THICK SUBCATEGORIES AND VIRTUALLY GORENSTEIN ALGEBRAS

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Abstract. An Artin algebra is by definition virtually Gorenstein if the class of modules which are right orthogonal (with respect to $\operatorname{Ext}^*(-,-)$) to all Gorenstein projective modules coincides with the class of modules which are left orthogonal to all Gorenstein injective modules. We provide a new characterization in terms of finitely generated modules. In addition, an example of an algebra is presented which is not virtually Gorenstein.

1. Introduction

Let Λ be an Artin algebra. We consider the category $\operatorname{Mod}\Lambda$ of right Λ-modules and its full subcategory $\operatorname{mod}\Lambda$ of finitely generated modules. Let $\operatorname{Proj}\Lambda$ denote the full subcategory of $\operatorname{Mod}\Lambda$ consisting of all projective modules and let $\operatorname{proj}\Lambda = \operatorname{Proj}\Lambda \cap \operatorname{mod}\Lambda$. Analogously the subcategories of injective Λ-modules $\operatorname{Inj}\Lambda$ and $\operatorname{inj}\Lambda$ are defined.

Given any exact category $\mathcal{A}$, we call a full subcategory $\mathcal{C}$ of $\mathcal{A}$ *thick* if it is closed under direct factors and has the following two out of three property: for every exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ with two terms in $\mathcal{C}$, the third term belongs to $\mathcal{C}$ as well. Let $\operatorname{Thick}(\mathcal{C})$ denote the smallest thick subcategory of $\mathcal{A}$ which contains $\mathcal{C}$. Take for example $\mathcal{A} = \operatorname{mod}\Lambda$. Then $\operatorname{Thick}(\operatorname{proj}\Lambda)$ consists of all finitely generated Λ-modules having finite projective dimension and $\operatorname{Thick}(\operatorname{inj}\Lambda)$ consists of all finitely generated Λ-modules having finite injective dimension.

The algebra Λ is by definition *right Gorenstein* if Λ viewed as a right module has finite injective dimension, equivalently if $\operatorname{Thick}(\operatorname{proj}\Lambda) \subseteq \operatorname{Thick}(\operatorname{inj}\Lambda)$. Note that it is an open problem, whether any left Gorenstein algebra is right Gorenstein. More specifically, in [2] the following connection with the finitistic dimension conjecture is pointed out.

For a left Gorenstein algebra Λ the following conditions are equivalent.

1. *The algebra Λ is right Gorenstein.*
2. *The finitistic dimension of Λ is finite.*
3. *The subcategory $\operatorname{Thick}(\operatorname{proj}\Lambda)$ of $\operatorname{mod}\Lambda$ is contravariantly finite.*

Recall that a subcategory $\mathcal{X}$ of $\operatorname{mod}\Lambda$ is *contravariantly finite* if every finitely generated Λ-module C has a *right $\mathcal{X}$-approximation* $X \to C$, that is, $X \in \mathcal{X}$ and the induced map $\operatorname{Hom}_\Lambda(X', X) \to \operatorname{Hom}_\Lambda(X', C)$ is surjective for every $X' \in \mathcal{X}$. *Covariantly finite* subcategories are defined dually.

The problem to understand the Gorenstein left-right symmetry provides a motivation for studying the following class of algebras which has been introduced in [5]. An algebra Λ is *virtually Gorenstein* if for every Λ-module X, the functor $\operatorname{Ext}^i_\Lambda(X, -)$ vanishes for all $i > 0$ on all Gorenstein injective Λ-modules if and only if $\operatorname{Ext}^i_\Lambda(-, X)$ vanishes for all

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$i > 0$ on all Gorenstein projective Λ-modules; see (2.2) for details. In this note we provide the following characterization in terms of finitely generated modules, complementing the discussion of virtually Gorenstein algebras in [4].

Theorem 1. *For an Artin algebra Λ the following are equivalent.*

1. *The algebra Λ is virtually Gorenstein.*
2. *The subcategory $\operatorname{Thick}(\operatorname{proj}\Lambda \cup \operatorname{inj}\Lambda)$ of $\operatorname{mod}\Lambda$ is contravariantly finite.*
3. *The subcategory $\operatorname{Thick}(\operatorname{proj}\Lambda \cup \operatorname{inj}\Lambda)$ of $\operatorname{mod}\Lambda$ is covariantly finite.*

An immediate consequence of this result is the fact that Λ is virtually Gorenstein if and only if Λ^{op} is virtually Gorenstein. The proof of the theorem relies on the interplay between cotorsion pairs for modules and torsion pairs for complexes of modules, building on previous work of both authors [4, 8]. The crucial step is a description of the subcategory $\operatorname{Thick}(\operatorname{proj}\Lambda \cup \operatorname{inj}\Lambda)$ which we formulate as a separate result.

Theorem 2. *For a finitely generated Λ-module X the following are equivalent.*

1. *The module X belongs to $\operatorname{Thick}(\operatorname{proj}\Lambda \cup \operatorname{inj}\Lambda)$.*
2. *$\operatorname{Ext}^i_\Lambda(X,-)$ vanishes for all $i > 0$ on all Gorenstein injective Λ-modules.*
3. *$\operatorname{Ext}^i_\Lambda(-,X)$ vanishes for all $i > 0$ on all Gorenstein projective Λ-modules.*

2. Gorenstein projective and injective modules

We recall the definitions of Gorenstein projective and Gorenstein injective modules. These classes of modules induce two cotorsion pairs which give rise to the property of an algebra to be virtually Gorenstein.

2.1. **Cotorsion pairs for abelian categories.** Let $\mathcal{A}$ be an abelian category, for instance $\mathcal{A} = \operatorname{Mod}\Lambda$ or $\mathcal{A} = \operatorname{mod}\Lambda$. Let $\mathcal{X}$ and $\mathcal{Y}$ be classes of objects in $\mathcal{A}$. We define

$$\mathcal{X}^\perp = \{Y \in \mathcal{A} \mid \operatorname{Ext}^i_{\mathcal{A}}(X,Y) = 0 \text{ for all } i > 0 \text{ and } X \in \mathcal{X}\},$$

$$^\perp\mathcal{Y} = \{X \in \mathcal{A} \mid \operatorname{Ext}^i_{\mathcal{A}}(X,Y) = 0 \text{ for all } i > 0 \text{ and } Y \in \mathcal{Y}\}.$$

Now let

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

be an exact sequence in $\mathcal{A}$. The map ψ is called a *special right $\mathcal{X}$-approximation* of C if $B \in \mathcal{X}$ and $A \in \mathcal{X}^\perp$. Dually, the map ϕ is called a *special left $\mathcal{Y}$-approximation* of A if $B \in \mathcal{Y}$ and $C \in {}^\perp\mathcal{Y}$. The following concept of a cotorsion pair originated in the work of Salce. Note that different terminology is used in this context; see for instance [10].

Definition. A *cotorsion pair* for $\mathcal{A}$ is a pair $(\mathcal{X},\mathcal{Y})$ of subcategories of $\mathcal{A}$ satisfying the following conditions:

1. $\mathcal{X} = {}^\perp\mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^\perp$;
2. every object in $\mathcal{A}$ admits a special right $\mathcal{X}$-approximation and a special left $\mathcal{Y}$-approximation.

2.2. **Gorenstein projective and Gorenstein injective modules.** Let $\mathcal{A}$ be an additive category. A complex

$$X = \quad \cdots \longrightarrow X^{i-1} \longrightarrow X^{i} \longrightarrow X^{i+1} \longrightarrow \cdots$$

in $\mathcal{A}$ is called *totally acyclic* if $\operatorname{Hom}_{\mathcal{A}}(A, X)$ and $\operatorname{Hom}_{\mathcal{A}}(X, A)$ are acyclic complexes of abelian groups for all A in $\mathcal{A}$. The following classes of modules have been introduced by Enochs and Jenda in [6], building on the notion of a finitely generated module of G-dimension zero introduced by Auslander in [1].

Definition. A Λ-module C is called

(1) *Gorenstein projective* if it is of the form $C = \operatorname{Coker}(X^{-1} \to X^{0})$ for some totally acyclic complex X of projective Λ-modules, and
(2) *Gorenstein injective* if it is of the fom $C = \operatorname{Ker}(X^{0} \to X^{1})$ for some totally acyclic complex X of injective Λ-modules.

Note that different terminology is used in the literature for Gorenstein projective and Gorenstein injective modules. For instance, Gorenstein projective modules are called modules of G-dimension zero or Cohen-Macaulay modules. In any case, for finitely generated modules we have Gorenstein projective = G-dimension zero; for finitely generated modules over a commutative Noetherian Gorenstein local ring we have G-dimension zero = (maximal) Cohen-Macaulay. However some authors use the three terms synonymously for any (finitely generated or not) module satisfying the first condition of Definition 2.2 (over any ring).

We denote by $\operatorname{GProj}\Lambda$ the full subcategory of $\operatorname{Mod}\Lambda$ which is formed by all Gorenstein projective Λ-modules, and $\operatorname{GInj}\Lambda$ denotes the full subcategory which is formed by all Gorenstein injective Λ-modules.

In [5], it is shown that there are cotorsion pairs

$$(\operatorname{GProj}\Lambda, (\operatorname{GProj}\Lambda)^{\perp}) \quad \text{and} \quad ({}^{\perp}(\operatorname{GInj}\Lambda), \operatorname{GInj}\Lambda)$$

for $\operatorname{Mod}\Lambda$ satisfying

$$\operatorname{GProj}\Lambda \cap (\operatorname{GProj}\Lambda)^{\perp} = \operatorname{Proj}\Lambda \quad \text{and} \quad {}^{\perp}(\operatorname{GInj}\Lambda) \cap \operatorname{GInj}\Lambda = \operatorname{Inj}\Lambda.$$

The algebra Λ is called *virtually Gorenstein* if

$$(\operatorname{GProj}\Lambda)^{\perp} = {}^{\perp}(\operatorname{GInj}\Lambda).$$

We refer to [4] for an extensive discussion of virtually Gorenstein algebras. For example, a left Gorenstein algebra is virtually Gorenstein if and only if it is right Gorenstein. In that case we have

$$(\operatorname{GProj}\Lambda)^{\perp} = \operatorname{Thick}(\operatorname{Proj}\Lambda) = \operatorname{Thick}(\operatorname{Inj}\Lambda) = {}^{\perp}(\operatorname{GInj}\Lambda).$$

Also, every algebra of finite representation type is virtually Gorenstein, and every algebra derived equivalent to a virtually Gorenstein algebra is again virtually Gorenstein.

2.3. **Resolving and coresolving subcategories.** We include for later reference some basic facts about resolving subcategories. Recall that a subcategory $\mathcal{X}$ of an abelian category $\mathcal{A}$ is *resolving* if it contains all projectives, and if for every exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, we have $A, C \in \mathcal{X}$ implies $B \in \mathcal{X}$, and $B, C \in \mathcal{X}$ implies $A \in \mathcal{X}$. *Coresolving* subcategories are defined dually.

Lemma ([4, Theorem 7.12])**.** *Let* $(\mathcal{X}, \mathcal{Y})$ *be a cotorsion pair for* $\operatorname{Mod}\Lambda$*.*

(1) *The subcategory* $\mathcal{X}$ *is resolving.*
(2) *The subcategory* $\mathcal{Y}$ *is resolving if and only* $\mathcal{X} \cap \mathcal{Y} = \operatorname{Proj}\Lambda$.
(3) *If* $\mathcal{Y}$ *is resolving, then* $\mathcal{X} \subseteq \operatorname{GProj}\Lambda$.

Proof. (1) is clear. To prove (2), suppose first that $\mathcal{Y}$ is resolving. Clearly, $\operatorname{Proj}\Lambda \subseteq \mathcal{X} \cap \mathcal{Y}$. Now let $Z \in \mathcal{X} \cap \mathcal{Y}$ and choose an epimorphism $\phi\colon P \to Z$ with P projective. We have $\operatorname{Ker}\phi \in \mathcal{X} \cap \mathcal{Y}$ and therefore ϕ splits. Thus Z is projective and we have $\mathcal{X} \cap \mathcal{Y} = \operatorname{Proj}\Lambda$. Now suppose that $\mathcal{X} \cap \mathcal{Y} = \operatorname{Proj}\Lambda$. Then every epimorphism $U \to V$ with $U, V \in \mathcal{X} \cap \mathcal{Y}$ splits. This implies that $\mathcal{Y}$ is a thick subcategory; see [9, Lemma 3.2]. In particular, $\mathcal{Y}$ is resolving.

To prove (3), suppose again that $\mathcal{Y}$ is resolving and fix $X \in \mathcal{X}$. We choose a projective resolution

$$\cdots \longrightarrow P^{-3} \xrightarrow{\delta^{-3}} P^{-2} \xrightarrow{\delta^{-2}} P^{-1} \longrightarrow X \longrightarrow 0$$

of X and complete this inductively to a totally acyclic complex (P, δ) of projective modules as follows. Choose for each $i \geq -1$ a special left $\mathcal{Y}$-approximation $\operatorname{Coker}\delta^{i-1} \to P^{i+1}$ and take for δ^i the composition $P^i \to \operatorname{Coker}\delta^{i-1} \to P^{i+1}$. Note that each P^i is projective since a special left $\mathcal{Y}$-approximation of an object in $\mathcal{X}$ belongs to $\mathcal{X} \cap \mathcal{Y}$. □

We mention that there is a dual result about coresolving subcategories and Gorenstein injective modules.

3. Categories of complexes and torsion pairs

3.1. **The category $\mathbf{K}(\operatorname{Inj}\Lambda)$.** Let $\mathcal{A}$ be an additive category. We denote by $\mathbf{K}(\mathcal{A})$ the homotopy category of complexes in $\mathcal{A}$. If $\mathcal{A}$ is abelian, then $\mathbf{D}(\mathcal{A})$ denotes the derived category of $\mathcal{A}$ which is obtained from $\mathbf{K}(\mathcal{A})$ by formally inverting all quasi-isomorphisms. Given any category $\mathcal{T}$ with small coproducts, we denote by $\mathcal{T}^c$ the full subcategory which is formed by all compact objects. Recall that an object X is *compact* if $\operatorname{Hom}_{\mathcal{T}}(X, -)$ preserves small coproducts, equivalently if every map $X \to \coprod_i Y_i$ into a small coproduct factors through a finite coproduct of Y_is.

Proposition ([8, Proposition 2.3])**.** *The triangulated category* $\mathbf{K}(\operatorname{Inj}\Lambda)$ *is compactly generated. The canonical functor* $\mathbf{K}(\operatorname{Inj}\Lambda) \to \mathbf{D}(\operatorname{Mod}\Lambda)$ *induces an equivalence*

$$\mathbf{K}(\operatorname{Inj}\Lambda)^c \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod}\Lambda).$$

We shall use the functor

$$\mathbf{i}\colon \operatorname{Mod}\Lambda \longrightarrow \mathbf{K}(\operatorname{Inj}\Lambda)$$

which sends a Λ-module to its injective resolution. More generally, the canonical functor $\mathbf{K}(\operatorname{Inj}\Lambda) \to \mathbf{D}(\operatorname{Mod}\Lambda)$ has a right adjoint sending a complex X to its semi-injective resolution $\mathbf{i}X$. This right adjoint induces an equivalence $\mathbf{D}^b(\operatorname{mod}\Lambda) \xrightarrow{\sim} \mathbf{K}(\operatorname{Inj}\Lambda)^c$ which is a quasi-inverse for the equivalence $\mathbf{K}(\operatorname{Inj}\Lambda)^c \xrightarrow{\sim} \mathbf{D}^b(\operatorname{mod}\Lambda)$.

The following functor takes complexes to modules. We define

$$\mathbf{s}\colon \mathbf{K}(\operatorname{Inj}\Lambda) \longrightarrow \overline{\operatorname{Mod}}\,\Lambda, \quad X \mapsto \operatorname{Ker}(X^0 \to X^1),$$

where $\overline{\operatorname{Mod}}\,\Lambda$ denotes the *stable category modulo injectives*. The objects of $\overline{\operatorname{Mod}}\,\Lambda$ are those of $\operatorname{Mod}\Lambda$ and the morphisms for a pair X, Y of Λ-modules are by definition

$$\overline{\operatorname{Hom}}_\Lambda(X, Y) = \operatorname{Hom}_\Lambda(X, Y)/\mathcal{I}(X, Y),$$

where $\mathcal{I}(X, Y)$ denotes the subgroup of morphisms which factor through an injective module. Analogously, the full subcategories $\overline{\operatorname{GInj}}\,\Lambda$ and $\overline{\operatorname{mod}}\,\Lambda$ of $\overline{\operatorname{Mod}}\,\Lambda$, and the stable category $\underline{\operatorname{Mod}}\,\Lambda$ modulo projectives are defined.

Remark. The category $\mathbf{K}(\operatorname{Proj}\Lambda)$ is compactly generated, simply because the equivalence $\operatorname{Proj}\Lambda \xrightarrow{\sim} \operatorname{Inj}\Lambda$ induces an equivalence $\mathbf{K}(\operatorname{Proj}\Lambda) \xrightarrow{\sim} \mathbf{K}(\operatorname{Inj}\Lambda)$. However, in general it is not true that the projective resolution $\mathbf{p}X$ of a finitely generated Λ-module X is a compact object in $\mathbf{K}(\operatorname{Proj}\Lambda)$. This is precisely the reason for concentrating on the category $\mathbf{K}(\operatorname{Inj}\Lambda)$ in this work.

3.2. **Totally acyclic complexes.** We denote by $\mathbf{K}_{\mathrm{tac}}(\mathcal{A})$ the full subcategory of $\mathbf{K}(\mathcal{A})$ which is formed by all totally acyclic complexes. The following proposition is the analogue of a result for the category $\mathbf{K}(\operatorname{Proj}\Lambda)$ due to Jørgensen [7].

Proposition ([8, §7])**.** (1) *The functor* $\mathbf{s}$ *induces an equivalence* $\mathbf{K}_{\mathrm{tac}}(\operatorname{Inj}\Lambda) \xrightarrow{\sim} \overline{\operatorname{GInj}}\,\Lambda$.

(2) *The inclusion functor* $\mathbf{K}_{\mathrm{tac}}(\operatorname{Inj}\Lambda) \to \mathbf{K}(\operatorname{Inj}\Lambda)$ *has a left adjoint which induces (via* $\mathbf{s}$*) a left adjoint* $F\colon \overline{\operatorname{Mod}}\,\Lambda \to \overline{\operatorname{GInj}}\,\Lambda$ *of the inclusion* $\overline{\operatorname{GInj}}\,\Lambda \to \overline{\operatorname{Mod}}\,\Lambda$.

(3) *Given* $X \in \operatorname{Mod}\Lambda$*, the adjunction morphism* $X \to FX$ *represents a special left* $\operatorname{GInj}\Lambda$*-approximation of* X.

Note that the equivalence $\mathbf{K}_{\mathrm{tac}}(\operatorname{Inj}\Lambda) \xrightarrow{\sim} \overline{\operatorname{GInj}}\,\Lambda$ restricts to an equivalence

$$\mathbf{K}_{\mathrm{tac}}(\operatorname{inj}\Lambda) \xrightarrow{\sim} \overline{\operatorname{GInj}}\,\Lambda \cap \overline{\operatorname{mod}}\,\Lambda.$$

3.3. **Torsion pairs for triangulated categories.** Let $\mathcal{T}$ be a triangulated category, for instance $\mathcal{T} = \mathbf{K}(\operatorname{Inj}\Lambda)$. A full subcategory $\mathcal{U}$ is *thick* if it is closed under shifts, mapping cones, and direct factors. We denote by $\operatorname{Thick}(\mathcal{U})$ the smallest thick subcategory of $\mathcal{T}$ which contains $\mathcal{U}$, and $\operatorname{Thick}^{\amalg}(\mathcal{U})$ denotes the smallest thick subcategory of $\mathcal{U}$ which contains $\mathcal{U}$ and is closed under small coproducts.

Definition. A pair $(\mathcal{U}, \mathcal{V})$ of thick subcategories of $\mathcal{T}$ forms a *torsion pair* for $\mathcal{T}$ if the following conditions are satisfied:

(1) $\mathcal{U} = \{X \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(X, Y) = 0 \text{ for all } Y \in \mathcal{V}\}$;
(2) $\mathcal{V} = \{Y \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(X, Y) = 0 \text{ for all } X \in \mathcal{U}\}$;
(3) every object $X \in \mathcal{T}$ fits into an exact triangle $X' \to X \to X'' \to$ with $X' \in \mathcal{U}$ and $X'' \in \mathcal{V}$.

3.4. **A torsion pair for $\mathbf{K}(\operatorname{Inj}\Lambda)$.** The totally acyclic complexes of injective Λ-modules form the torsion free part of a torsion pair. This follows from the fact that a complex Y of injective Λ-modules is totally acyclic if and only if

$$\operatorname{Hom}_{\mathbf{K}(\operatorname{Inj}\Lambda)}(\mathbf{i}X, Y[n]) \cong \operatorname{Hom}_{\mathbf{K}(\operatorname{Mod}\Lambda)}(X, Y[n]) \cong H^n \operatorname{Hom}_{\Lambda}(X, Y) = 0$$

for all $n \in \mathbb{Z}$ and all $X \in \operatorname{Mod}\Lambda$ which are projective or injective.

Proposition ([8, §7])**.** *We have a torsion pair*

$$(\mathcal{U}, \mathcal{V}) = \big(\operatorname{Thick}^{\amalg}(\mathbf{i}(\operatorname{proj}\Lambda \cup \operatorname{inj}\Lambda)), \mathbf{K}_{\mathrm{tac}}(\operatorname{Inj}\Lambda)\big)$$

for $\mathbf{K}(\operatorname{Inj}\Lambda)$ *which has the following properties.*

(1) *The pair* $(\mathcal{U}, \mathcal{V})$ *induces a cotorsion pair*

$$(\mathbf{s}\mathcal{U}, \mathbf{s}\mathcal{V}) = ({}^{\perp}(\operatorname{GInj}\Lambda), \operatorname{GInj}\Lambda)$$

for $\operatorname{Mod}\Lambda$ *with*

$$^{\perp}(\operatorname{GInj}\Lambda)\cap\operatorname{GInj}\Lambda=\operatorname{Inj}\Lambda.$$

(2) *Let* $X\in\operatorname{Mod}\Lambda$ *and choose an exact triangle* $X'\xrightarrow{\phi}\mathbf{i}X\xrightarrow{\psi}X''\to$ *in* $\mathbf{K}(\operatorname{Inj}\Lambda)$ *with* $X'\in\mathcal{U}$ *and* $X''\in\mathcal{V}$*. Then* $\mathbf{s}\phi$ *represents a special right* $^{\perp}(\operatorname{GInj}\Lambda)$*-approximation and* $\mathbf{s}\psi$ *represents a special left* $\operatorname{GInj}\Lambda$*-approximation of* X*.*

Let us mention the following dual result (which will not be used in this work). There exists a torsion pair

$$\big(\mathbf{K}_{\mathrm{tac}}(\operatorname{Proj}\Lambda),\operatorname{Thick}^{\Pi}(\mathbf{p}(\operatorname{proj}\Lambda\cup\operatorname{inj}\Lambda))\big)$$

for $\mathbf{K}(\operatorname{Proj}\Lambda)$ which induces via the canonical functor $\mathbf{K}(\operatorname{Proj}\Lambda)\to\underline{\operatorname{Mod}}\,\Lambda$ the cotorsion pair $(\operatorname{GProj}\Lambda,(\operatorname{GProj}\Lambda)^{\perp})$ for $\operatorname{Mod}\Lambda$.

Remark. If a finitely generated Λ-module X admits a special left $\operatorname{GInj}\Lambda$-approximation in $\operatorname{mod}\Lambda$, then X admits a special right $^{\perp}(\operatorname{GInj}\Lambda)$-approximation in $\operatorname{mod}\Lambda$. This follows from the fact that both approximations are connected via an exact triangle $X'\to\mathbf{i}X\to X''\to$ which can be chosen to lie in $\mathbf{K}(\operatorname{inj}\Lambda)$ if $\mathbf{s}X''$ is finitely generated.

3.5. **The finitely generated orthogonal complement of** $\operatorname{GInj}\Lambda$**.**

Theorem. *We have*

$$(\operatorname{GProj}\Lambda)^{\perp}\cap\operatorname{mod}\Lambda=\operatorname{Thick}(\operatorname{proj}\Lambda\cup\operatorname{inj}\Lambda)={}^{\perp}(\operatorname{GInj}\Lambda)\cap\operatorname{mod}\Lambda.$$

Proof. Fix a finitely generated Λ-module X. We use the cotorsion pair $(\mathcal{U},\mathcal{V})$ from Proposition 3.4. Observe that X belongs to $^{\perp}(\operatorname{GInj}\Lambda)$ if and only if $\mathbf{i}X$ belongs to $\mathcal{U}$. This follows from part (2) of Proposition 3.4 if we consider the exact triangle $X'\to\mathbf{i}X\to X''\to$ with $X'\in\mathcal{U}$ and $X''\in\mathcal{V}$. We have $X\in{}^{\perp}(\operatorname{GInj}\Lambda)$ if and only if $\mathbf{s}X''\in{}^{\perp}(\operatorname{GInj}\Lambda)\cap\operatorname{GInj}\Lambda=\operatorname{Inj}\Lambda$, since $\mathbf{si}X=X$ (up to injective direct factors). The complex X'' is totally acyclic and therefore Proposition 3.2 implies that $\mathbf{s}X''\in\operatorname{Inj}\Lambda$ if and only if $X''=0$ if and only if $\mathbf{i}X\in\mathcal{U}$. Next we use the identification

$$\mathbf{D}^b(\operatorname{mod}\Lambda)=\mathbf{K}(\operatorname{Inj}\Lambda)^c$$

via the functor $\mathbf{i}$ sending a complex to its semi-injective resolution; see (3.1). This implies

$$\mathcal{U}\cap\mathbf{K}(\operatorname{Inj}\Lambda)^c=\operatorname{Thick}(\mathbf{i}(\operatorname{proj}\Lambda\cup\operatorname{inj}\Lambda))$$

and we see that $X\in{}^{\perp}(\operatorname{GInj}\Lambda)$ if and only if $\mathbf{i}X\in\operatorname{Thick}(\mathbf{i}(\operatorname{proj}\Lambda\cup\operatorname{inj}\Lambda))$. Applying the correspondence between thick subcategories of $\operatorname{mod}\Lambda$ and $\mathbf{D}^b(\operatorname{mod}\Lambda)$ from the appendix, we conclude that $X\in{}^{\perp}(\operatorname{GInj}\Lambda)$ if and only if $X\in\operatorname{Thick}(\operatorname{proj}\Lambda\cup\operatorname{inj}\Lambda)$.

To complete the proof, observe that we have

$$(\operatorname{GProj}\Lambda)^{\perp}\cap\operatorname{mod}\Lambda={}^{\perp}(\operatorname{GInj}\Lambda)\cap\operatorname{mod}\Lambda$$

by [4, Lemma 8.6]. □

4. Virtually Gorenstein algebras

4.1. **A characterization via thick subcategories.** We apply the results from the preceding section and provide a characterization of virtually Gorenstein algebras in terms of finitely generated modules.

Theorem. *For an Artin algebra* Λ *the following are equivalent.*

(1) *The algebra* Λ *is virtually Gorenstein.*
(2) *The subcategory* $\operatorname{Thick}(\operatorname{proj}\Lambda\cup\operatorname{inj}\Lambda)$ *of* $\operatorname{mod}\Lambda$ *is contravariantly finite.*
(3) *The subcategory* $\operatorname{Thick}(\operatorname{proj}\Lambda\cup\operatorname{inj}\Lambda)$ *of* $\operatorname{mod}\Lambda$ *is covariantly finite.*

Proof. (1) $\Rightarrow$ (2): We adapt the proof of Theorem 8.2 in [4], using Theorem 3.5 as a new ingredient. In fact, the material which we need has been collected in (3.2) and (3.4). The proof is based on the following elementary fact. Given an adjoint pair of functors, the left adjoint preserves compactness provided the right adjoint preserves small coproducts. Note that $(\underline{\operatorname{Mod}}\,\Lambda)^c=\underline{\operatorname{mod}}\,\Lambda$ and $(\overline{\operatorname{Mod}}\,\Lambda)^c=\overline{\operatorname{mod}}\,\Lambda$; see [4, Lemma 6.3]. We apply this fact to the inclusion $\underline{\operatorname{GProj}}\,\Lambda\to\underline{\operatorname{Mod}}\,\Lambda$. This functor has a right adjoint $G\colon\underline{\operatorname{Mod}}\,\Lambda\to\underline{\operatorname{GProj}}\,\Lambda$ such that the adjunction morphism $GX\to X$ represents a special right $\operatorname{GProj}\Lambda$-approximation for every Λ-module X; see (3.2). The equality $(\operatorname{GProj}\Lambda)^\perp={}^\perp(\operatorname{GInj}\Lambda)$ implies that $(\operatorname{GProj}\Lambda)^\perp$ is closed under small coproducts. Thus a small coproduct of special right $\operatorname{GProj}\Lambda$-approximation is again a special right $\operatorname{GProj}\Lambda$-approximation. It follows that G preserves coproducts and therefore every compact object in $\underline{\operatorname{GProj}}\,\Lambda$ belongs to $\underline{\operatorname{mod}}\,\Lambda$. The equivalence $\operatorname{Proj}\Lambda\xrightarrow{\sim}\operatorname{Inj}\Lambda$ induces an equivalence

$$\underline{\operatorname{Mod}}\,\Lambda\supseteq\underline{\operatorname{GProj}}\,\Lambda\xleftarrow{\sim}\mathbf{K}_{\mathrm{tac}}(\operatorname{Proj}\Lambda)\xrightarrow{\sim}\mathbf{K}_{\mathrm{tac}}(\operatorname{Inj}\Lambda)\xrightarrow{\sim}\overline{\operatorname{GInj}}\,\Lambda\subseteq\overline{\operatorname{Mod}}\,\Lambda$$

which sends compact objects to compact objects and objects in $\underline{\operatorname{mod}}\,\Lambda$ to objects in $\overline{\operatorname{mod}}\,\Lambda$. Thus every compact object in $\overline{\operatorname{GInj}}\,\Lambda$ belongs to $\overline{\operatorname{mod}}\,\Lambda$. On the other hand, the left adjoint $F\colon\overline{\operatorname{Mod}}\,\Lambda\to\overline{\operatorname{GInj}}\,\Lambda$ of the inclusion preserves compactness, since the inclusion preserves small coproducts. Given $X\in\operatorname{Mod}\Lambda$, the adjunction morphism $X\to FX$ represents a special left $\operatorname{GInj}\Lambda$-approximation of X. We conclude that this approximation can be chosen in $\operatorname{mod}\Lambda$ if X is finitely generated. The Remark 3.4 shows that in this case X admits a special right ${}^\perp(\operatorname{GInj}\Lambda)$-approximation in $\operatorname{mod}\Lambda$. Thus

$$\operatorname{Thick}(\operatorname{proj}\Lambda\cup\operatorname{inj}\Lambda)={}^\perp(\operatorname{GInj}\Lambda)\cap\operatorname{mod}\Lambda$$

is a contravariantly finite subcategory of $\operatorname{mod}\Lambda$, thanks to Theorem 3.5.

(2) $\Leftrightarrow$ (3): In [9, Corollary 2.6], it is shown that every resolving and contravariantly finite subcategory of $\operatorname{mod}\Lambda$ is covariantly finite. Dually, every coresolving and covariantly finite subcategory of $\operatorname{mod}\Lambda$ is contravariantly finite.

(2) & (3) $\Rightarrow$ (1): Let $\mathcal{D}=\operatorname{Thick}(\operatorname{proj}\Lambda\cup\operatorname{inj}\Lambda)$. We obtain for $\operatorname{mod}\Lambda$ a cotorsion pair $(\mathcal{C},\mathcal{D})$ because $\mathcal{D}$ is a covariantly finite and coresolving subcategory. This is a well-known fact; see for instance [2] or [9, Lemma 2.1]. Analogously, we obtain a cotorsion pair $(\mathcal{D},\mathcal{E})$ for $\operatorname{mod}\Lambda$ since $\mathcal{D}$ is a contravariantly finite and resolving subcategory. Given any subcategory $\mathcal{X}$ of $\operatorname{mod}\Lambda$, we denote by $\varinjlim\mathcal{X}$ the full subcategory of $\operatorname{Mod}\Lambda$ consisting of all filtered colimits of modules in $\mathcal{X}$. In [9, Theorem 2.4], it is shown that there are cotorsion pairs $(\varinjlim\mathcal{C},\varinjlim\mathcal{D})$ and $(\varinjlim\mathcal{D},\varinjlim\mathcal{E})$ for $\operatorname{Mod}\Lambda$. We claim that

$$(\varinjlim\mathcal{C},\varinjlim\mathcal{D})=(\operatorname{GProj}\Lambda,(\operatorname{GProj}\Lambda)^\perp)\quad\text{and}\quad(\varinjlim\mathcal{D},\varinjlim\mathcal{E})=({}^\perp(\operatorname{GInj}\Lambda),\operatorname{GInj}\Lambda).$$

To prove this claim, first observe that $\varinjlim\mathcal{D}$ is resolving and coresolving. Therefore $\varinjlim\mathcal{C}\subseteq\operatorname{GProj}\Lambda$ and $\varinjlim\mathcal{E}\subseteq\operatorname{GInj}\Lambda$ by Lemma 2.3. Now fix a special right $\mathcal{D}$-approximation $S_\mathcal{D}\to S$ of $S=\Lambda/\operatorname{rad}\Lambda$ and observe that $S_\mathcal{D}$ belongs to ${}^\perp(\operatorname{GInj}\Lambda)$ since $\mathcal{D}\subseteq{}^\perp(\operatorname{GInj}\Lambda)$ by Theorem 3.5. This implies $\varinjlim\mathcal{D}\subseteq{}^\perp(\operatorname{GInj}\Lambda)$ because each object in $\varinjlim \mathcal{D}$ is obtained from $S_{\mathcal{D}}$ by taking small coproducts of copies of $S_{\mathcal{D}}$, forming finitely many extensions, and taking direct factors; see [9, Theorem 2.4]. Analogously, $\varinjlim \mathcal{D} \subseteq (\operatorname{GProj}\Lambda)^{\perp}$ because each object in $\varinjlim \mathcal{D}$ is obtained from a special left $\mathcal{D}$-approximation $S \to S^{\mathcal{D}}$ by taking small products of copies of $S^{\mathcal{D}}$, finitely many extensions, and direct factors. Thus our claim follows and therefore Λ is virtually Gorenstein. □

Let Λ be a virtually Gorenstein algebra. Then we obtain from the preceding proof a description of the subcategory $(\operatorname{GProj}\Lambda)^{\perp} = {}^{\perp}(\operatorname{GInj}\Lambda)$. To formulate this, we use the notation $\operatorname{Thick}^{\Pi}$ and $\operatorname{Thick}^{\amalg}$ to denote thick subcategories of $\operatorname{Mod}\Lambda$ which are closed under small products and coproducts, respectively.

Corollary. *Let Λ be a virtually Gorenstein algebra. Then we have*

$$(\operatorname{GProj}\Lambda)^{\perp} = \operatorname{Thick}^{\Pi}(\operatorname{Proj}\Lambda \cup \operatorname{Inj}\Lambda) = \operatorname{Thick}^{\amalg}(\operatorname{Proj}\Lambda \cup \operatorname{Inj}\Lambda) = {}^{\perp}(\operatorname{GInj}\Lambda).$$

4.2. **A characterization via filtered colimits.** We provide another characterization of virtually Gorenstein algebras in terms of filtered colimits of finitely generated modules. It is convenient to define the subcategories

$$\operatorname{Gproj}\Lambda := \operatorname{GProj}\Lambda \cap \operatorname{mod}\Lambda \quad \text{and} \quad \operatorname{Ginj}\Lambda := \operatorname{GInj}\Lambda \cap \operatorname{mod}\Lambda.$$

Theorem. *For an Artin algebra Λ the following are equivalent.*

(1) *Λ is virtually Gorenstein.*
(2) *Any Gorenstein projective module is a filtered colimit of finitely generated Gorenstein projective modules.*
(3) *Any Gorenstein injective module is a filtered colimit of finitely generated Gorenstein injective modules.*

If Λ is virtually Gorenstein, then $\operatorname{Gproj}\Lambda$ *and* $\operatorname{Ginj}\Lambda$ *are both covariantly and contravariantly finite subcategories of* $\operatorname{mod}\Lambda$.

Proof. (1) ⇒ (2) & (3): See the proof of Theorem 4.1.

(2) ⇔ (3): Let D denote the duality between right and left Λ-modules. The adjoint pair of functors $-\otimes_{\Lambda} D(\Lambda)$ and $\operatorname{Hom}_{\Lambda}(D(\Lambda),-)$ induces an equivalence between $\operatorname{Proj}\Lambda$ and $\operatorname{Inj}\Lambda$, which restricts to an equivalence between $\operatorname{proj}\Lambda$ and $\operatorname{inj}\Lambda$. Therefore the adjoint pair induces an equivalence between $\operatorname{GProj}\Lambda$ and $\operatorname{GInj}\Lambda$, and between $\operatorname{Gproj}\Lambda$ and $\operatorname{Ginj}\Lambda$, respectively, see [4, Proposition 3.4]. Now use the fact that both functors preserves filtered colimits.

(3) ⇒ (1): A standard argument shows that $\operatorname{Ginj}\Lambda$ is a covariantly finite subcategory of $\operatorname{mod}\Lambda$. To see this, fix X in $\operatorname{mod}\Lambda$ and let $X \to Y_{\alpha}$ be a representative family of maps into objects from $\operatorname{Ginj}\Lambda$. Then $\prod_{\alpha} Y_{\alpha}$ is by our assumption a filtered colimit of objects in $\operatorname{Ginj}\Lambda$, and therefore the map $X \to \prod_{\alpha} Y_{\alpha}$ factors through a map $X \to Y$ with Y in $\operatorname{Ginj}\Lambda$. By our construction, the map $X \to Y$ is a left $\operatorname{Ginj}\Lambda$-approximation of X, since every map $X \to Y_{\alpha}$ factors through $X \to Y$. Thus $\operatorname{Ginj}\Lambda$ is covariantly finite.

Let $\mathcal{D} = \operatorname{Ginj}\Lambda$. As in the proof of Theorem 4.1, we obtain cotorsion pairs $(\mathcal{C},\mathcal{D})$ and $(\varinjlim \mathcal{C}, \varinjlim \mathcal{D})$ for $\operatorname{mod}\Lambda$ and $\operatorname{Mod}\Lambda$, respectively. We claim that $\mathcal{C} = {}^{\perp}(\operatorname{GInj}\Lambda) \cap \operatorname{mod}\Lambda$. We have $\mathcal{C} \subseteq {}^{\perp}(\operatorname{GInj}\Lambda) \cap \operatorname{mod}\Lambda$, because $\operatorname{GInj}\Lambda \subseteq \varinjlim \mathcal{D}$ by our assumption. To show the other inclusion, let X be in ${}^{\perp}(\operatorname{GInj}\Lambda) \cap \operatorname{mod}\Lambda$ and choose an exact sequence $0 \to D \to C \to X \to 0$ such that $C \to X$ is a special right $\mathcal{C}$-approximation. This sequence splits since D is Gorenstein injective, and therefore X belongs to $\mathcal{C}$. It remains to recall from Theorem 3.5 that

$$^{\perp}(\operatorname{GInj}\Lambda) \cap \operatorname{mod}\Lambda = \operatorname{Thick}(\operatorname{proj}\Lambda \cup \operatorname{inj}\Lambda).$$

Thus $\operatorname{Thick}(\operatorname{proj}\Lambda \cup \operatorname{inj}\Lambda)$ is contravariantly finite, and therefore the characterization from Theorem 4.1 implies that Λ is virtually Gorenstein.

To conclude this proof, assume that Λ is virtually Gorenstein. Thus Λ^{op} is virtually Gorenstein as well. We have seen that $\operatorname{Ginj}\Lambda$ is a covariantly finite subcategory of $\operatorname{mod}\Lambda$. Recall that any covariantly finite and coresolving subcategory of $\operatorname{mod}\Lambda$ is contravariantly finite, by [9, Corollary 2.6]. Thus $\operatorname{Ginj}\Lambda$ is contravariantly finite. The duality between right and left Λ-modules identifies $\operatorname{Ginj}\Lambda^{\mathrm{op}}$ with $\operatorname{Gproj}\Lambda$. We conclude that $\operatorname{Gproj}\Lambda$ is both covariantly and contravariantly finite. □

4.3. **An example.** We provide an example of an Artin algebra which is not virtually Gorenstein. This is based on work of Yoshino [12]. In fact, he constructs a class of algebras Λ such that the subcategory $\operatorname{Gproj}\Lambda$ of $\operatorname{mod}\Lambda$ is not contravariantly finite. Then we use the fact from Theorem 4.2 that $\operatorname{Gproj}\Lambda$ is contravariantly finite whenever Λ is virtually Gorenstein.

Proposition. *Let K be a field. Then the 6-dimensional K-algebra*

$$\Lambda = K[x,y,z]/\langle x^2, yz, y^2 - xz, z^2 - yx\rangle$$

is not virtually Gorenstein.

Proof. Consider the K-algebra

$$\Gamma = K[x,y,z]/\langle xz - y^2, yx - z^2, zy - x^2\rangle.$$

This is a one-dimensional Cohen-Macaulay non-Gorenstein homogeneous ring. We have $\Lambda = \Gamma/x^2\Gamma$ and this algebra has radical cubed zero and is non-Gorenstein. It is shown in [12] (in a more general setting) that the trivial Λ-module K has no right $\operatorname{Gproj}\Lambda$-approximation, hence $\operatorname{Gproj}\Lambda$ fails to be contravariantly finite in $\operatorname{mod}\Lambda$. The proof uses the graded structure of Λ and its Hilbert series. Consequently, by Theorem 4.2, Λ is an example of a finite-dimensional K-algebra which is not virtually Gorenstein. □

Appendix A. Thick subcategories

Let $\mathcal{A}$ be an exact category and suppose that $\mathcal{A}$ is idempotent complete (that is, $\mathcal{A}$ has split idempotents). A full subcategory $\mathcal{C}$ of $\mathcal{A}$ is called *thick* if it is closed under direct factors and has the following two out of three property: for every exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ with two terms in $\mathcal{C}$, the third term belongs to $\mathcal{C}$ as well.

Now consider the bounded derived category $\mathbf{D}^b(\mathcal{A})$ of $\mathcal{A}$ and identify $\mathcal{A}$ with the full subcategory of $\mathbf{D}^b(\mathcal{A})$ consisting of all complexes concentrated in degree zero. Recall that a full subcategory of a triangulated category is *thick* if it is closed under shifts, mapping cones, and direct factors.

We discuss the relation between thick subcategories of exact and triangulated categories. To this end, let us call a thick subcategory $\mathcal{C}$ of $\mathcal{A}$ *cofinal* if every admissable epimorphism $Y \to Z$ into an object $Z \in \mathcal{C}$ admits an admissable epimorphism $Y' \to Z$ with $Y' \in \mathcal{C}$ and factoring through $Y \to Z$.

The following result is used in the proof of Theorem 3.5.

Proposition. *Let $\mathcal{C}$ be a full subcategory of $\mathcal{A}$. If $\mathcal{C}$ is a cofinal thick subcategory of $\mathcal{A}$, then $\mathcal{C}$ is of the form $\mathcal{D} \cap \mathcal{A}$ for some thick subcategory $\mathcal{D}$ of $\mathbf{D}^b(\mathcal{A})$. Conversely, if $\mathcal{C} = \mathcal{D} \cap \mathcal{A}$ for some thick subcategory $\mathcal{D}$ of $\mathbf{D}^b(\mathcal{A})$, then $\mathcal{C}$ is a thick subcategory of $\mathcal{A}$.*

Proof. Suppose that $\mathcal{C}$ is a cofinal thick subcategory. This assumption on $\mathcal{C}$ implies that the inclusion $\mathcal{C} \to \mathcal{A}$ induces a fully faithful and exact functor $\mathbf{D}^b(\mathcal{C}) \to \mathbf{D}^b(\mathcal{A})$; see for instance [11, Proposition III.2.4.1]. Note also that idempotents in $\mathbf{D}^b(\mathcal{C})$ split since $\mathcal{C}$ has this property; see [3, Theorem 2.8]. Thus the full subcategory $\mathcal{D}$ of $\mathbf{D}^b(\mathcal{A})$ consisting of complexes quasi-isomorphic to a complex of objects in $\mathcal{C}$ is a thick subcategory. We claim that $\mathcal{C} = \mathcal{D} \cap \mathcal{A}$. Clearly, $\mathcal{C} \subseteq \mathcal{D} \cap \mathcal{A}$. Thus we fix $X \in \mathcal{D} \cap \mathcal{A}$. Then X is in $\mathbf{D}^b(\mathcal{A})$ isomorphic to a bounded complex C with differential δ such that $C^n \in \mathcal{C}$ for all n and C is acyclic in all degrees $n \neq 0$. Now we use that $\mathcal{C}$ is thick. Thus $\operatorname{Coker} \delta^{-2}$ and $\operatorname{Ker} \delta^0$ belong to $\mathcal{C}$, and we have an admissable monomorphism $\operatorname{Coker} \delta^{-2} \to \operatorname{Ker} \delta^0$ such that the cokernel is isomorphic to X. We conclude that X belongs to $\mathcal{C}$ and therefore $\mathcal{C} = \mathcal{D} \cap \mathcal{A}$.

The converse is clear since each exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ gives rise to an exact triangle $X \to Y \to Z \to$ in $\mathbf{D}^b(\mathcal{A})$. □

Example. Let $\mathcal{A}$ be an exact category having enough projective objects. Then every thick subcategory $\mathcal{C}$ containing all projective objects is cofinal.

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