ROUQUIER'S THEOREM ON REPRESENTATION DIMENSION

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Abstract. Based on work of Rouquier, some bounds for Aulander's representation dimension are discussed. More specifically, if X is a reduced projective scheme of dimension n over some field, and T is a tilting complex of coherent $\mathcal{O}_X$-modules, then the representation dimension of the endomorphism algebra $\operatorname{End}_{\mathcal{O}_X}(T)$ is at least n.

1. Introduction

Let Λ be an artin algebra and denote by $\operatorname{mod}\Lambda$ the category of right Λ-modules. Auslander defined in [1] the *representation dimension* of Λ as

$$\operatorname{rep.dim}\Lambda = \min\{\operatorname{gl.dim}\operatorname{End}_\Lambda(M) \mid M \text{ generates and cogenerates } \operatorname{mod}\Lambda\}.$$

It is known from work of Iyama [7] that $\operatorname{rep.dim}\Lambda < \infty$. Recently, Rouquier [9] has shown that there is no upper bound for the representation dimension of an artin algebra. More precisely, given a field k and an integer $n \geqslant 1$, the exterior algebra $\Lambda(k^n)$ has representation dimension $n+1$. In this note, we use ideas from Rouquier's work and prove the following result.

Theorem. *Let k be a field and $n \geqslant 1$ be an integer. Denote by Λ_n the k-algebra given by the following quiver with relations:*

$$\begin{tikzcd} 0 \arrow[r, bend left, "x_0"] \arrow[r, bend right, "x_n"'] \arrow[r, phantom, "\vdots"] & 1 \arrow[r, bend left, "x_0"] \arrow[r, bend right, "x_n"'] \arrow[r, phantom, "\vdots"] & 2 & \cdots & n-1 \arrow[r, bend left, "x_0"] \arrow[r, bend right, "x_n"'] \arrow[r, phantom, "\vdots"] & n \end{tikzcd} \quad \text{and} \quad x_ix_j = x_jx_i \quad (0 \leqslant i, j \leqslant n).$$

If M is a finitely generated Λ_n-module which generates the category of finitely generated Λ_n-modules, then $\operatorname{gl.dim}\operatorname{End}_{\Lambda_n}(M) \geqslant n$. *In particular,* $\operatorname{rep.dim}\Lambda_n \geqslant n$.

Our aim is to give an elementary exposition of the fact that the representation dimension is unbounded. Instead of taking the exterior algebras, we have chosen another class of algebras in order to simplify Rouquier's original proof. Let us stress again that most ideas are taken from [9]. However, along the way some of Rouquier's arguments have been changed and some of the results seem to be new.

2. Dimensions of triangulated categories

Let $\mathcal{T}$ be a triangulated category and fix subcategories $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2 \subseteq \mathcal{T}$. Denote by $\mathcal{X}_1 * \mathcal{X}_2$ the full subcategory of $\mathcal{T}$ consisting of objects X which admit an exact triangle $X_1 \to X \to X_2 \to \Sigma X_1$ in $\mathcal{T}$ with $X_i \in \mathcal{X}_i$. Denote by $\langle\mathcal{X}\rangle$ the smallest full subcategory of $\mathcal{T}$ which contains $\mathcal{X}$ and is closed under taking coproducts, direct factors, and all shifts. Let $\mathcal{X}_1 \diamond \mathcal{X}_2 = \langle \mathcal{X}_1 * \mathcal{X}_2\rangle$. Inductively one defines $\langle\mathcal{X}\rangle_0 = 0$ and $\langle\mathcal{X}\rangle_n = \langle\mathcal{X}\rangle_{n-1} \diamond \langle\mathcal{X}\rangle$

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for $n \geqslant 1$. Note that the operations $*$ and $\diamond$ are associative, thanks to the octahedral axiom; see [4, Sec. 2].

Definition 2.1 ([9, Definition 3.1]). Let $\mathcal{T}$ be a triangulated category. The *dimension* of $\mathcal{T}$ is

$$\dim \mathcal{T} = \min\{n \geqslant 0 \mid \text{there exists } X \in \mathcal{T} \text{ such that } \langle X \rangle_{n+1} = \mathcal{T}\}.$$

Lemma 2.2 ([9, Lemma 3.3]). *Let $F\colon \mathcal{S} \to \mathcal{T}$ be an exact functor such that each object in $\mathcal{T}$ is isomorphic to an object in the image of F. Then $\dim \mathcal{S} \geqslant \dim \mathcal{T}$.*

Proof. If $\mathcal{S} = \langle X \rangle_n$, then $\mathcal{T} = \langle FX \rangle_n$. □

Lemma 2.3 ([9, Lemma 4.11]). *Let $\mathcal{T}$ be a triangulated category and let*

$$H_1 \xrightarrow{f_1} H_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} H_n \xrightarrow{f_n} H_{n+1}$$

be a sequence of morphisms between cohomological functors $\mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$. For each i, let $\mathcal{X}_i$ be a subcategory of $\mathcal{T}$ such that f_i vanishes on $\mathcal{X}_i$ and $\mathcal{X}_i = \langle \mathcal{X}_i \rangle$. Then the composite $f_n \circ \dots \circ f_1$ vanishes on $\mathcal{X}_1 \diamond \dots \diamond \mathcal{X}_n$.

Proof. Using induction, it is sufficient to prove the assertion for $n = 2$. Let $X_1 \to X \to X_2 \to \Sigma X_1$ be an exact triangle with $X_i \in \mathcal{X}_i$. We obtain the following commutative diagram with exact rows.

$$\begin{tikzcd}
H_1X_2 \arrow[r] \arrow[d] & H_1X \arrow[r] \arrow[d] & H_1X_1 \arrow[d, "0"] \\
H_2X_2 \arrow[r] \arrow[d, "0"'] & H_2X \arrow[r] \arrow[d] & H_2X_1 \arrow[d] \\
H_3X_2 \arrow[r] & H_3X \arrow[r] & H_3X_1
\end{tikzcd}$$

A simple diagram chase shows that $f_2 \circ f_1$ vanishes on X. Now observe that $\mathcal{X}_1 \diamond \mathcal{X}_2$ consists of direct factors of objects in $\mathcal{X}_1 * \mathcal{X}_2$. □

Let $\mathcal{A}$ be an abelian category. We denote by $\mathbf{D}^b(\mathcal{A})$ the derived category of bounded complexes in $\mathcal{A}$ and identify $\mathcal{A}$ with the full subcategory consisting of complexes concentrated in degree zero. The category $\mathbf{D}^b(\mathcal{A})$ is obtained from the homotopy category $\mathbf{K}^b(\mathcal{A})$ by formally inverting all quasi-isomorphisms. Note that

$$\mathrm{Ext}^n_{\mathcal{A}}(X, Y) \cong \mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(X, \Sigma^n Y)$$

for all $X, Y \in \mathcal{A}$ and $n \geqslant 0$.

Lemma 2.4. *Let $\mathcal{A}$ be an abelian category and $X \in \mathcal{A}$ satisfying $\mathrm{Ext}^n_{\mathcal{A}}(X, -) \neq 0$. Then we have $X \notin \langle \mathcal{P} \rangle_n \subseteq \mathbf{D}^b(\mathcal{A})$ for the subcategory $\mathcal{P} \subseteq \mathcal{A}$ of projective objects.*

Proof. Let $X = X_0$ and fix an extension

$$\xi\colon\ 0 \to X_n \to E_n \to E_{n-1} \to \dots \to E_1 \to X_0 \to 0$$

in $\mathcal{A}$. We view ξ in $\mathrm{Ext}^n_{\mathcal{A}}(X_0, X_n)$ as the composite of extensions

$$\xi_i\colon\ 0 \to X_i \to E_i \to X_{i-1} \to 0.$$

Each ξ_i induces a connecting morphism

$$f_i\colon \operatorname{Hom}_{\mathbf{D}^b(\mathcal{A})}(-,\Sigma^{i-1}X_{i-1}) \longrightarrow \operatorname{Hom}_{\mathbf{D}^b(\mathcal{A})}(-,\Sigma^{i}X_{i})$$

vanishing on $\langle\mathcal{P}\rangle$. The composite $f_n\circ\ldots\circ f_1$ sends the identity morphism of X_0 to $\xi=\xi_n\circ\ldots\circ\xi_1$. Thus $\xi\neq 0$ implies $X_0\notin\langle\mathcal{P}\rangle_n$, by Lemma 2.3. □

Lemma 2.5 ([5, Theorem 8.3])**.** *Let $\mathcal{A}$ be an abelian category and suppose that $\mathcal{A}$ has enough projective objects. Let X be a bounded complex in $\mathcal{A}$ such that B_iX and H_iX have projective dimension at most n for all i. Then $X\in\langle\mathcal{P}\rangle_{n+1}\subseteq\mathbf{D}^b(\mathcal{A})$ for the subcategory $\mathcal{P}\subseteq\mathcal{A}$ of projective objects.*

Proof. For each i, choose epimorphisms $P^{B_iX}\twoheadrightarrow B_iX$ and $P^{H_iX}\twoheadrightarrow H_iX$ such that P^{B_iX} and P^{H_iX} are projective. We obtain commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & P^{B_iX} \arrow[r] \arrow[d, two heads] & P^{Z_iX} \arrow[r] \arrow[d, two heads] & P^{H_iX} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & B_iX \arrow[r] & Z_iX \arrow[r] & H_iX \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Similarly, we obtain commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & P^{Z_iX} \arrow[r] \arrow[d, two heads] & P^{X_i} \arrow[r] \arrow[d, two heads] & P^{B_{i-1}X} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & Z_iX \arrow[r] & X_i \arrow[r] & B_{i-1}X \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Defining $P_i=P^{X_i}$ and taking the composite

$$P_{i+1}=P^{X_{i+1}}\twoheadrightarrow P^{B_iX}\rightarrowtail P^{Z_iX}\rightarrowtail P^{X_i}=P_i$$

as differential, we obtain a complex P. Note that $B_iP=P^{B_iX}$, $Z_iP=P^{Z_iX}$, $H_iP=P^{H_iX}$ for all i, and therefore $P\in\langle\mathcal{P}\rangle$. Now let $X^0=X$ and $P^0=P$. We have an epimorphism $P^0\to X^0$ and denote by X^1 the shift of the degreewise kernel. This yields an exact triangle $P^0\to X^0\to X^1\to\Sigma P^0$ in $\mathbf{D}^b(\mathcal{A})$. Note that B_iX^1 and H_iX^1 have projective dimension at most $n-1$ for all i. We inductively continue this construction, and by our assumption on B_iX and H_iX, we have $X^n\in\langle\mathcal{P}\rangle$. Thus $X^{n-1}\in\langle\mathcal{P}\rangle_2$, and inductively, $X\in\langle\mathcal{P}\rangle_{n+1}$. □

Recall that a ring Λ is *right coherent* if the category $\operatorname{mod}\Lambda$ of finitely presented right Λ-modules is abelian.

Proposition 2.6. *Let Λ be a right coherent ring. Then*

$$\dim\mathbf{D}^b(\operatorname{mod}\Lambda)\leqslant\sup\{\operatorname{pd}X\mid X\in\operatorname{mod}\Lambda\}\leqslant\operatorname{gl.dim}\Lambda.$$

Proof. We apply Lemma 2.5. Take the abelian category $\mathcal{A}=\operatorname{mod}\Lambda$ and observe that $\langle\mathcal{P}\rangle_n=\langle\Lambda\rangle_n$ for all $n\geqslant 0$. □

Remark 2.7. The result of the preceding proposition answers a question in [9, Rem. 7.27].

3. Representation generators

Given an object M of an additive category, we denote by $\operatorname{add} M$ the smallest full subcategory closed under finite coproducts and direct factors.

Definition 3.1. Let $\mathcal{A}$ be an abelian category and fix an object $M \in \mathcal{A}$. An M-*resolution* of an object $X \in \mathcal{A}$ is an exact sequence

$$\ldots \to M_2 \to M_1 \to M_0 \to X \to 0$$

in $\mathcal{A}$ such that $M_i \in \operatorname{add} M$ for all i and the induced sequence

$$\ldots \to \operatorname{Hom}_{\mathcal{A}}(M, M_2) \to \operatorname{Hom}_{\mathcal{A}}(M, M_1) \to \operatorname{Hom}_{\mathcal{A}}(M, M_0) \to \operatorname{Hom}_{\mathcal{A}}(M, X) \to 0$$

of abelian groups is exact. The resolution has *finite length* if $M_i = 0$ for $i \gg 0$. We call M a *representation generator* if every object $X \in \mathcal{A}$ admits an M-resolution of finite length.

Remark 3.2. Note that we do not assume any common bound for the length of an M-resolution in our definition of a representation generator M. However, the finite bound $\operatorname{gl.dim} \operatorname{End}_{\mathcal{A}}(M)$ is automatic in some interesting cases; see Lemma 3.5.

Lemma 3.3. *Let $\mathcal{A}$ be an abelian category. An object $M \in \mathcal{A}$ is a representation generator if and only if the inclusion $\mathbf{K}^b(\operatorname{add} M) \to \mathbf{K}^b(\mathcal{A})$ admits a right adjoint $F \colon \mathbf{K}^b(\mathcal{A}) \to \mathbf{K}^b(\operatorname{add} M)$ such that the adjunction morphism $FX \to X$ is a quasi-isomorphism for all $X \in \mathbf{K}^b(\mathcal{A})$.*

Proof (cf. [9, Proposition 8.3]). Let M be a representation generator. For each $X \in \mathbf{K}^b(\mathcal{A})$, we need to construct an *approximation* $M^X \to X$ such that

1. $M^X \in \mathbf{K}^b(\operatorname{add} M)$,
2. the induced morphism $\operatorname{Hom}_{\mathbf{K}^b(\operatorname{add} M)}(Y, M^X) \to \operatorname{Hom}_{\mathbf{K}^b(\mathcal{A})}(Y, X)$ is bijective for all $Y \in \mathbf{K}^b(\operatorname{add} M)$, and
3. $M^X \to X$ is a quasi-isomorphism.

Then we define $F \colon \mathbf{K}^b(\mathcal{A}) \to \mathbf{K}^b(\operatorname{add} M)$ by sending X to M^X. We construct $M^X \to X$ by induction on the width of X. Suppose first that X is concentrated in degree zero. Take a finite length M-resolution

$$0 \to M_n \to \ldots \to M_1 \to M_0 \to X \to 0$$

which exists by assumption, and define $M^X_i = M_i$ for $0 \leqslant i \leqslant n$ and $M^X_i = 0$ otherwise. The morphism $M_0 \to X$ induces a quasi-isomorphism $\phi \colon M^X \to X$. Moreover, $\operatorname{Hom}_{\mathcal{A}}(M, \phi)$ is a quasi-isomorphism and therefore $\operatorname{Hom}_{\mathbf{K}^b(\mathcal{A})}(M, \phi)$ is bijective. Using induction on the width of a complex Y in $\mathbf{K}^b(\operatorname{add} M)$, one sees that $\operatorname{Hom}_{\mathbf{K}^b(\mathcal{A})}(Y, \phi)$ is bijective. Now suppose that X fits into an exact triangle $X' \to X'' \to X \to \Sigma X'$ where approximations $\phi' \colon M^{X'} \to X'$ and $\phi'' \colon M^{X''} \to X''$ have been constructed. Using the bijection

$$\operatorname{Hom}_{\mathbf{K}^b(\operatorname{add} M)}(M^{X'}, M^{X''}) \to \operatorname{Hom}_{\mathbf{K}^b(\mathcal{A})}(M^{X'}, X''),$$

we obtain a morphism $M^{X'} \to M^{X''}$ and complete it to an exact triangle

$$M^{X'} \to M^{X''} \to M^X \to \Sigma M^{X'}.$$

Moreover, we can complete ϕ' and ϕ'' to a morphism of triangles and obtain an approximation $\phi \colon M^X \to X$.

Suppose now that the inclusion $\mathbf{K}^b(\operatorname{add} M) \to \mathbf{K}^b(\mathcal{A})$ admits a right adjoint F such that the adjunction morphism $FX \to X$ is a quasi-isomorphism for all X in $\mathbf{K}^b(\mathcal{A})$. Given an object $X \in \mathcal{A}$, we view X as a complex concentrated in degree zero. The properties of the morphism $FX \to X$ imply that we obtain an M-resolution

$$0 \to M_n \to \ldots \to M_1 \to M_0 \to X \to 0$$

by taking

$$M_i = \begin{cases} (FX)_i & i > 0, \\ Z_0(FX) & i = 0, \\ 0 & i < 0. \end{cases}$$

It is easily checked that $M_0 \in \operatorname{add} M$, and $FX \to X$ induces the morphism $M_0 \to X$. □

Proposition 3.4. *Let $\mathcal{A}$ be an abelian category with a representation generator M. Then $\Gamma = \operatorname{End}_{\mathcal{A}}(M)$ is right coherent and every finitely presented Γ-module has finite projective dimension. Moreover, we have*

$$\dim \mathbf{D}^b(\mathcal{A}) \leqslant \dim \mathbf{D}^b(\operatorname{mod} \Gamma) \leqslant \sup\{\operatorname{pd} X \mid X \in \operatorname{mod} \Gamma\} \leqslant \operatorname{gl.dim} \Gamma.$$

Proof. We use that $\operatorname{Hom}_{\mathcal{A}}(M, -)$ induces an equivalence $\operatorname{add} M \to \operatorname{proj} \Gamma$ onto the category of finitely generated projective Γ-modules. Recall that Γ is right coherent if the kernel of any morphism ϕ in $\operatorname{proj} \Gamma$ is finitely presented. We have $\phi = \operatorname{Hom}_{\mathcal{A}}(M, \psi)$ for some morphism ψ in $\operatorname{add} M$. Then $X = \operatorname{Ker} \psi$ admits a finite length M-resolution

$$0 \to M_n \to \ldots \to M_1 \to M_0 \to X \to 0$$

by our assumption on M. The morphism $M_1 \to M_0$ induces a projective presentation

$$\operatorname{Hom}_{\mathcal{A}}(M, M_1) \to \operatorname{Hom}_{\mathcal{A}}(M, M_0) \to \operatorname{Hom}_{\mathcal{A}}(M, X) \to 0$$

of $\operatorname{Hom}_{\mathcal{A}}(M, X) = \operatorname{Ker} \phi$ in $\operatorname{mod} \Gamma$. Thus Γ is right coherent. Now let Z be in $\operatorname{mod} \Gamma$ with projective presentation

$$P_1 \xrightarrow{\phi} P_0 \to Z \to 0.$$

Again, we have $\phi = \operatorname{Hom}_{\mathcal{A}}(M, \psi)$ for some morphism ψ in $\operatorname{add} M$, and $X = \operatorname{Ker} \psi$ admits an M-resolution

$$0 \to M_n \to \ldots \to M_1 \to M_0 \to X \to 0.$$

Applying $\operatorname{Hom}_{\mathcal{A}}(M, -)$, we obtain a projective resolution of length n of $\operatorname{Hom}_{\mathcal{A}}(M, X) = \operatorname{Ker} \phi$. Thus $\operatorname{pd} Z \leqslant n + 2$.

For the bounds on $\dim \mathbf{D}^b(\mathcal{A})$, observe that we have exact equivalences

$$\mathbf{K}^b(\operatorname{add} M) \xrightarrow{\sim} \mathbf{K}^b(\operatorname{proj} \Gamma) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod} \Gamma).$$

The functor $\mathbf{K}^b(\operatorname{add} M) \to \mathbf{D}^b(\mathcal{A})$ is essentially surjective by Lemma 3.3, and we obtain

$$\dim \mathbf{D}^b(\mathcal{A}) \leqslant \dim \mathbf{D}^b(\operatorname{mod} \Gamma)$$

from Lemma 2.2. For the rest, apply Proposition 2.6. □

For the module category of an artin algebra, we have the following characterization of a representation generator.

Lemma 3.5. *Let Λ be an artin algebra. Then $M \in \operatorname{mod} \Lambda$ is a representation generator of $\operatorname{mod} \Lambda$ if and only if M generates $\operatorname{mod} \Lambda$ and $\operatorname{End}_{\Lambda}(M)$ has finite global dimension.*

Proof. Suppose first that M is a representation generator of $\operatorname{mod}\Lambda$ with $\Gamma = \operatorname{End}_\Lambda(M)$. For each $X \in \operatorname{mod}\Lambda$, we have an epimorphism $M_0 \to X$ with M_0 in $\operatorname{add} M$. Thus M generates $\operatorname{mod}\Lambda$. Every finitely presented Γ-module has finite projective dimension, by Proposition 3.4. It follows that Γ has finite global dimension, because the global dimension equals the projective dimension of $\Gamma/\mathfrak{r}$, where $\mathfrak{r}$ denotes the Jacobson radical.

Now suppose that M generates $\operatorname{mod}\Lambda$ and $\Gamma = \operatorname{End}_\Lambda(M)$ has finite global dimension. Every finitely presented Λ-module X admits an epimorphism $\phi\colon M^X \to X$ with M^X in $\operatorname{add} M$ such that $\operatorname{Hom}_\Lambda(M, \phi)$ is an epimorphism. This is clear since $\operatorname{Hom}_\Lambda(M, X)$ is finitely generated over Γ. Let $M_0 = M^X$ and $X_1 = \operatorname{Ker}\phi$. Inductively, we define $M_i = M^{X_i}$ and obtain morphisms $M_i \to X_i \to M_{i-1}$ which induce a projective resolution

$$\ldots \to \operatorname{Hom}_\Lambda(M, M_1) \to \operatorname{Hom}_\Lambda(M, M_0) \to \operatorname{Hom}_\Lambda(M, X) \to 0$$

in $\operatorname{mod}\Gamma$. We have $X_i \in \operatorname{add} M$ for $i \geqslant \operatorname{gl.dim}\Gamma$ and therefore the construction terminates, giving an M-resolution of finite length

$$0 \to M_n \to \ldots \to M_1 \to M_0 \to X \to 0$$

as required. ☐

Corollary 3.6 ([9, Proposition 8.3])**.** *Let Λ be an artin algebra and $M \in \operatorname{mod}\Lambda$. If M generates $\operatorname{mod}\Lambda$, then*

$$\dim \mathbf{D}^b(\operatorname{mod}\Lambda) \leqslant \operatorname{gl.dim} \operatorname{End}_\Lambda(M).$$

Proof. Combine Lemma 3.5 and Proposition 3.4. ☐

Example 3.7 ([2, Theorem 10.2]). Let Λ be an artin algebra with Jacobson radical $\mathfrak{r}$ and $\mathfrak{r}^n = 0$. Then $M = \coprod_{i=1}^n \Lambda/\mathfrak{r}^i$ is a representation generator of $\operatorname{mod}\Lambda$ with $\operatorname{gl.dim}\operatorname{End}_\Lambda(M) \leqslant n$.

4. Some geometry

We compute explicit bounds for the dimension of certain derived categories, using some elementary facts from algebraic geometry. Given a scheme X, the category of coherent $\mathcal{O}_X$-modules is denoted by $\operatorname{coh} X$.

Lemma 4.1. *Let X be a reduced projective scheme over a field, and let $F \in \operatorname{coh} X$. Then every irreducible component of X contains a closed point $x \in X$ such that F_x is a free $\mathcal{O}_x$-module.*

Proof. Assume first that X is irreducible. The local ring $\mathcal{O}_\xi$ at the generic point $\xi \in X$ is a field, since X is reduced. Thus F_ξ is a free $\mathcal{O}_\xi$-module, and there is a neighbourhood U such that $F|_U$ is free [6, Ex. II.5.7]. We find a closed $x \in U$ since the closed points of X are dense [6, Ex. II.3.14].

Now assume that X is arbitrary and fix an irreducible component $Y \subseteq X$. We pass to Y and the same argument as before works if we choose the point $x \in Y$ such that it is not contained in any other irreducible component. ☐

Lemma 4.2. *Let X be a reduced projective scheme over a field, and $M \in \mathbf{D}^b(\operatorname{coh} X)$. Then every irreducible component of X contains a closed point $x \in X$ such that*

$$M_x \in \langle \mathcal{O}_x \rangle \subseteq \mathbf{D}^b(\operatorname{mod} \mathcal{O}_x).$$

Proof. We apply Lemma 4.1 and find a closed point x such that B_iM_x and H_iM_x are free $\mathcal{O}_x$-modules for all i. Thus $M_x \in \langle \mathcal{O}_x \rangle$ by Lemma 2.5. $\square$

Lemma 4.3 ([9, Lemma 7.14])**.** *Let Λ be a commutative local noetherian ring with maximal ideal $\mathfrak{m}$. If Λ has Krull dimension n, then $\Lambda/\mathfrak{m} \notin \langle \Lambda \rangle_n$.*

Proof. The projective dimension of $\Lambda/\mathfrak{m}$ is at least n. Now apply Lemma 2.4. $\square$

Proposition 4.4 ([9, Proposition 7.17])**.** *Let X be a reduced projective scheme over a field. Then $\dim \mathbf{D}^b(\operatorname{coh} X) \geqslant \dim X$.*

Proof. We may identify $X = \operatorname{Proj} S$ for some graded ring S [6, Cor. II.5.16]. Let $M \in \mathbf{D}^b(\operatorname{coh} X)$ and $n \geqslant 0$ such that $\mathbf{D}^b(\operatorname{coh} X) = \langle M \rangle_{n+1}$. Choose an irreducible component $Y \subseteq X$ of maximal dimension. Using Lemma 4.2, there exists a closed $x \in Y$ such that $M_x \in \langle \mathcal{O}_x \rangle$. Let $I(x) \subseteq S$ be the homogeneous prime ideal corresponding to x and F be the sheaf associated to $S/I(x)$. Then $F \in \langle M \rangle_{n+1}$ implies for the residue field

$$k(x) = F_x \in \langle M_x \rangle_{n+1} \subseteq \langle \mathcal{O}_x \rangle_{n+1}.$$

Thus $n \geqslant \dim \mathcal{O}_x = \dim Y = \dim X$, by Lemma 4.3. $\square$

5. Representation dimensions

We are now in a position to give the proof of the main result.

Proof of the Main Theorem. Let k be a field and $n \geqslant 1$ an integer. We consider the projective scheme $X = \mathbf{P}^n_k$ and fix the sheaf $T = \coprod_{i=0}^n \mathcal{O}_X(i)$. We have $\operatorname{End}_{\mathcal{O}_X}(T) \cong \Lambda_n$ and an equivalence

$$\mathbf{R}\operatorname{Hom}_{\mathcal{O}_X}(T, -) \colon \mathbf{D}^b(\operatorname{coh} X) \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod} \Lambda_n)$$

by work of Beilinson [3], because T is a tilting sheaf. Now let M be a finitely generated Λ_n-module which generates $\operatorname{mod} \Lambda_n$. Applying Proposition 4.4 and Corollary 3.6, we obtain

$$n = \dim X \leqslant \dim \mathbf{D}^b(\operatorname{coh} X) = \dim \mathbf{D}^b(\operatorname{mod} \Lambda_n) \leqslant \operatorname{gl.dim} \operatorname{End}_\Lambda(M).$$

From the definition of the representation dimension, it follows that $\operatorname{rep.dim} \Lambda_n \geqslant n$. $\square$

Remark 5.1. We have actually $\dim \mathbf{D}^b(\operatorname{coh} X) = n$ for $X = \mathbf{P}^n_k$ since

$$\dim \mathbf{D}^b(\operatorname{coh} X) = \dim \mathbf{D}^b(\operatorname{mod} \Lambda_n) \leqslant \operatorname{gl.dim} \Lambda_n = n$$

by Proposition 2.6.

The proof of the main theorem provides the following method for constructing algebras with representation dimension bounded from below.

Theorem 5.2. *Let X be a reduced projective scheme over a field, and let T be a tilting complex in $\mathbf{D}^b(\operatorname{coh} X)$ with $\Lambda = \operatorname{End}_{\mathbf{D}^b(\operatorname{coh} X)}(T)$. Given a generator M of $\operatorname{mod} \Lambda$, we have $\operatorname{gl.dim} \operatorname{End}_\Lambda(M) \geqslant \dim X$. In particular, $\operatorname{rep.dim} \Lambda \geqslant \dim X$.*

The actual computation of the representation dimension of $\Lambda = \operatorname{End}_{\mathbf{D}^b(\operatorname{coh} X)}(T)$ seems to be difficult. However, there is the following bound from above. The argument for the first part is due to Lenzing. We fix a field k and let $D = \operatorname{Hom}_k(-, k)$.

Proposition 5.3. *Let X be a smooth projective scheme of dimension $n \geqslant 1$ over a field, and let $T \in \operatorname{coh} X$ be a tilting sheaf with $\Lambda = \operatorname{End}_{\mathcal{O}_X}(T)$.*

(1) *Let Q be an injective and P be a projective Λ-module. Then* $\operatorname{Hom}_\Lambda(Q,P)=0$.
(2) *We have* $\operatorname{rep.dim}\Lambda \leqslant 2\operatorname{gl.dim}\Lambda+1<\infty$.

Proof. (1) Let X be projective over the field k and denote by $\omega=\omega_{X/k}$ the dualizing sheaf. Then Serre duality gives

$$D\operatorname{Hom}_{\mathbf{D}^b(\operatorname{coh}X)}(E,F)\cong\operatorname{Hom}_{\mathbf{D}^b(\operatorname{coh}X)}(F,E\otimes_{\mathcal{O}_X}\omega[n])\quad\text{for}\quad E,F\in\mathbf{D}^b(\operatorname{coh}X)\,.$$

The equivalence

$$\mathbf{R}\operatorname{Hom}_{\mathcal{O}_X}(T,-)\colon \mathbf{D}^b(\operatorname{coh}X)\xrightarrow{\sim}\mathbf{D}^b(\operatorname{mod}\Lambda)$$

identifies the full subcategory

$$\mathcal{A}=\{F\in\mathbf{D}^b(\operatorname{coh}X)\mid \operatorname{Hom}_{\mathbf{D}^b(\operatorname{coh}X)}(T,F[i])=0\text{ for }i\neq 0\}$$

with $\operatorname{mod}\Lambda$. Clearly, T is a projective generator of $\mathcal{A}$ and we claim that $U=T\otimes_{\mathcal{O}_X}\omega[n]$ is an injective cogenerator of $\mathcal{A}$. In deed, we have $U\in\mathcal{A}$ since

$$\operatorname{Hom}_{\mathbf{D}^b(\operatorname{coh}X)}(T,U[i])\cong D\operatorname{Hom}_{\mathbf{D}^b(\operatorname{coh}X)}(T,T[-i])=0$$

for $i\neq 0$. Moreover, U is an injective cogenerator of $\mathcal{A}$ since Serre duality implies

$$\operatorname{Hom}_{\mathcal{A}}(F,U)\cong D\operatorname{Hom}_{\mathcal{A}}(T,F)\quad\text{for}\quad F\in\mathcal{A}$$

and T is a projective generator. Finally, we have

$$\operatorname{Hom}_{\mathcal{A}}(U,T)=\operatorname{Hom}_{\mathbf{D}^b(\operatorname{coh}X)}(T\otimes_{\mathcal{O}_X}\omega[n],T)\cong\operatorname{Ext}^{-n}_X(T\otimes_{\mathcal{O}_X}\omega,T)=0\,.$$

(2) Let $M=\Lambda\amalg D\Lambda$. Clearly, M generates and cogenerates $\operatorname{mod}\Lambda$. Using that $\operatorname{Hom}_\Lambda(D\Lambda,\Lambda)=0$, we obtain

$$\operatorname{rep.dim}\Lambda\leqslant\operatorname{gl.dim}\operatorname{End}_\Lambda(M)\leqslant 2\operatorname{gl.dim}\Lambda+1,$$

for instance by [8, Prop. 7.5.1]. Finally, observe that $\operatorname{gl.dim}\Lambda<\infty$, since $\mathbf{D}^b(\operatorname{mod}\Lambda)$ admits a Serre functor. This follows from Lemma 5.4 below. ☐

Lemma 5.4. *Let Λ be a finite dimensional algebra over a field k. Suppose there is an exact functor $F\colon\mathbf{D}^b(\operatorname{mod}\Lambda)\to\mathbf{D}^b(\operatorname{mod}\Lambda)$ such that*

$$D\operatorname{Hom}_{\mathbf{D}^b(\operatorname{mod}\Lambda)}(X,Y)\cong\operatorname{Hom}_{\mathbf{D}^b(\operatorname{mod}\Lambda)}(Y,FX)\quad\textit{for}\quad X,Y\in\mathbf{D}^b(\operatorname{mod}\Lambda).$$

Then $\operatorname{gl.dim}\Lambda<\infty$.

Proof. Denote by $S_1,S_2,\dots,S_r$ the simple Λ-modules. We need to check that for each pair i,j, we have $\operatorname{Ext}^l_\Lambda(S_i,S_j)=0$ for $l\gg 0$. We compute

$$D\operatorname{Ext}^l_\Lambda(S_i,S_j)\cong D\operatorname{Hom}_{\mathbf{D}^b(\operatorname{mod}\Lambda)}(S_i,S_j[l])\cong\operatorname{Hom}_{\mathbf{D}^b(\operatorname{mod}\Lambda)}(S_j,FS_i[-l]).$$

Now use that for any pair X,Y of bounded complexes, we have

$$\operatorname{Hom}_{\mathbf{D}^b(\operatorname{mod}\Lambda)}(X,Y[-l])=0\quad\text{for}\quad l\gg 0.$$

☐

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