THICK SUBCATEGORIES OF MODULES OVER COMMUTATIVE NOETHERIAN RINGS

(WITH AN APPENDIX BY SRIKANTH IYENGAR)

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Abstract. For a commutative noetherian ring A, we compare the support of a complex of A-modules with the support of its cohomology. This leads to a classification of all full subcategories of A-modules which are thick (that is, closed under taking kernels, cokernels, and extensions) and closed under taking arbitrary direct sums. In addition, subcategories of A-modules that are closed under taking submodules, extensions, and direct unions are classified via associated prime ideals.

1. Introduction

Let A be a commutative noetherian ring. We consider the category $\operatorname{Mod} A$ of A-modules and the spectrum $\operatorname{Spec} A$ of prime ideals of A. Given a complex X of A-modules, we wish to relate its support (in the sense of Foxby [6])

$$\operatorname{Supp} X = \{\mathfrak{p} \in \operatorname{Spec} A \mid X \otimes^{\mathbf{L}}_A k(\mathfrak{p}) \neq 0\}$$

to the support of its cohomology

$$\operatorname{Supp} H^* X = \bigcup_{i \in \mathbb{Z}} \operatorname{Supp} H^i X.$$

In some cases we have the equality

$$\operatorname{Supp} X = \operatorname{Supp} H^* X,$$

for example when A is a Dedekind domain or when H^*X is finitely generated over A. The failure of this equality is the main theme of recent joint work with Benson and Iyengar [2], which is motivated by the study of support varieties of modular representations. In this paper we establish a closely related classification of thick subcategories of $\operatorname{Mod} A$ and prove the following result.

Theorem 1.1. *For a subset Φ of $\operatorname{Spec} A$ the following conditions are equivalent:*

1. *For every complex X of A-modules we have*
$$\operatorname{Supp} X \subseteq \Phi \iff \operatorname{Supp} H^* X \subseteq \Phi.$$
2. *The A-modules M with $\operatorname{Supp} M \subseteq \Phi$ form a thick subcategory of $\operatorname{Mod} A$.*
3. *Every map $I^0 \to I^1$ between injective A-modules with $\operatorname{Ass} I^i \subseteq \Phi$ $(i = 0, 1)$ can be completed to an exact sequence $I^0 \to I^1 \to I^2$ such that I^2 is injective and $\operatorname{Ass} I^2 \subseteq \Phi$.*

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Recall that a classical result of Gabriel [8] provides a bijection between the set of localizing subcategories of $\operatorname{Mod} A$ and the set of specialization closed subsets of $\operatorname{Spec} A$. More recently, a number of authors studied subcategories of $\operatorname{Mod} A$ in terms of subsets of $\operatorname{Spec} A$; see [9, 11, 16]. In this paper we generalize Gabriel's result in two directions. The first direction (via associated primes) is discussed in Section 2; it is fairly elementary but seems to be new. The second direction (via support) leads to a classification of thick subcategories of $\operatorname{Mod} A$ and corrects a result of Hovey in [11]; see Theorem 3.1. This classification is formulated in terms of subsets $\Phi \subseteq \operatorname{Spec} A$ satisfying the equivalent conditions of Theorem 1.1. We call such subsets *coherent* and establish the following.

Theorem 1.2. *For a commutative noetherian ring A the following conditions are equivalent:*

1. *The Krull dimension of A is at most one.*
2. *Every subset of* $\operatorname{Spec} A$ *is coherent.*
3. $\operatorname{Supp} X = \operatorname{Supp} H^* X$ *for every complex X of A-modules.*

The proof of this theorem is illustrated by some explict examples of subsets of $\operatorname{Spec} A$ which are not coherent. It would be interesting to have a geometric interpretation of coherent subsets in terms of the Zariski topology on $\operatorname{Spec} A$.

2. Subcategories via associated primes

Let M be an A-module. Recall that a prime ideal $\mathfrak{p}$ is *associated* to M, if $A/\mathfrak{p}$ is isomorphic to a submodule of M. We denote by $\operatorname{Ass} M$ the set of all prime ideals which are associated to M.

Theorem 2.1. *The map sending a subcategory $\mathcal{C}$ of* $\operatorname{Mod} A$ *to*

$$\operatorname{Ass} \mathcal{C} = \bigcup_{M \in \mathcal{C}} \operatorname{Ass} M$$

induces an inclusion preserving bijection between the set of full subcategories of $\operatorname{Mod} A$, *which are closed under taking submodules, extensions, and direct unions, and the set of subsets of* $\operatorname{Spec} A$. *The inverse map sends a subset Φ of* $\operatorname{Spec} A$ *to*

$$\operatorname{Ass}^{-1} \Phi = \{M \in \operatorname{Mod} A \mid \operatorname{Ass} M \subseteq \Phi\}.$$

Let us reformulate this result in elementary terms.

Theorem 2.2. *Let M and N be a A-modules. Then N can be generated from M via taking submodules, extensions, and direct unions if and only if* $\operatorname{Ass} N \subseteq \operatorname{Ass} M$.

Here, the term *direct union* refers to a direct limit of a system of submodules. We prove Theorem 2.1; then Theorem 2.2 is an immediate consequence. The proof uses some basic facts about associated primes and the structure of injective modules. The injective envelope of a module M is denoted by $E(M)$ and we observe that $\operatorname{Ass} E(M) = \operatorname{Ass} M$.

Lemma 2.3. *Let M be an A-module. Given a submodule $N \subseteq M$ and a family (M_i) of submodules satisfying $M = \bigcup_i M_i$, we have*

$$\operatorname{Ass} N \subseteq \operatorname{Ass} M \subseteq \operatorname{Ass} N \cup \operatorname{Ass} M/N \quad \textit{and} \quad \operatorname{Ass} M = \bigcup_i \operatorname{Ass} M_i.$$

Proof. See [4, Chap. IV, §1]. □

For a prime ideal $\mathfrak{p}$ we denote by $k(\mathfrak{p})$ its residue field.

Lemma 2.4. *Let I be an indecomposable injective A-module and $\mathfrak{p}$ its associated prime ideal. Then I is obtained from $A/\mathfrak{p}$ by taking extensions and direct unions. More precisely,*

1. *I is obtained from $k(\mathfrak{p})$ by taking extensions and direct unions, and*
2. *$k(\mathfrak{p})$ is a direct union of copies of $A/\mathfrak{p}$.*

Proof. We sketch the argument and refer to [5, Chap. X, §8] for details. For each integer $n \geq 0$ let I_n denote the submodule of I consisting of all elements annihilated by $\mathfrak{p}^n$. Then we have $I = \bigcup_{n\geq 0} I_n$ and each factor I_{n+1}/I_n is isomorphic to a finite direct sum of copies of the residue field $k(\mathfrak{p})$. Now observe that $k(\mathfrak{p})$ is the field of fractions of $A/\mathfrak{p}$ and therefore a direct union of the form

$$k(\mathfrak{p}) = \bigcup_{0\neq x\in A/\mathfrak{p}} x^{-1}A/\mathfrak{p}. \qquad \square$$

Proof of Theorem 2.1. Let Φ be a subset of $\operatorname{Spec} A$. Then the subcategory $\operatorname{Ass}^{-1}\Phi$ is closed under taking submodules, extensions, and direct unions, by Lemma 2.3. Clearly, we have

$$\operatorname{Ass}(\operatorname{Ass}^{-1}\Phi) = \Phi.$$

Now let $\mathcal{C}$ be a subcategory of $\operatorname{Mod} A$, which is closed under taking submodules, extensions, and direct unions. We claim that

$$\operatorname{Ass}^{-1}(\operatorname{Ass}\mathcal{C}) = \mathcal{C}.$$

The inclusion $\operatorname{Ass}^{-1}(\operatorname{Ass}\mathcal{C}) \supseteq \mathcal{C}$ is clear. Now suppose that M is a module contained in $\operatorname{Ass}^{-1}(\operatorname{Ass}\mathcal{C})$. Then its injective envelope $E(M)$ is a direct sum of copies of the form $E(A/\mathfrak{p})$ with $\mathfrak{p} \in \operatorname{Ass}\mathcal{C}$, since $\operatorname{Ass} E(M) = \operatorname{Ass} M$. But $\mathfrak{p} \in \operatorname{Ass}\mathcal{C}$ implies $A/\mathfrak{p} \in \mathcal{C}$, and therefore $E(A/\mathfrak{p})$ belongs to $\mathcal{C}$, by Lemma 2.4. It follows that $E(M)$ belongs to $\mathcal{C}$ and therefore $M \in \mathcal{C}$. This finishes the proof. $\square$

We state a number of consequences of Theorem 2.1.

Corollary 2.5. *For a full subcategory $\mathcal{C}$ of $\operatorname{Mod} A$ the following conditions are equivalent.*

1. *$\mathcal{C}$ is closed under taking submodules, extensions, and direct unions.*
2. *There exists a subset Φ of $\operatorname{Spec} A$ such that $\mathcal{C}$ consists of all A-modules M satisfying $\operatorname{Ass} M \subseteq \Phi$.*
3. *There exists an injective A-module I such that $\mathcal{C}$ consists of all A-modules which admit a monomorphism into a direct sum of copies of I.*

Proof. (1) ⇔ (2): This is an immediate consequence of Theorem 2.1.

(2) ⇒ (3): Take $I = \bigoplus_{\mathfrak{p}\in\Phi} E(A/\mathfrak{p})$. Then $\operatorname{Ass} M \subseteq \Phi$ for every submodule M of a direct sum of copies of I. On the other hand, if $\operatorname{Ass} M \subseteq \Phi$, then $\operatorname{Ass} E(M) \subseteq \Phi$ and therefore $E(M)$ is a submodule of a direct sum of copies of I.

(3) ⇒ (1): Clear. $\square$

Next we restrict the map $\mathcal{C} \mapsto \operatorname{Ass}\mathcal{C}$ to the category $\operatorname{mod} A$ of all finitely generated A-modules.

Corollary 2.6 ([16, Theorem 4.1]). *The map* $\mathcal{D} \mapsto \operatorname{Ass} \mathcal{D}$ *induces an inclusion preserving bijection between the set of full subcategories of* $\operatorname{mod} A$*, which are closed under taking submodules and extensions, and the set of subsets of* $\operatorname{Spec} A$*.*

Proof. We sketch the proof and leave details to the interested reader. Consider the map $\mathcal{C} \mapsto \mathcal{C} \cap \operatorname{mod} A$ between

(i) the set of full subcategories of $\operatorname{Mod} A$, which are closed under taking submodules, extensions, and direct unions, and

(ii) the set of full subcategories of $\operatorname{mod} A$, which are closed under taking submodules and extensions.

This map is bijective; its inverse sends a subcategory $\mathcal{D}$ from (ii) to the full subcategory of $\operatorname{Mod} A$ consisting of all direct unions of modules in $\mathcal{D}$. The composition of the first map $\mathcal{C} \mapsto \mathcal{C} \cap \operatorname{mod} A$ with $\mathcal{D} \mapsto \operatorname{Ass} \mathcal{D}$ is the bjection from Theorem 2.1. Thus $\mathcal{D} \mapsto \operatorname{Ass} \mathcal{D}$ is bijective. ☐

Recall that a full subcategory $\mathcal{C}$ of $\operatorname{Mod} A$ is *localizing* if $\mathcal{C}$ is closed under taking submodules, factor modules, extensions, and arbitrary direct sums. A subset Φ of $\operatorname{Spec} A$ is *specialization closed* if for any pair $\mathfrak{p} \subseteq \mathfrak{q}$ of prime ideals, $\mathfrak{p} \in \Phi$ implies $\mathfrak{q} \in \Phi$.

Corollary 2.7 ([8, p. 425]). *The map* $\mathcal{C} \mapsto \operatorname{Ass} \mathcal{C}$ *induces an inclusion preserving bijection between the set of localizing subcategories of* $\operatorname{Mod} A$ *and the set of specialization closed subsets of* $\operatorname{Spec} A$*.*

Proof. Suppose that $\mathcal{C}$ is localizing and let $\mathfrak{p} \subseteq \mathfrak{q}$ be prime ideals with $\mathfrak{p}$ in $\Phi = \operatorname{Ass} \mathcal{C}$. Then $A/\mathfrak{p} \in \mathcal{C}$ and therefore $A/\mathfrak{q} \in \mathcal{C}$, because $A/\mathfrak{q}$ is a factor module of $A/\mathfrak{p}$. Thus $\mathfrak{q} \in \Phi$, and we have that Φ is specialization closed.

Now suppose that $\Phi \subseteq \operatorname{Spec} A$ is specialization closed and let $N \subseteq M$ be A-modules with M in $\mathcal{C} = \operatorname{Ass}^{-1} \Phi$. Then M/N belongs to $\mathcal{C}$, since

$$\operatorname{Ass} M/N \subseteq \{\mathfrak{p} \in \operatorname{Spec} A \mid (M/N)_{\mathfrak{p}} \neq 0\} \subseteq \{\mathfrak{p} \in \operatorname{Spec} A \mid M_{\mathfrak{p}} \neq 0\} \subseteq \Phi$$

where the last inclusion uses that Φ is specialization closed. Thus $\mathcal{C}$ is localizing. ☐

3. Subcategories via support

A full subcategory $\mathcal{C}$ of $\operatorname{Mod} A$ is called *thick* if for each exact sequence

$$M_1 \to M_2 \to M_3 \to M_4 \to M_5$$

of A-modules with M_i in $\mathcal{C}$ for $i = 1, 2, 4, 5$, the module M_3 belongs to $\mathcal{C}$. It is clear that $\mathcal{C}$ is thick if and only if $\mathcal{C}$ is closed under taking kernels, cokernels, and extensions. A thick subcategory is an abelian category and the inclusion functor is exact.

We wish to classify all thick subcategories of $\operatorname{Mod} A$ which are closed under arbitrary taking direct sums. For this a few definitions are needed.

Let M be an A-module. Following [6], the *support* of M is by definition

$$\operatorname{Supp} M = \{\mathfrak{p} \in \operatorname{Spec} A \mid \operatorname{Tor}^A_*(M, k(\mathfrak{p})) \neq 0\}.$$

Note that $\operatorname{Supp} M \subseteq \{\mathfrak{p} \in \operatorname{Spec} A \mid M_{\mathfrak{p}} \neq 0\}$ and equality holds if M is finitely generated. For example, $\operatorname{Supp} I = \operatorname{Ass} I$ for every injective A-module I; see Lemma 3.3 for a generalization.

Let Φ be a subset of $\operatorname{Spec} A$ and define the full subcategory

$$\operatorname{Inj}_{\Phi} A = \{I \in \operatorname{Mod} A \mid I \text{ is injective and } \operatorname{Ass} I \subseteq \Phi\}.$$

We call Φ *coherent*[1] if each morphism $I^0 \to I^1$ in $\operatorname{Inj}_\Phi A$ can be completed to an exact sequence $I^0 \to I^1 \to I^2$ with I^2 in $\operatorname{Inj}_\Phi A$. For example, each specialization closed subset of $\operatorname{Spec} A$ is coherent.

Theorem 3.1. *The map sending a subcategory $\mathcal{C}$ of $\operatorname{Mod} A$ to*

$$\operatorname{Supp} \mathcal{C} = \bigcup_{M \in \mathcal{C}} \operatorname{Supp} M$$

induces an inclusion preserving bijection between the set of full subcategories of $\operatorname{Mod} A$, which are thick and closed under taking arbitrary direct sums, and the set of coherent subsets of $\operatorname{Spec} A$. The inverse map sends a subset Φ of $\operatorname{Spec} A$ to

$$\operatorname{Supp}^{-1} \Phi = \{M \in \operatorname{Mod} A \mid \operatorname{Supp} M \subseteq \Phi\}.$$

Let us give an example of a set of prime ideals which is not coherent. This is based on an example from [2] and provides a counterexample to Hovey's classification of thick subcategories closed under arbitrary direct sums in [11, Theorem 5.2].

Example 3.2. Let k be a field and $A = k[\![x, y]\!]$. Then $\Phi = \operatorname{Spec} A \setminus \{\mathfrak{m}\}$ with $\mathfrak{m} = (x, y)$ is not coherent. To see this, let

$$0 \to A \to E(A) \to \bigoplus_{\operatorname{ht}\mathfrak{p}=1} E(A/\mathfrak{p}) \to E(A/\mathfrak{m}) \to 0$$

be a minimal injective resolution of A. Then the morphism

$$E(A) \to \bigoplus_{\operatorname{ht}\mathfrak{p}=1} E(A/\mathfrak{p})$$

cannot be completed to an exact sequence lying in $\operatorname{Inj}_\Phi A$, because its cokernel $E(A/\mathfrak{m})$ does not belong to $\operatorname{Inj}_\Phi A$.

It should be clear that one can construct such examples more generally for commutative noetherian rings of Krull dimension at least two. On the other hand, if A is a Dedekind domain, then all subsets of $\operatorname{Spec} A$ are coherent, because every factor module of an injective A-module is injective and therefore the cokernel of a morphism $I^0 \to I^1$ between injective A-modules is up to isomorphism a direct summand of I^1. We refer to Section 4 for details about coherent subsets.

The proof of Theorem 3.1 uses an alternative description of the support of a module. This involves the derived category $\mathbf{D}(\operatorname{Mod} A)$ of $\operatorname{Mod} A$. Given two complexes X and Y of A-modules, we write $X \otimes_A^{\mathbf{L}} Y$ for their tensor product in $\mathbf{D}(\operatorname{Mod} A)$. Note that for every A-module M, we have

$$\operatorname{Supp} M = \{\mathfrak{p} \in \operatorname{Spec} A \mid M \otimes_A^{\mathbf{L}} k(\mathfrak{p}) \neq 0\}.$$

The following lemma is due to Foxby. The proof given here is inspired by Neeman's work [14, §2]; see Proposition 5.1 for a more general statement.

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[1] The term *coherent* refers to the characterizing property of a coherent ring that every morphism $P_1 \to P_0$ between finitely generated projective modules can be completed to an exact sequence $P_2 \to P_1 \to P_0$ such that P_2 is finitely generated projective.

Lemma 3.3 ([6, Remark 2.9])**.** *Let M be an A-module. Given a minimal injective resolution I^* of M, we have*

$$\operatorname{Supp} M = \bigcup_{i \geq 0} \operatorname{Ass} I^i.$$

Proof. First observe that we can pass from M to the complex I, because the morphism $M \to I$ induces an isomorphism in $\mathbf{D}(\operatorname{Mod} A)$. Fix a prime ideal $\mathfrak{p}$. Recall that each injective A-module J admits a unique decomposition

$$J = \bigoplus_{\mathfrak{q} \text{ prime}} \Gamma_{\mathfrak{q}} J$$

such that $\operatorname{Ass} \Gamma_{\mathfrak{q}} J \subseteq \{\mathfrak{q}\}$ for all $\mathfrak{q}$. We denote by $\Gamma_{\mathfrak{p}} I$ the complex which is obtained from I by taking in each degree the component with associated prime $\mathfrak{p}$. To be precise, $\Gamma_{\mathfrak{p}} I$ is the subcomplex of $I_{\mathfrak{p}} = I \otimes_A A_{\mathfrak{p}}$ supported at the closed point $\mathfrak{p}$. Note that the sequence $I \to I_{\mathfrak{p}} \leftarrow \Gamma_{\mathfrak{p}} I$ of canonical morphisms is degreewise a split epimorphism, followed by a split monomorphism. In particular, it induces an isomorphism

$$I \otimes_A^{\mathbf{L}} k(\mathfrak{p}) \cong I_{\mathfrak{p}} \otimes_A^{\mathbf{L}} k(\mathfrak{p}) \cong \Gamma_{\mathfrak{p}} I \otimes_A^{\mathbf{L}} k(\mathfrak{p}),$$

since $I' \otimes_A^{\mathbf{L}} k(\mathfrak{p}) = 0$ for the kernel I' of $I \to I_{\mathfrak{p}}$ and $I'' \otimes_A^{\mathbf{L}} k(\mathfrak{p}) = 0$ for the cokernel I'' of $\Gamma_{\mathfrak{p}} I \to I_{\mathfrak{p}}$.

Suppose first that $I \otimes_A^{\mathbf{L}} k(\mathfrak{p}) \neq 0$. Then $\Gamma_{\mathfrak{p}} I \neq 0$ and therefore $\mathfrak{p} \in \operatorname{Ass} I^i$ for some i. Now suppose that $I \otimes_A^{\mathbf{L}} k(\mathfrak{p}) = 0$. Then $\Gamma_{\mathfrak{p}} I = 0$ by [14, Lemma 2.14]. We want to conclude that $\Gamma_{\mathfrak{p}}(I^i) = (\Gamma_{\mathfrak{p}} I)^i = 0$ for all i. Here we need to use the minimality of I. Recall that a complex J of injective A-modules is *minimal* if for all i the kernel of the differential $J^i \to J^{i+1}$ is an essential submodule of J^i. If J is minimal and $J^i = 0$ for $i \ll 0$, then $H^i J = 0$ for all i implies $J^i = 0$ for all i. Observe that $\Gamma_{\mathfrak{p}}$ preserves minimality. Thus $\Gamma_{\mathfrak{p}} I = 0$ in $\mathbf{D}(\operatorname{Mod} A)$ implies $\mathfrak{p} \notin \operatorname{Ass} I^i$ for all i, because I is minimal. ☐

Lemma 3.4. *Let Φ be a coherent subset of* $\operatorname{Spec} A$*. Then*

$$\operatorname{Supp}^{-1} \Phi = \{M \in \operatorname{Mod} A \mid \operatorname{Supp} M \subseteq \Phi\}$$

is a thick subcategory of $\operatorname{Mod} A$*.*

Proof. We consider the full subcategory $\mathcal{C}$ consisting of all A-modules M which fit into an exact sequence $0 \to M \to I^0 \to I^1$ with I^i injective and $\operatorname{Ass} I^i \subseteq \Phi$ for $i = 0, 1$. It is clear that $\mathcal{C}$ is an additive subcategory of $\operatorname{Mod} A$ which is closed under taking kernels. An application of the horseshoe lemma shows that $\mathcal{C}$ is closed under forming extensions. Next observe that $\mathcal{C}$ is closed under taking cokernels. Here we use that Φ is coherent. By definition, the cokernel of a morphism between injective modules in $\mathcal{C}$ belongs to $\mathcal{C}$. A standard argument then shows that this property extends to arbitrary morphisms in $\mathcal{C}$. It follows from Lemma 3.3 that $\operatorname{Supp}^{-1} \Phi = \mathcal{C}$, and therefore $\operatorname{Supp}^{-1} \Phi$ is thick. ☐

Lemma 3.5. *Let $\mathcal{C}$ be a subcategory of* $\operatorname{Mod} A$ *which is thick and closed under taking arbitrary direct sums. Then the injective envelope $E(M)$ belongs to $\mathcal{C}$ for every M in $\mathcal{C}$.*

Proof. Fix M in $\mathcal{C}$. First observe that $\operatorname{Tor}_i^A(M, N)$ belongs to $\mathcal{C}$ for every A-module N and every integer i. This is clear, because for any projective resolution P of N, the terms of the complex $M \otimes_A P$ and therefore its cohomology lies in $\mathcal{C}$. Given $\mathfrak{p} \in \operatorname{Supp} M$, it follows that $k(\mathfrak{p})$ belongs to $\mathcal{C}$, since $\operatorname{Tor}_i^R(M, k(\mathfrak{p}))$ is a direct sum of copies of $k(\mathfrak{p})$.

Then Lemma 2.4 implies that $E(A/\mathfrak{p})$ belongs to $\mathcal{C}$, and we conclude from Lemma 3.3 that $E(M)$ belongs to $\mathcal{C}$. □

Proof of Theorem 3.1. Let Φ be a coherent subset of $\operatorname{Spec} A$. Then the subcategory $\operatorname{Supp}^{-1}\Phi$ is thick and closed under taking arbitrary direct sums, by Lemma 3.4. Clearly, we have

$$\operatorname{Supp}(\operatorname{Supp}^{-1}\Phi) = \Phi.$$

Now let $\mathcal{C}$ be a subcategory of $\operatorname{Mod} A$, which is thick and closed under taking arbitrary direct sums. Let $\Phi = \operatorname{Supp}\mathcal{C}$. First observe that $\operatorname{Inj}_\Phi A \subseteq \mathcal{C}$, by Lemma 3.5. We claim that Φ is coherent. In deed, each morphism $I^0 \to I^1$ in $\operatorname{Inj}_\Phi A$ can be completed to an exact sequence $I^0 \to I^1 \to I^2$ in $\operatorname{Inj}_\Phi A$ by taking for I^2 the injective envelope of a cokernel of $I^0 \to I^1$. Next we claim that

$$\operatorname{Supp}^{-1}(\operatorname{Supp}\mathcal{C}) = \mathcal{C}.$$

The inclusion $\operatorname{Supp}^{-1}(\operatorname{Supp}\mathcal{C}) \supseteq \mathcal{C}$ is clear. Now suppose that M is a module contained in $\operatorname{Supp}^{-1}(\operatorname{Supp}\mathcal{C})$ and choose a minimal injective resolution I^*. Then $\operatorname{Supp} I^i \subseteq \operatorname{Supp}\mathcal{C}$ for all i. Thus I^0 and I^1 belong to $\mathcal{C}$, and we conclude that M belongs to $\mathcal{C}$. □

The classification of thick subcategories specializes to Gabriel's classification of localizing subcategories.

Corollary 3.6 ([8, p. 425])**.** *The map* $\mathcal{C} \mapsto \operatorname{Supp}\mathcal{C}$ *induces an inclusion preserving bijection between the set of localizing subcategories of* $\operatorname{Mod} A$ *and the set of specialization closed subsets of* $\operatorname{Spec} A$.

4. Coherent subsets of Spec A

In this section we collect some basic properties of coherent subsets of $\operatorname{Spec} A$. Let us fix some notation. Given a multiplicatively closed subset S of A, let $\pi\colon A \to S^{-1}A$ denote the localization. Then we identify $\operatorname{Spec} S^{-1}A$ via π^{-1} with the subset of all prime ideals $\mathfrak{p}$ of A satisfying $S \cap \mathfrak{p} = \emptyset$.

Proposition 4.1. *Let* Φ *be a subset of* $\operatorname{Spec} A$.

1. *Let* (Φ_i) *be a family of coherent subsets of* $\operatorname{Spec} A$. *Then* $\bigcap_i \Phi_i$ *is coherent.*
2. *If* Φ *is specialization closed, then* Φ *is coherent.*
3. *If* $\mathfrak{q} \subseteq \bigcup_{\mathfrak{p}\in\Phi}\mathfrak{p}$ *implies* $\mathfrak{q} \in \Phi$ *for every prime ideal* $\mathfrak{q}$, *then* Φ *is coherent.*
4. *The subset* Φ *is coherent if and only if* $\Phi \cap \operatorname{Spec} A_\mathfrak{p}$ *is a coherent subset of* $\operatorname{Spec} A_\mathfrak{p}$ *for each prime ideal* $\mathfrak{p}$.

Proof. We use that Φ is coherent if and only if the cokernel C of each morphism $I^0 \to I^1$ between injective A-modules with $\operatorname{Ass} I^i \subseteq \Phi$ $(i = 0, 1)$ satisfies $\operatorname{Ass} C \subseteq \Phi$.

(1) Clear.

(2) The assumption on Φ implies that for each pair $N \subseteq M$ of A-modules with $\operatorname{Ass} M \subseteq \Phi$, we have that $\operatorname{Ass} M/N \subseteq \Phi$.

(3) The set

$$S = A \setminus \bigcup_{\mathfrak{p}\in\Phi}\mathfrak{p} = \bigcap_{\mathfrak{p}\in\Phi} A \setminus \mathfrak{p}$$

is multiplicatively closed. The assumption on Φ implies that the localization $A \to S^{-1}A$ identifies all injective $S^{-1}A$-modules with the injective A-modules I satisfying $\operatorname{Ass} I \subseteq \Phi$.

(4) We write $\Phi_\mathfrak{p} = \Phi \cap \operatorname{Spec} A_\mathfrak{p}$ for each prime ideal $\mathfrak{p}$. Suppose first that Φ is coherent and fix a prime ideal $\mathfrak{p}$. Let $I^0 \to I^1$ be a map in $\operatorname{Inj}_{\Phi_\mathfrak{p}} A$. There exist an exact sequence $I^0 \to I^1 \to I^2$ in $\operatorname{Inj}_\Phi A$ and localization at $\mathfrak{p}$ induces an exact sequence in $\operatorname{Inj}_{\Phi_\mathfrak{p}} A$. Thus $\Phi_\mathfrak{p}$ is coherent. Now suppose that Φ is not coherent. It follows that there exists an exact sequence

$$I^0 \to I^1 \to C \to 0$$

of A-modules with $I^i \in \operatorname{Inj}_\Phi A$ $(i = 0, 1)$ but $\operatorname{Ass} C \not\subseteq \Phi$. Let $\mathfrak{p} \in \operatorname{Ass} C \setminus \Phi$. Then we localize at $\mathfrak{p}$ and obtain an exact sequence

$$I^0_\mathfrak{p} \to I^1_\mathfrak{p} \to C_\mathfrak{p} \to 0$$

of $A_\mathfrak{p}$-modules with $I^i_\mathfrak{p} \in \operatorname{Inj}_{\Phi_\mathfrak{p}} A$ $(i = 0, 1)$ but

$$\operatorname{Ass} C_\mathfrak{p} = (\operatorname{Ass} C) \cap \operatorname{Spec} A_\mathfrak{p} \not\subseteq \Phi_\mathfrak{p}.$$

Thus $\Phi_\mathfrak{p}$ is a subset of $\operatorname{Spec} A_\mathfrak{p}$ which is not coherent. □

Remark 4.2. (1) A subset Φ of $\operatorname{Spec} A$ satisfies the condition (3) of Proposition 4.1 if and only if it is of the form $\operatorname{Spec} S^{-1}A$ for some multiplicatively closed subset S.

(2) The union of two coherent subsets need not to be coherent. For instance, Example 3.2 provides a subset which is not coherent but Zariski open. Each Zariski open subset U can be written as the finite union of basic open subsets. However, a basic open set is coherent because it is of the form $\operatorname{Spec} S^{-1}A$ for some multiplicatively closed subset S.

Corollary 4.3. *If the Krull dimension of A is at most one, then every subset of* $\operatorname{Spec} A$ *is coherent.*

The converse of this statement is proved in the appendix of this paper.

Proof. We may assume that A is local, by part (4) of Proposition 4.1, and we denote by $\mathfrak{m}$ the maximal ideal. Let Φ be a subset of $\operatorname{Spec} A$. If Φ contains $\mathfrak{m}$, then Φ is specialization closed and therefore coherent, by part (2) of Proposition 4.1. If $\mathfrak{m}$ is not contained in Φ, then all prime ideals in Φ are minimal and therefore the prime avoidance theorem implies that the condition in part (3) of Proposition 4.1 is satisfied. Thus Φ is coherent. □

Given a prime ideal $\mathfrak{p}$ of A, let

$$V(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} A \mid \mathfrak{p} \subseteq \mathfrak{q}\} \quad \text{and} \quad \Lambda(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} A \mid \mathfrak{q} \subseteq \mathfrak{p}\}.$$

Subsets of the from $V(\mathfrak{p})$ and $\Lambda(\mathfrak{p})$ are coherent. They can be used to build new coherent subsets.

Corollary 4.4. *Let Φ and Ψ be subsets of* $\operatorname{Spec} A$ *and suppose that Ψ is finite. Then*

$$\bigcup_{\mathfrak{p}\in\Phi,\mathfrak{q}\in\Psi} V(\mathfrak{p}) \cap \Lambda(\mathfrak{q})$$

is a coherent subset of $\operatorname{Spec} A$.

Proof. We can express the set as the intersection of two coherent subsets:

$$\bigcup_{\mathfrak{p}\in\Phi,\mathfrak{q}\in\Psi} V(\mathfrak{p}) \cap \Lambda(\mathfrak{q}) = \Big(\bigcup_{\mathfrak{p}\in\Phi} V(\mathfrak{p})\Big) \cap \Big(\bigcup_{\mathfrak{q}\in\Psi} \Lambda(\mathfrak{q})\Big).$$ □

Example 4.5. Let $\mathfrak{p}_1, \mathfrak{p}_2$ be prime ideals. If $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$, then $\{\mathfrak{q} \mid \mathfrak{p}_1 \subseteq \mathfrak{q} \subseteq \mathfrak{p}_2\}$ is coherent. If $\mathfrak{p}_1 \not\subseteq \mathfrak{p}_2$ and $\mathfrak{p}_2 \not\subseteq \mathfrak{p}_1$, then $\{\mathfrak{p}_1, \mathfrak{p}_2\}$ is coherent. In both cases the set is of the form

$$\bigcup_{\mathfrak{p},\mathfrak{q}\in\Phi} V(\mathfrak{p}) \cap \Lambda(\mathfrak{q}) \quad \text{with} \quad \Phi = \{\mathfrak{p}_1, \mathfrak{p}_2\}.$$

5. Support of complexes

Let X be a complex of A-modules. Following [6], the *support* of X is by definition

$$\operatorname{Supp} X = \{\mathfrak{p} \in \operatorname{Spec} A \mid X \otimes^{\mathbf{L}}_A k(\mathfrak{p}) \neq 0\}.$$

We use an alternative description of the support of X. A complex I of injective A-modules together with a quasi-isomorphism $X \to I$ is called a *minimal K-injective resolution* of X, if I is *K-injective* (that is, every morphism from an acyclic complex to I is null homotopic) and I is *minimal* (that is, for all i the kernel of the differential $I^i \to I^{i+1}$ is an essential submodule of I^i).

One can show that each complex of A-modules admits a minimal K-injective resolution; see [15, Theorem 4.5] or [3, Application 2.4] for the existence of a K-injective resolution and [12, Proposition B.2] for the minimality. Note that each acyclic and K-injective complex is null homotopic. Moreover, if a minimal complex I of injective A-modules is null homotopic, then $I^i = 0$ for all i.

The next proposition is the obvious generalization of Lemma 3.3 from modules to complexes of modules. The proof requires only minor modifications; it follows closely [14, §2].

Proposition 5.1. *Let X be a complex of A-modules and $X \to I$ a minimal K-injective resolution of X. Then we have*

$$\operatorname{Supp} X = \bigcup_{i\in\mathbb{Z}} \operatorname{Ass} I^i.$$

Proof. First observe that we can pass from X to the complex I, because the morphism $X \to I$ induces an isomorphism in $\mathbf{D}(\operatorname{Mod} A)$. Fix a prime ideal $\mathfrak{p}$. Recall that each injective A-module J admits a unique decomposition

$$J = \bigoplus_{\mathfrak{q}\text{ prime}} \Gamma_{\mathfrak{q}} J$$

such that $\operatorname{Ass} \Gamma_{\mathfrak{q}} J \subseteq \{\mathfrak{q}\}$ for all $\mathfrak{q}$. We denote by $\Gamma_{\mathfrak{p}} I$ the complex which is obtained from I by taking in each degree the component with associated prime $\mathfrak{p}$. To be precise, $\Gamma_{\mathfrak{p}} I$ is the subcomplex of $I_{\mathfrak{p}} = I \otimes_A A_{\mathfrak{p}}$ supported at the closed point $\mathfrak{p}$. Note that the sequence $I \to I_{\mathfrak{p}} \leftarrow \Gamma_{\mathfrak{p}} I$ of canonical morphisms is degreewise a split epimorphism, followed by a split monomorphism. In particular, it induces an isomorphism

$$I \otimes^{\mathbf{L}}_A k(\mathfrak{p}) \cong I_{\mathfrak{p}} \otimes^{\mathbf{L}}_A k(\mathfrak{p}) \cong \Gamma_{\mathfrak{p}} I \otimes^{\mathbf{L}}_A k(\mathfrak{p}),$$

since $I' \otimes^{\mathbf{L}}_A k(\mathfrak{p}) = 0$ for the kernel I' of $I \to I_{\mathfrak{p}}$ and $I'' \otimes^{\mathbf{L}}_A k(\mathfrak{p}) = 0$ for the cokernel I'' of $\Gamma_{\mathfrak{p}} I \to I_{\mathfrak{p}}$.

Suppose first that $I \otimes^{\mathbf{L}}_A k(\mathfrak{p}) \neq 0$. Then $\Gamma_{\mathfrak{p}} I \neq 0$ and therefore $\mathfrak{p} \in \operatorname{Ass} I^i$ for some i. Now suppose that $I \otimes^{\mathbf{L}}_A k(\mathfrak{p}) = 0$. Then $\Gamma_{\mathfrak{p}} I = 0$ by [14, Lemma 2.14], that is, $\Gamma_{\mathfrak{p}} I$ is acyclic. We want to conclude that $\Gamma_{\mathfrak{p}}(I^i) = (\Gamma_{\mathfrak{p}} I)^i = 0$ for all i. Here we need to use the minimality of I. We observe that $\Gamma_{\mathfrak{p}}$ preserves minimality. Also, $\Gamma_{\mathfrak{p}} I$ is a K-injective complex of injective A-modules. Thus $\Gamma_{\mathfrak{p}} I = 0$ in $\mathbf{D}(\operatorname{Mod} A)$ implies $\mathfrak{p} \notin \operatorname{Ass} I^i$ for all i, because I is minimal. ☐

Theorem 5.2. *For a subset* Φ *of* $\operatorname{Spec} A$ *the following conditions are equivalent:*

(1) Φ *is coherent.*

(2) *For every complex* X *of* A*-modules we have*

$$\operatorname{Supp} X \subseteq \Phi \quad \Longleftrightarrow \quad \operatorname{Supp} H^i X \subseteq \Phi \text{ for all } i \in \mathbb{Z}.$$

(3) *For every complex* X *of* A*-modules we have*

$$\operatorname{Supp} X \subseteq \Phi \quad \Longrightarrow \quad \operatorname{Supp} H^i X \subseteq \Phi \text{ for all } i \in \mathbb{Z}.$$

Proof. (1) $\Rightarrow$ (2): Suppose that Φ is coherent. We use that the A-modules M with $\operatorname{Supp} M \subseteq \Phi$ form a thick subcategory, by Lemma 3.4. Now fix a complex X of A-modules. If $\operatorname{Supp} X \subseteq \Phi$, then we have a quasi-isomorphic complex I of injective A-modules with $\operatorname{Ass} I^i \subseteq \Phi$ for all i, by Proposition 5.1. It follows that

$$\operatorname{Supp} H^i X = \operatorname{Supp} H^i I \subseteq \Phi$$

for all i. Now let $\operatorname{Supp} H^* X \subseteq \Phi$ and $\mathfrak{p} \in \operatorname{Supp} X$. A dévissage argument shows that $\operatorname{Supp} H^*(X \otimes_A^{\mathbf{L}} Y) \subseteq \Phi$ for every complex Y. In particular,

$$\{\mathfrak{p}\} = \operatorname{Supp} H^*(X \otimes_A^{\mathbf{L}} k(\mathfrak{p})) \subseteq \Phi$$

because $X \otimes_A^{\mathbf{L}} k(\mathfrak{p})$ is a direct sum of shifted copies of $k(\mathfrak{p})$. Thus $\operatorname{Supp} X \subseteq \Phi$.

(2) $\Rightarrow$ (3): Clear.

(3) $\Rightarrow$ (1): Let $I^0 \to I^1$ be a morphism of injective A-modules with $\operatorname{Ass} I^i \subseteq \Phi$ for $i = 0, 1$. Viewing I as a complex, we have $\operatorname{Supp} I \subseteq \Phi$ by Proposition 5.1, and therefore $\operatorname{Supp} H^0 I \subseteq \Phi$. It follows from Lemma 3.3 that we can complete $I^0 \to I^1$ to an injective resolution

$$0 \to H^0 I \to I^0 \to I^1 \to I^2 \to \cdots$$

of $H^0 I$ with $\operatorname{Ass} I^i \subseteq \Phi$ for all i. Thus Φ is coherent. ☐

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Appendix A. Noncoherent subsets of $\operatorname{Spec} A$

By Srikanth Iyengar

In this appendix, we establish the converse of Corollary 4.3. To this end, we recall some standard notions from commutative algebra; this serves also to fix notation.

Definition A.1. Let A be a commutative noetherian ring, $\mathfrak{a}$ an ideal in A, and let M be an A-module. The $\mathfrak{a}$-*depth* of M is the number

$$\operatorname{depth}_A(\mathfrak{a}, M) = \inf\{n \mid \operatorname{Ext}_A^n(A/\mathfrak{a}, M) \neq 0\}.$$

This invariant of M can also be detected from its Koszul homology on a finite set of elements generating $\mathfrak{a}$, and also its local cohomology with respect to $\mathfrak{a}$; see, for instance, [7, Theorem 2.1]. When M is finitely generated and $\mathfrak{a}M \neq M$, this number coincides with the length of the longest M-regular sequence in $\mathfrak{a}$; see [13, Theorem 28].

As usual, if A is local, with maximal ideal $\mathfrak{m}$, we write $\operatorname{depth}_A M$ for the $\mathfrak{m}$-depth of M, and call it the *depth* of M.

We record the following standard properties of depth, for ease of reference.

Lemma A.2. *Let $\mathfrak{a}$ be an ideal in a commutative noetherian ring A, and let M be an A-module. The following statements hold.*

1. *One has an equality,* $\operatorname{depth}_A(\sqrt{\mathfrak{a}}, M) = \operatorname{depth}_A(\mathfrak{a}, M)$.
2. *With I^* the minimal injective resolution of M, for each prime ideal $\mathfrak{p}$ one has*
$$\operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p} = \inf\{n \mid E(A/\mathfrak{p}) \textit{ is a direct summand in } I^n\}\,.$$
3. *If $A \to B$ is a homomorphism of rings and N is a B-module, then viewing N as an A-module by restriction of scalars, one has*
$$\operatorname{depth}_A(\mathfrak{a}, N) = \operatorname{depth}_B(\mathfrak{a}B, N)\,.$$

Proof. For (1) see, for instance, [7, Proposition 2.11], while (3) is evident, if one computes depth using Koszul complexes, or via local cohomology; see [7, Theorem 2.1]. Part (2) holds as $(I^*)_\mathfrak{p}$ is the minimal injective resolution of $M_\mathfrak{p}$ over $A_\mathfrak{p}$, so the complex $\operatorname{Hom}_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}, (I^*)_\mathfrak{p})$, whose first nonzero cohomology module occurs in degree $\operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p}$, has zero differential. ☐

We need the following result of Auslander and Buchsbaum, which is implicit in [1].

Proposition A.3. *Let A be a commutative noetherian ring with* $\dim A \geq 1$. *There exists a prime $\mathfrak{p}$ in* $\operatorname{Spec} A$ *such that*
$$\operatorname{depth} A_\mathfrak{p} = \dim A_\mathfrak{p} = \dim A - 1.$$

Proof. The proof uses an induction on $\dim A$. When $\dim A = 1$, for $\mathfrak{p}$ one may take any minimal prime of A. This is the basis of the induction.

Assume $\dim A \geq 2$, and let $\mathfrak{m}$ be the maximal ideal of A. Since $\operatorname{Ass} A$ is finite, the prime avoidance theorem implies that the following set in nonempty:
$$\mathfrak{m} \setminus \bigcup_{\substack{\mathfrak{p}\in \operatorname{Ass} A\\ \mathfrak{p}\neq\mathfrak{m}}} \mathfrak{p}\,.$$
Choose an element x in it. One then has that $\dim(A/Ax) = \dim A - 1$, so the induction hypothesis yields a prime $\mathfrak{p}$ in A, containing x, such that
$$\operatorname{depth}(A_\mathfrak{p}/A_\mathfrak{p}x) = \dim(A_\mathfrak{p}/A_\mathfrak{p}x) = \dim(A/Ax) - 1 = \dim A - 2\,.$$
Observe that $\mathfrak{p} \neq \mathfrak{m}$, so the choice of x ensures that it is a nonzero divisor in $A_\mathfrak{p}$. Thus one has $\operatorname{depth} A_\mathfrak{p} = \dim A_\mathfrak{p} = \dim A - 1$. This completes the induction argument. ☐

The gist of the result below is well-known; we provide a proof for lack of a suitable reference for this formulation.

Theorem A.4. *Let A be a commutative noetherian ring with* $\dim A \geq 2$. *There exists an A-module M and a prime $\mathfrak{p}$ in* $\operatorname{Spec} A$ *such that* $\operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p} = \max\{2, \dim A - 1\}$.

Proof. When $\dim A \geq 3$, we apply Proposition A.3, and take for $\mathfrak{p}$ any prime such that $\operatorname{depth} A_{\mathfrak{p}} = \dim A - 1$ and set $M = A$. Henceforth, we assume $\dim A = 2$. Choosing a prime ideal $\mathfrak{p}$ in A with $\dim A_{\mathfrak{p}} = 2$, and replacing A with $A_{\mathfrak{p}}$, we may assume also that A is local; the goal then is to find an A-module M such that $\operatorname{depth}_A M = 2$. At this point, one may refer to, for instance, Hochster's article [10], especially Section 3. We provide details, for completeness.

Let $\widehat{A}$ be the completion of A at its maximal ideal, say $\mathfrak{m}$. One has $\dim \widehat{A} = 2$, so there exists a prime ideal $\mathfrak{a}$ in $\widehat{A}$ with $\dim(\widehat{A}/\mathfrak{a}) = 2$. Consider the canonical homomorphisms of rings $A \to \widehat{A} \to \widehat{A}/\mathfrak{a}$. Observer that $\mathfrak{m}(\widehat{A}/\mathfrak{a})$ is the maximal ideal of $\widehat{A}/\mathfrak{a}$, so, by Lemma A.2(3), for any module M over $\widehat{A}/\mathfrak{a}$, one has

$$\operatorname{depth}_A M = \operatorname{depth}_{\widehat{A}/\mathfrak{a}} M\,.$$

Replacing A with $\widehat{A}/\mathfrak{a}$ one may assume A is a complete local domain, with $\dim A = 2$.

Let B be the integral closure of A in its field of fractions; the conditions on A imply that B is finite as an A-module, by [13, Corollary 2, pp. 234], so also a two dimensional noetherian ring, by [13, Theorem 20]. The finiteness of the extension $A \subseteq B$ implies that there exists a prime ideal $\mathfrak{q}$ in B with $\dim B_{\mathfrak{q}} = 2$ and $\mathfrak{q} \cap A = \mathfrak{m}$, the maximal ideal of A; see [13, Theorem 5(iii)]. The choice of $\mathfrak{q}$ ensures that $\sqrt{\mathfrak{m}B_{\mathfrak{q}}} = \mathfrak{q}B_{\mathfrak{p}}$, so part (3) and (2) of Lemma A.2 yield the first and second equalities below:

$$\begin{aligned}\operatorname{depth}_A B_{\mathfrak{q}} &= \operatorname{depth}_{B_{\mathfrak{q}}}(\mathfrak{m}B_{\mathfrak{q}}, B_{\mathfrak{q}})\\ &= \operatorname{depth} B_{\mathfrak{q}}\\ &= \dim B_{\mathfrak{q}}\\ &= 2\end{aligned}$$

The penultimate equality holds because $B_{\mathfrak{q}}$ is Cohen-Macaulay, by Serre's criterion for normality; see [13, Theorem 39]. The A-module $B_{\mathfrak{q}}$ has thus the desired depth. □

The result below is a perfect converse to Corollary 4.3.

Corollary A.5. *If A is a commutative noetherian ring with $\dim A \geq 2$, then there exists a subset Φ of $\operatorname{Spec} A$ that is not coherent.*

Proof. By Theorem A.4, there exists an A-module M and a prime ideal $\mathfrak{p}$ in A such that $d = \operatorname{depth}_{A_{\mathfrak{p}}} M_{\mathfrak{p}} \geq 2$. Let I^* be a minimal injective resolution of M, and set

$$\Phi = \operatorname{Ass}_A(I^{d-2}) \cup \operatorname{Ass}_A(I^{d-1})$$

Observe that $\mathfrak{p}$ is in $\operatorname{Ass}_A(I^d)$ but not in Φ, by Lemma A.2(2). Thus, the set Φ is not coherent, as the associated primes of $\operatorname{Coker}(I^{d-2} \to I^{d-1})$ coincide with those of I^d. □

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