = \{\alpha^*[r] \mid \alpha \in Q_1, r \in \mathbb{Z}\} \cup \{\alpha_*[r] \mid \alpha \in Q_1, r \in \mathbb{Z}\}.$$

Observe that there exists a chain of irreducible morphisms

$$C^{r_1}P(i_1) \longrightarrow C^{r_2}P(i_2) \longrightarrow \cdots \longrightarrow C^{r_m}P(i_m)$$

if and only if there is a path

$$i_1[r_1] \longrightarrow i_2[r_2] \longrightarrow \cdots \longrightarrow i_m[r_m]$$

in Γ. Here we assume that $C^{r_j}P(i_j) \neq 0$ for $1 \leq j \leq m$. This observation follows from Proposition 6.6.3 and motivates the following construction.

We define for each pair of vertices $i[r]$ and $j[s]$ in Γ a linear map

$$\pi\colon k[\Gamma(i[r], j[s])] \longrightarrow \operatorname{Hom}(C^rP(i), C^sP(j)) \tag{6.7.1}$$

by induction on the path length. For a path $\xi\colon i[r] \to j[s]$ in Γ let

$$\pi(\xi) = \begin{cases} \mathrm{id}_{C^rP(i)} & \text{if } \xi = \varepsilon_{i[r]}, \\ C^r\alpha^* & \text{if } \xi = \alpha^*[r], \\ C^r\alpha_* & \text{if } \xi = \alpha_*[r], \\ \pi(\xi_l)\dots\pi(\xi_1) & \text{if } \xi = \xi_l\dots\xi_1, l > 1. \end{cases}$$

Proposition 6.7.1. *Let $C^rP(i)$ and $C^sP(j)$ be two indecomposable preprojective representations. Then the linear map π (6.7.1) is an epimorphism. The kernel of π is spanned by all elements of the form*

$$\sum_{\substack{\alpha\in Q_1\\ t(\alpha)=l}} \tau\alpha_*[t]\alpha^*[t]\sigma + \sum_{\substack{\alpha\in Q_1\\ s(\alpha)=l}} \tau\alpha^*[t-1]\alpha_*[t]\sigma$$

where $l[t]$ runs through all vertices of Γ and σ, τ run through all paths $\sigma\colon i[r] \to l[t]$ and $\tau\colon l[t-1] \to j[s]$ in Γ.

Proof. We do not give the proof but refer instead to the discussion of preprojective components in [9], in particular to 2.3.3. ☐

6.8. **The Dynkin case.** The following theorem summarises the structure of the morphisms between representations of quivers of Dynkin type. Note that in this case each indecomposable representation is preprojective and therefore of the form $C^rP(i)$ for some vertex i and some $r \leq 0$.

Theorem 6.8.1. *Let Q be a quiver whose underlying graph is a Dynkin diagram. Suppose $X \cong C^rP(i)$ and $Y \cong C^sP(j)$ are two indecomposable representations.*

1. *We have* $\operatorname{End}(X) \cong k$ *and every non-isomorphism $X \to Y$ is a sum of compositions of irreducible morphisms between indecomposable representations.*
2. *There exists an irreducible morphism $X \to Y$ if and only if*
   (a) *$r = s$ and there exists an arrow $j \to i$, or*

(b) $r = s + 1$ *and there exists an arrow* $i \to j$.

Given two irreducible morphisms $\phi, \phi' \colon X \to Y$, *there exists* $\lambda \in k$ *with* $\phi' = \lambda\phi$.

(3) *There exists an integer* $d = d(X, Y) \geq 0$ *such that*

$$\operatorname{Hom}(X, Y) = \operatorname{Rad}^0(X, Y) = \ldots = \operatorname{Rad}^d(X, Y) \supsetneq \operatorname{Rad}^{d+1}(X, Y) = 0$$

Proof. (1) We have $\operatorname{End}(X) \cong k$ by Proposition 6.6.2. There are only finitely many isomorphism classes of indecomposable representations by Theorem 5.1.1. Thus the length of the indecomposable representations is bounded by some n and we have therefore $\operatorname{Rad}^{2^n-1}(X, Y) = 0$ by Proposition 6.3.2. It follows from Proposition 6.2.4 that every non-isomorphism $X \to Y$ is a sum of compositions of irreducible morphisms between indecomposable representations.

(2) See Proposition 6.6.3. The last assertion follows from (3), because $\operatorname{Irr}(X, Y) \neq 0$ implies $\operatorname{Rad}^2(X, Y) = 0$.

(3) Let $d = d(i[r], j[s])$ be the length of a path $\xi \colon i[r] \to j[s]$ in Γ and put $d = 0$ if there is no such path. Observe that d does not depend on the choice of ξ because any two parallel paths in Γ have the same length. Here one uses that the underlying graph of Q is a tree. Now we apply (1). The assertion is clear if $\operatorname{Rad}(X, Y) = 0$. A non-zero morphism $\phi \in \operatorname{Rad}(X, Y)$ can be written as a finite sum $\phi = \sum_l \phi_{ld} \ldots \phi_{l2}\phi_{l1}$ of compositions of irreducible morphisms between indecomposable representations, and each chain has length d since it corresponds to a path $i[r] \to j[s]$ in Γ. Thus we have $\phi \in \operatorname{Rad}^d(X, Y)$ but $\phi \notin \operatorname{Rad}^{d+1}(X, Y)$. □

7. The infinite radical

In this section we characterize the quivers of finite representation in terms of morphisms between their representations. We use some global properties, in particular infinite chains of radical morphisms.

Throughout this section we fix a quiver Q.

7.1. **Reflection functors and exact sequences.**

A sequence of morphisms between representations

$$X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} \ldots \xrightarrow{\phi_r} X_{r+1}$$

is called *exact* if $\operatorname{Im} \phi_i = \operatorname{Ker} \phi_{i+1}$ for all $1 \leq i < r$.

Lemma 7.1.1. *Let* i *be a vertex of* Q *and* $0 \to X' \to X \to X'' \to 0$ *an exact sequence.*

(1) *Suppose* i *is a sink. Then* S_i^+ *induces an exact sequence*

$$0 \to S_i^+X' \to S_i^+X \to S_i^+X'' \to \operatorname{Coker} \iota_i X' \to \operatorname{Coker} \iota_i X \to \operatorname{Coker} \iota_i X'' \to 0.$$

(2) *Suppose* i *is a source. Then* S_i^- *induces an exact sequence*

$$0 \to \operatorname{Ker} \pi_i X' \to \operatorname{Ker} \pi_i X \to \operatorname{Ker} \pi_i X'' \to S_i^-X' \to S_i^-X \to S_i^-X'' \to 0.$$

Proof. Apply the snake lemma. □

The lemma can be simplified. An elementary argument shows that any exact sequence $0 \to X' \to X \to X'' \to 0$ induces exact sequences

$$0 \to S_i^+X' \to S_i^+X \to S_i^+X'' \to X^+ \to 0 \quad (i \text{ a sink})$$

$$0 \to X^- \to S_i^-X' \to S_i^-X \to S_i^-X'' \to 0 \quad (i \text{ a source})$$

such that X^+ and X^- are direct sums of copies of $S(i)$.

Lemma 7.1.2. *Suppose Q has no oriented cycle and fix a vertex i. Then we have the following exact sequences*

$$0 \longrightarrow \bigoplus_{\alpha\colon i\to j} P(j) \xrightarrow{(\alpha^*)} P(i) \longrightarrow S(i) \longrightarrow 0 \tag{7.1.1}$$

$$0 \longrightarrow P(i) \xrightarrow{(\alpha^*)} \bigoplus_{\alpha\colon j\to i} P(j) \xrightarrow{(\alpha_*)} C^-P(i) \longrightarrow 0 \quad (i \text{ a sink}) \tag{7.1.2}$$

$$0 \longrightarrow P(i) \xrightarrow{\mu(i)} \Big(\bigoplus_{\alpha\colon j\to i} P(j)\Big) \oplus \Big(\bigoplus_{\beta\colon i\to j} C^-P(j)\Big) \xrightarrow{\nu(i)} C^-P(i) \longrightarrow 0 \tag{7.1.3}$$

where $\mu(i)_\alpha = \alpha^*$, $\mu(i)_\beta = \beta_*$, $\nu(i)_\alpha = \alpha_*$, and $\nu(i)_\beta = C^-\beta^*$.

Proof. For the first sequence see Lemma 6.4.1.

The second sequence is obtained from the first as follows. Let i be a sink of Q. Then i is a source of $\tilde{Q} = \sigma_i Q$. Now use Lemma 7.1.1 and apply S_i^- to the sequence (7.1.1) for $\tilde{Q}$ to obtain the sequence (7.1.2) for Q.

The third sequence is obtained from the second as follows. Assume for simplicity that $1, \dots, n$ is an admissable labeling of the vertices of Q. Then i is a sink for $\tilde{Q} = \sigma_{i-1} \dots \sigma_1 Q$ and we recall that $P(i) \cong S_1^- \dots S_{i-1}^- S(i)$ by Lemma 3.4.2. Now use again Lemma 7.1.1 and apply $S_1^- \dots S_{i-1}^-$ to the sequence (7.1.2) for $\tilde{Q}$ to obtain the sequence (7.1.3) for Q. ☐

7.2. **Infinite chains of morphisms.**

Proposition 7.2.1. *Let Q be a quiver of Euclidean type. Then there exists an infinite family of non-isomorphisms $\phi_p\colon X_p \to X_{p+1}$, $p \geq 1$, between indecomposable representations such that $\phi_n \dots \phi_1 \neq 0$ for all $n \geq 1$.*

Proof. Suppose first Q is of type $\tilde{A}_n$. We allow any orientation, in particular an oriented cycle. We have an infinite family of indecomposable representations $X(p)$, $p \geq 1$, by Proposition 5.3.2. The construction of $X(p)$ shows that the canonical inclusion $k^p \to k^{p+1}$ induces a monomorphism $\phi_p\colon X(p) \to X(p+1)$ for each $p \geq 1$.

Now suppose Q has no oriented cycle. We use for all $i \in Q_0$ the monomorphism $\mu(i)\colon P(i) \to E(i)$ in (7.1.3) and observe that $C^r\mu(i)$ is a monomorphism for all $r \leq 0$. This follows from Lemma 7.1.1 and the fact that $C^rP(i) \neq 0$ for all $r \leq 0$; see Theorem 5.3.1. Now choose a vertex i_1 of Q. Let $X_1 = P(i_1)$ and denote by $\chi_1\colon X_1 \to I$ a non-zero morphism to the indecomposable injective representation $I = I(i_1)$. Then χ_1 factors through $\mu(i_1)$ by Remark 1.7.4, because $\mu(i_1)$ is a monomorphism. Thus we can choose an indecomposable direct summand $X_2 = C^{r_2}P(i_2)$ of $E(i_1)$ corresponding to an arrow α_1 and a morphism $\chi_2\colon X_2 \to I$ such that $\chi_2\phi_1 \neq 0$ where $\phi_1 = \mu(i_1)_{\alpha_1}$. The morphism χ_2 factors through $C^{r_2}\mu(i_2)$ because $C^{r_2}\mu(i_2)$ is a monomorphism. Thus we can choose an indecomposable direct summand $X_3 = C^{r_3}P(i_3)$ of $C^{r_2}E(i_2)$ corresponding to an arrow α_2 and a morphism $\chi_3\colon X_3 \to I$ such that $\chi_3\phi_2\phi_1 \neq 0$ where $\phi_2 = C^{r_2}\mu(i_2)_{\alpha_2}$. We continue and obtain an infinite family of morphisms $\phi_p\colon X_p \to X_{p+1}$ such that $\phi_n \dots \phi_1 \neq 0$ for all $n \geq 1$. ☐

7.3. **A characterization of finite representation type.**

Theorem 7.3.1. *For a quiver Q the following are equivalent:*

1. *The number of isomorphism classes of indecomposable representations is finite.*
2. *There is a global bound for the length of every indecomposable representation.*
3. *We have* $\bigcap_{n\geq 0}\operatorname{Rad}^n(X,Y)=0$ *for every pair* X,Y *of representations.*
4. *Given an infinite family of non-isomorphism* $\phi_i\colon X_i\to X_{i+1}$, $i\geq 1$, *between indecomposable representations, there exists* $n\geq 1$ *such that* $\phi_n\ldots\phi_1=0$.

Proof. (1) ⇒ (2): Clear.

(2) ⇒ (3): Use Proposition 6.3.2.

(3) ⇒ (4): Suppose $\psi_n=\phi_n\ldots\phi_1\neq 0$ for all $n\geq 1$. Then there exists $r\geq 1$ such that $\bigcup_{n\geq 1}\operatorname{Ker}\psi_n=\operatorname{Ker}\psi_r\neq X_1$ because X_1 is finite dimensional. We let $X=X_1/\operatorname{Ker}\psi_r$ and denote by $\phi\colon X\to X_{r+1}$ the canonical monomorphism. Then the composition $\phi_n\ldots\phi_{r+1}\phi$ is a monomorphism for all $n>r$. Choose an indecomposable injective representations I and a non-zero morphism $\chi\colon X\to I$ which exist by Lemma 1.7.1. Then Remark 1.7.4 implies that χ factors through $\phi_n\ldots\phi_{r+1}\phi$ for all $n>r$. Thus χ belongs to $\bigcap_{n\geq 0}\operatorname{Rad}^n(X,I)$.

(4) ⇒ (1): Suppose there are infinitely many indecomposable representations. Then Q contains an Euclidean subquiver Q' by Corollary 5.3.4. We obtain from Proposition 7.2.1 an infinite chain $\phi_i\colon X_i\to X_{i+1}$ of non-isomorphisms between indecomposable representations of Q', which we can extend to a chain of morphisms for Q as in the proof of Corollary 5.3.4. ☐

8. Notes

In [6] Gabriel proved that a connected quiver admits only finitely many indecomposable representations if and only if it is of Dynkin type. This result and the corresponding classification of the indecomposable representations in terms of their dimension vectors is known as Gabriel's theorem; see also [7]. The proof using reflection functors is due to Bernšteĭn, Gel'fand, and Ponomarev [3]. For the quivers of Euclidean type and the complete classification of their indecomposable representations we refer to [5]. Irreducible morphisms were introduced by Auslander and Reiten in [2] and their structure for algebras of finite representation type is discussed in [8]. Infinite chains of morphisms were used by Auslander when he characterised algebras of finite representation type [1].

The exposition given here follows mostly [3]. However, we avoid the use of algebraic group actions and Tit's argument. Section 2 is based on some notes of Hubery and Section 4 on notes of Crawley-Boevey [4].

References

[1] M. Auslander, *Representation theory of artin algebras II*, Comm. in Algebra **1** (1974), 269–310.

[2] M. Auslander and I. Reiten, *Representation theory of artin algebras IV. Invariants given by almost split sequences*, Comm. in Algebra **5** (1977), 443–518.

[3] I. N. Bernšteĭn, I. M. Gel'fand, and V. A. Ponomarev, *Coxeter functors and Gabriel's theorem*, Uspehi Mat. Nauk **28** (1973), no. 2(170), 19–33.

[4] W. W. Crawley-Boevey, *Lectures on representations of quivers*, `www.amsta.leeds.ac.uk/~pmtwc/`

[5] V. Dlab, C. M. Ringel, *Indecomposable representations of graphs and algebras*, Mem. Amer. Math. Soc. **173** (1976).

[6] P. Gabriel, *Unzerlegbare Darstellungen I*, Manuscripta Math. **6** (1972), 71–103.

[7] P. Gabriel, *Indecomposable representations II*, Symposia Mathematica **11** (1973), 81–104.

[8] P. Gabriel, *Auslander-Reiten sequences and representation finite algebras*, Lect. Notes Math. **831**, Springer-Verlag (1980), 1–71.

[9] C. M. Ringel, *Tame algebras and integral quadratic forms*, Lect. Notes Math. **1099**, Springer-Verlag (1984).

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