THICK SUBCATEGORIES OF THE DERIVED CATEGORY OF A HEREDITARY ALGEBRA

KRISTIAN BRÜNING

ABSTRACT. We classify thick subcategories of the bounded derived category of an abelian category $\mathcal{A}$ in terms of subcategories of $\mathcal{A}$. The proof can be applied to characterize the localizing subcategories of the full derived category of $\mathcal{A}$. As an application we prove an algebraic analogon of the telescope conjecture for the derived category of a representation finite hereditary artin algebra.

CONTENTS

1. INTRODUCTION

A full subcategory in a triangulated category is *thick* if it is closed under forming suspensions, triangles and retracts. A classification can be used to gain other structural information about the ambient triangulated category, as in [HS98]. Thick subcategories were studied in stable homotopy theory, commutative algebra and representation theory of groups: The first classification theorem was obtained by Hopkins and Smith for the p-local finite stable homotopy category [HS98]. They showed that a thick subcategory is equivalent to the $K(n)_*$-acyclics of the cohomology theory represented by some Morava K-theory spectrum $K(n)$. Hopkins and Neeman showed that the thick subcategories in the derived category of a commutative noetherian ring R correspond to the specialization closed subsets of the prime

2000 *Mathematics Subject Classification.* 18E30,16G20.

Key words and phrases. Thick subcategory, hereditary, smashing conjecture, localizing subcategory, derived category.

ideal spectrum of R [Hop87, Nee92]. Later on Thomason generalized this result to schemes [Tho97]. Benson, Carlson and Rickard classified the thick subcategories of the stable module category of the group algebra kG of a p-group G in terms of closed subvarieties of the maximal ideal spectrum of the group cohomology ring $H^*(G;k)$ [BCR97].

In the main theorem of this paper we classify the thick subcategories of the bounded derived category of a hereditary abelian category.

Theorem 1.1. *For a hereditary abelian category $\mathcal{A}$ the zeroth homology group functor induces a one to one correspondence between the thick subcategories of the bounded derived category $\mathcal{D}^b(\mathcal{A})$ and the thick subcategories in $\mathcal{A}$.*

This result includes for instance the bounded derived category of finitely presented right modules $\mathcal{D}^b(\text{mod}A)$, for a finite dimensional algebra over a field k and therefore enhances the study of thick subcategories to the field of representation theory of algebras.

We start by fixing some notations in section two. In the third section we define hereditary categories and describe the structure of the derived category. Furthermore thick subcategories in an abelian category are defined and studied in section four. The classification result is proved in the fifth section and is illustrated by an explicit description of the thick subcategories for two representation finite k-algebras. In the sixth section we adapt the proof of our main theorem to characterize localizing subcategories of the full derived category. Finally we use a result of Auslander and Ringel-Tachikawa to deduce a finiteness result for the localizing subcategories which implies that an algebraic analogon of the telescope conjecture for the derived category of a hereditary artin algebra of finite representation type is true.

This paper is part of the author's PhD-thesis [Brü07]. He would like to thank Henning Krause for his support and enduring interest, and the Deutsche Forschungsgemeinschaft for the financial support.

2. Preliminaries

Throughout this paper unless otherwise stated $\mathcal{A}$ denotes an abelian category and $\mathcal{D}^b(\mathcal{A})$ the stands for the bounded derived category of $\mathcal{A}$. If $\mathcal{A}$ is in addition a Grothendieck category, then the unbounded derived category exists [Bek00]. We identify $\mathcal{A}$ with the complexes concentrated in degree zero in the derived (or bounded derived) category of $\mathcal{A}$ via the inclusion $i\colon \mathcal{A} \to \mathcal{D}^{(b)}(A)$ and, by abuse of notation, do not distinguish between objects in $\mathcal{A}$ and $\text{im}(i)$. All modules in this paper are right modules. If R is a ring then a complex is called *perfect* if it is a complex of finitely generated projective R-modules. Let $\text{mod}(R)$ denote the category of finitely presented R-modules. Recall that the full subcategory of compact objects $\mathcal{D}(A)^c$ is equivalent to the full subcategory of perfect complexes $\mathcal{D}^{\text{per}}(A)$ in $\mathcal{D}(\text{Mod}(R)))$. Furthermore if the global dimension of A is finite then $\mathcal{D}^{\text{per}}(A)$ is equivalent to $\mathcal{D}^b(\text{mod}(A))$.

3. The derived category of hereditary abelian categories

In this section we describe the structure of the derived category of a hereditary abelian category which serves as the main tool to obtain the classification result in Paragraph 5.

Definition 3.1. An abelian category $\mathcal{A}$ is called hereditary if $\operatorname{Ext}^i_{\mathcal{A}}(M, N)$ vanishes for all $M, N \in \mathcal{A}$ and all $i \geq 2$.

Throughout this paragraph let $\mathcal{A}$ be a hereditary abelian category.

Example 3.2. If A is a hereditary ring, i.e. a ring such that every right ideal is a projective A-module, then the module category $\operatorname{mod}(A)$ is hereditary.

The derived category $\mathcal{D}(\mathcal{A})$ of a hereditary abelian category $\mathcal{A}$ is closely related to $\mathcal{A}$ itself since every complex of $\mathcal{D}(\mathcal{A})$ is isomorphic to a direct sum (and direct product) of stalk complexes:

Lemma 3.3. *For every $X \in \mathcal{D}^b(\mathcal{A})$ there are isomorphisms in $\mathcal{D}^b(\mathcal{A})$*

$$\prod_{n\in\mathbb{Z}} H^n X[-n] \cong X \cong \bigoplus_{m\in\mathbb{Z}} H^m X[-m]. \tag{1}$$

If $\mathcal{A}$ is in addition a Grothendieck category and X is an object in $\mathcal{D}(A)$, then the isomorphism (1) exists.

A proof of this well known lemma can be found in [Kra04]. The homomorphisms in $\mathcal{D}(\mathcal{A})$ therefore reduce to

$$\operatorname{Hom}^*_{\mathcal{D}(\mathcal{A})}(M, N) \cong \operatorname{Hom}_{\mathcal{A}}(M, N) \oplus \operatorname{Ext}^1_{\mathcal{A}}(M, N)$$

for $M, N \in \mathcal{A}$. So the derived category consists of shifted copies of $\mathcal{A}$, and the morphisms are given by extensions and homomorphisms in $\mathcal{A}$. This structure is visualized in Figure 1.

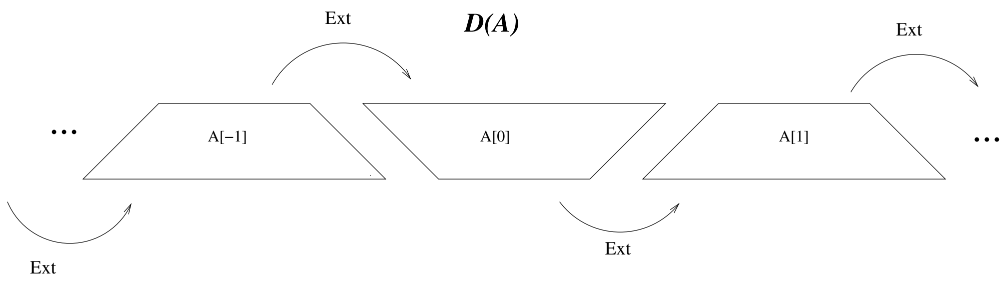


Figure 1

Non-equivalent hereditary abelian categories can give rise to the same derived category:

Theorem 3.4. [Hap88, I.5.5, 5.6] *Let k be a field. If Q and Q' are quivers with the same underlying graph of Dynkin type but of different orientation, then $\mathcal{D}(\text{mod}(kQ))$ and $\mathcal{D}(\text{mod}(kQ'))$ are equivalent as triangulated categories.*

The structure of the derived category motivates why the thick subcategories in $\mathcal{D}(\mathcal{A})$ should be determined by data in $\mathcal{A}$. If in addition $\mathcal{A} = \text{mod}(kQ)$ is the module category of a path algebra of a Dynkin quiver then we should be able to describe the thick subcategories combinatorially.

4. Thick subcategories of abelian categories

We define and investigate thick subcategories of an abelian category $\mathcal{A}$ and discuss Hovey's classification of the thick subcategories in the category of modules over a regular coherent commutative ring.

Throughout this paragraph let $\mathcal{A}$ be an abelian category.

Definition 4.1. A full subcategory $\mathcal{M}$ of $\mathcal{A}$ is called *thick* if for every exact sequence

$$M_1 \to M_2 \to M_3 \to M_4 \to M_5$$

the object M_3 is in $\mathcal{M}$ if the objects M_1, M_2, M_4, M_5 are in $\mathcal{M}$.

Hovey calls these subcategories "wide" [Hov01]. In the following two lemmas some easy properties of thick subcategories are deduced. For the convenience of the reader the proof [Hov01] is reproduced here.

Lemma 4.2. *A full subcategory $\mathcal{M}$ in $\mathcal{A}$ is thick, if and only if it is closed under forming of extensions, kernels and cokernels.*

Proof. Let $\mathcal{M} \subset \mathcal{A}$ be thick and

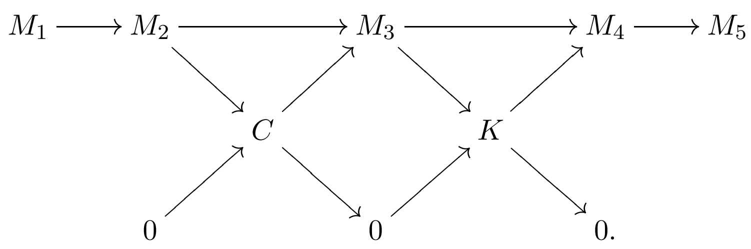
$$M_1 \to M_2 \to M_3 \to M_4 \to M_5$$

be exact in $\mathcal{A}$. If in the exact sequence above $M_1 = M_5 = 0$ and M_2 and M_4 are in $\mathcal{M}$, then M_3 is in $\mathcal{M}$ since $\mathcal{M}$ is thick. Therefore $\mathcal{M}$ is closed under extensions. If we set $M_1 = M_2 = 0$, respectively $M_4 = M_5 = 0$ it follows that $\mathcal{M}$ is closed under kernels and cokernels, respectively.

Conversely let $\mathcal{M} \subset \mathcal{A}$ be closed under extensions, kernels and cokernels and let

$$M_1 \to M_2 \to M_3 \to M_4 \to M_5$$

be exact with $M_1, M_2, M_4, M_5 \in \mathcal{M}$. Since $\mathcal{M}$ is closed under cokernels and kernels, $C := \text{coker}(M_1 \to M_2)$ and $K := \ker(M_4 \to M_5)$ are in $\mathcal{M}$. Hence we obtain a diagram:



Therefore M_3 is an extension of C and K and hence it is in $\mathcal{M}$. □

As an additional property we have

Lemma 4.3. *A thick category in $\mathcal{A}$ is closed under direct summands.*

Proof. Let $M \oplus N$ be in the thick category $\mathcal{M}$. The kernel of the map $M \oplus N \to M \oplus N$ which sends (m, n) to $(0, n)$ is M. □

So a thick subcategory in $\mathcal{A}$ is an abelian subcategory in $\mathcal{A}$ that is closed under retracts such that the inclusion functor is exact. This property motivates its name.

There are geometric examples of thick subcategories.

Example 4.4. The category of coherent modules over the structure sheaf $\mathcal{O}_{\mathbb{X}}$ of a scheme $\mathbb{X}$ is thick [Gro60, 5.3.5].

Other examples of thick subcategories arise from the category $\operatorname{add}(M)$ of direct sums of direct summands of M.

Lemma 4.5. *Let k be a field.*

(i) *Let A be an arbitrary k-algebra. If M is an indecomposable finitely presented A-module with $\operatorname{Hom}_A(M, M) = k$ and $\operatorname{Ext}^1_A(M, M) = 0$, then $\operatorname{add}(M)$ is thick.*

(ii) *If A is a finite dimensional hereditary k-algebra of finite representation type and M is an indecomposable A-module, then $\operatorname{add}(M)$ is thick.*

Proof. Since M is indecomposable the equation $\operatorname{add}(M) = \{\bigoplus_{i=1}^n M \mid n \geq 0\}$ holds. The functor $\operatorname{Ext}^1_A(-, -)$ is additive in both variables. Therefore $\operatorname{add}(M)$ is closed under extensions because M has no non-trivial self-extensions. For positive integers n and m let $f\colon M^n \to M^m$ be A-linear. Every non-trivial component of f is of the form $x \cdot \mathrm{id}_M$ for some $x \in k \setminus \{0\}$ because $\operatorname{Hom}_A(M, M) = k$. Since k is a field every element $x \in k \setminus \{0\}$ is invertible. Therefore the kernel and the cokernel of $x \cdot \mathrm{id}_M$ are trivial and the kernel and the cokernel of f are in $\operatorname{add}(M)$. Therefore (i) follows.

If M is an indecomposable module over a finite dimensional hereditary k-algebra A of finite type, then by [ASS06, VII 5.14] $\operatorname{Hom}_A(M, M) = k$ and $\operatorname{Ext}^1_A(M, M) = 0$. Hence (ii) follows from (i). □

Theorem 4.6. [Hov01, Theorem 3.6] *Let R be a commutative regular coherent ring. There is a one-to-one correspondence between the thick subcategories in $\mathcal{D}^b(\mathrm{mod}(R))$ and the thick subcategories of* $\mathrm{mod}(R)$.

If R is regular noetherian, then a thick subcategory is also closed under subobjects, quotient-objects and extensions [Hov01, 3.7] and is therefore a *Serre subcategory*. Garkusha and Prest generalized Theorem 4.6 in the following way: if R is a commutative coherent ring, then the thick subcategories in $\mathcal{D}^{\mathrm{per}}(R)$ correspond bijectively to the Serre subcategories in $\mathrm{mod}(R)$ [GP07, Theorem C]. In these theorems the classifications [Nee92, Tho97] of the thick subcategories of $\mathcal{D}^b(\mathrm{mod}(R))$ are used to determine the thick subcategories of $\mathrm{mod}(R)$. We go the other way around and describe thick subcategories of the triangulated category in terms of the abelian category.

5. Classification of thick subcategories

In this section we prove the classification result and determine all thick subcategories in two examples combinatorially.

Theorem 5.1. *Let $\mathcal{A}$ be a hereditary abelian category. The assignments*

$$f\colon \mathcal{C} \mapsto \{H^0 C \mid C \in \mathcal{C}\} \qquad \text{and} \qquad g\colon \mathcal{M} \mapsto \{C \in \mathcal{D}^b(\mathcal{A}) \mid H^n C \in \mathcal{M}\, \forall\, n \in \mathbb{Z}\}$$

induce mutually inverse bijections between

- *the class of thick subcategories in $\mathcal{D}^b(\mathcal{A})$ and*
- *the class of thick subcategories in $\mathcal{A}$.*

Proof. The proof mainly uses Lemma 3.3. First note that g is well defined because $\mathcal{M}$ is thick and closed under direct summands by Lemma 4.3. The map f is well defined because of the following lemma:

Lemma 5.2. *Let $\mathcal{C} \subset \mathcal{D}^b(\mathcal{A})$ be thick. The full subcategory $f(\mathcal{C}) \subset \mathcal{A}$ is thick.*

It remains to show that f and g are mutually inverse. The inclusion $f(g(\mathcal{M})) \subset \mathcal{M}$ is obvious. Any object $M \in \mathcal{M}$ is in $f(g(\mathcal{M}))$ since the stalk complex $\cdots \to 0 \to M \to 0 \to \dots$ is in $g(\mathcal{M})$. Since a complex is determined by its homology (Lemma 3.3) the equality $g(f(\mathcal{C})) = \mathcal{C}$ holds. □

In order to prove Lemma 5.2 we need the following

Lemma 5.3. *If $g : C \to D$ is a map of complexes such that the differentials of C and D are zero and $g_m = 0$ for all $m \neq n$, then* $\ker(g)$ *and* $\mathrm{coker}(g)$ *are retracts of* $H^*(\mathrm{cone}(g))$.

Proof. The only non-zero differential in cone(g) is cone$(g)^{n-1} \to$ cone$(g)^n$:

$$\begin{array}{ccccccc}
\mathrm{cone}(g)^{n-2} & & C^{n-1} & \oplus & D^{n-2} \\
\downarrow d & & \downarrow 0 & \overset{0}{\searrow} & \downarrow 0 \\
\mathrm{cone}(g)^{n-1} & & C^{n} & \oplus & D^{n-1} \\
\downarrow d & & \downarrow 0 & \overset{g}{\searrow} & \downarrow 0 \\
\mathrm{cone}(g)^{n} & & C^{n+1} & \oplus & D^{n} \\
\downarrow d & & \downarrow 0 & \overset{0}{\searrow} & \downarrow 0 \\
\mathrm{cone}(g)^{n+1} & & C^{n+2} & \oplus & D^{n+1}.
\end{array}$$

Thus we can compute the homology:

$$H^m(\mathrm{cone}(g)) = \begin{cases} C^{m+1} \oplus D^m & m \leq n-2 \text{ or } m \geq n+1 \\ \ker(g) \oplus D^{n-1} & m = n-1 \\ C^{n+1} \oplus \mathrm{coker}(g) & m = n. \end{cases}$$

□

Proof of Lemma 5.2. We show that $f(\mathcal{C})$ is closed under extensions, kernels and cokernels. So let C_1, C_2 be in $\mathcal{C}$ and $M \in f(\mathcal{C})$ such that there is a short exact sequence

$$0 \to H^0C_1 \to M \to H^0C_2 \to 0.$$

This sequence corresponds to a triangle

$$H^0C_1 \to M \to H^0C_2 \to \Sigma H^0C_1$$

in $\mathcal{D}^b(\mathcal{A})$. Each homology group of a complex $C \in \mathcal{C}$ is again contained in $\mathcal{C}$ since by Lemma 3.3 H^nC is a retract of C up to isomorphism and $\mathcal{C}$ is thick. Therefore H^0C_1 and H^0C_2 are in $\mathcal{C}$ and because $\mathcal{C}$ is closed under suspensions, $\Sigma H^0C_1 \in \mathcal{C}$. Since $\mathcal{C}$ is closed under extensions, we conclude that M is in $\mathcal{C}$. Hence $M \in f(\mathcal{C})$ because the zeroth homology of $\dots \to 0 \to M \to 0 \to \dots$ is M.
So it only remains to show that $f(\mathcal{C})$ is closed under kernels and cokernels. Let C_1, C_2 be in $\mathcal{C}$ and f be a morphism in the exact sequence in $\mathcal{A}$:

$$0 \to \ker(f) \to H^0C_1 \xrightarrow{f} H^0C_2 \to \mathrm{coker}(f) \to 0.$$

Now extend f to a map of complexes

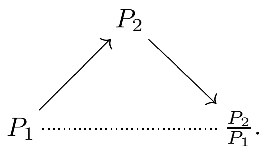
$$\bigoplus_{n \in \mathbb{Z}} H^nC_1[-n] \to \bigoplus_{m \in \mathbb{Z}} H^mC_2[-m]$$

which is f in degree 0 and zero in all other degrees. We call it again f. Since $C_i \cong \bigoplus_{n\in\mathbb{Z}} H^nC_i[-n]$ for $i = 1, 2$, the map f belongs to $\mathcal{C}$. The cone of f

is in $\mathcal{C}$. By Lemma 5.3 $\ker(f)$ and $\operatorname{coker}(f)$ are retracts of $H^0(\operatorname{cone}(f))$ and are hence (considered as stalk complexes) in $\mathcal{C}$. Therefore the kernel and cokernel of f, considered as objects in $\mathcal{A}$, are in $f(\mathcal{C})$. □

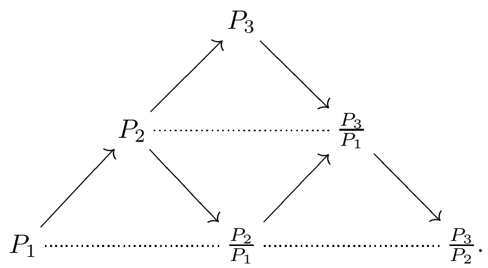
With this theorem we have reduced the classification of thick subcategories in the triangulated category $\mathcal{D}^b(\mathcal{A})$ to the task of understanding thick subcategories in $\mathcal{A}$. In easy examples it is possible to determine them combinatorially. Let k be a field and A be a representation finite hereditary k-algebra. As a consequence of Lemma 4.5 there are examples of thick subcategories of the category of finitely presented modules $\operatorname{mod}(A)$. As an immediate consequence we are able to determine the thick subcategories of finite dimensional representations of an A_2- and an A_3-quiver. For the two examples let k be an algebraically closed field, Q the respective quiver, $A = kQ$ the path algebra and $\mathcal{A} = \operatorname{mod}(kQ)$ the category of finitely presented modules over A. We use the Auslander-Reiten quiver to describe the category $\mathcal{A}$ combinatorially.

Example 5.4. Let Q be the quiver $1 \longleftarrow 2$. The Auslander-Reiten quiver is the following graph:



By Lemma 4.5 there are exactly four non-trivial thick subcategories: $\operatorname{add}(P_1)$, $\operatorname{add}(P_2)$, $\operatorname{add}(\frac{P_2}{P_1})$ and $\operatorname{mod}(kQ)$.

Example 5.5. Let Q be the quiver $1 \longleftarrow 2 \longleftarrow 3$. The Auslander-Reiten quiver has the following shape:



Lemma 4.5 tells us that there are six thick subcategories containing exactly one indecomposable. Furthermore there are two thick subcategories that contain two indecomposable modules, four with three indecomposables and the whole module category with six indecomposables.

The left column of Table 1 shows the thick subcategories in terms of the contained indecomposable modules. E.g. $\langle P_1, P_3\rangle$ is the smallest thick subcategory containing P_1 and P_3. The right column displays the part of the corresponding Auslander-Reiten quiver that is contained in the thick subcategory $\mathcal{C}$. Modules in $\mathcal{C}$ are labelled with fat bullets and morphisms in $\mathcal{C}$ with full arrows.

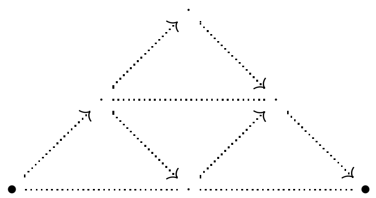
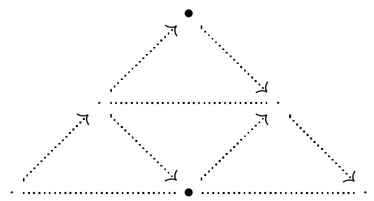
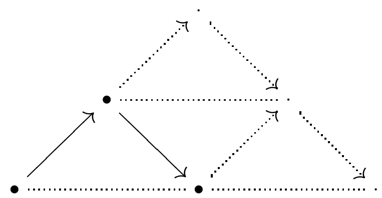
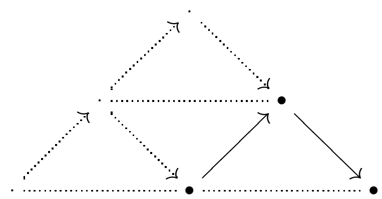
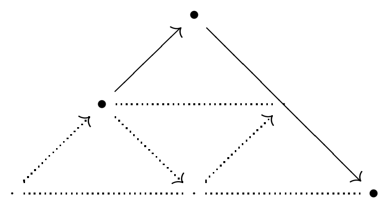
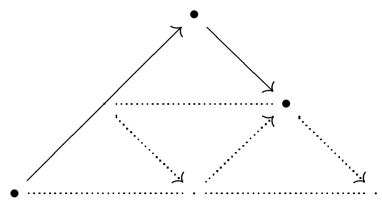
$\langle P_1\rangle, \ldots, \langle P_3/P_1\rangle$	•
$\langle P_1, P_3/P_2\rangle$	
$\langle P_3, P_2/P_1\rangle$	
$\langle P_1, P_2, P_2/P_1\rangle$	
$\langle P_2/P_1, P_3/P_1, P_3/P_2\rangle$	
$\langle P_2, P_3, P_3/P_2\rangle$	
$\langle P_1, P_3, P_3/P_1\rangle$	
$\langle P_1, P_2, P_3\rangle$	$\mathrm{mod}(A)$

TABLE 1

The thick subcategories are symmetric with respect to reflection at the axis going through P_3 and $\frac{P_2}{P_1}$ in the Auslander-Reiten quiver. The categories $\operatorname{add}(P_3)$, $\operatorname{add}(\frac{P_2}{P_1})$, $\langle P_3, \frac{P_2}{P_1}\rangle$, $\langle P_1, \frac{P_3}{P_2}\rangle$ and $\operatorname{mod}(kQ)$ are invariant under the reflection. Under the reflection $\operatorname{add}(P_3)$ corresponds with $\operatorname{add}(\frac{P_3}{P_1})$, $\operatorname{add}(P_1)$ corresponds with $\operatorname{add}(\frac{P_3}{P_2})$, $\langle P_1, P_2, \frac{P_2}{P_1}\rangle$ corresponds with $\langle \frac{P_3}{P_1}, \frac{P_2}{P_1}, \frac{P_3}{P_2}\rangle$ and $\langle P_1, P_3, \frac{P_3}{P_1}\rangle$ corresponds with $\langle P_2, P_3, \frac{P_3}{P_2}\rangle$.

It would be interesting to work out all thick subcategories for all representation finite algebras.

Corollary 5.6. *Let $\mathcal{A}$ and $\mathcal{A}'$ be hereditary abelian categories. If $\mathcal{D}^b(\mathcal{A})$ and $\mathcal{D}^b(\mathcal{A}')$ are triangle equivalent, then there is an isomorphism of lattices between the thick subcategories in $\mathcal{A}$ and the thick subcategories in $\mathcal{A}'$.*

Example 5.7. Let k be a field and let Q be the Kronecker quiver. The category of coherent sheaves $\operatorname{Coh}(\mathbb{P}^1_k)$ on the projective line $\mathbb{P}^1_k$ is hereditary. Beĭlinson showed that $\mathcal{D}^b(\operatorname{Coh}(\mathbb{P}^1_k)$ is triangle equivalent to $\mathcal{D}^b(\operatorname{mod}(kQ))$ [Beĭ78]. Therefore Corollary 5.6 tells us that there is an isomorphism between the lattice of thick subcategories in $\operatorname{Coh}(\mathbb{P}^1_k)$ and the lattice of thick subcategories in $\operatorname{mod}(kQ)$.

Example 5.8. By Theorem 3.4 and Corollary 5.6 the lattice of thick subcategories in the category of finitely generated representations of a Dynkin quiver does not depend on the orientation.

If the algebra A is not of global dimension one, then Lemma 3.3 does not remain true. But if the global dimension of A is finite the Happel functor $\mathcal{D}^b(\operatorname{mod}(A)) \to \underline{\operatorname{mod}}(\hat{A})$ is an equivalence [Hap88, II.4.9]. Here $\hat{A}$ denotes the repetitive algebra of A. A generalization of the classification Theorem 5.1 may possibly be achieved by characterizing the thick subcategories of $\underline{\operatorname{mod}}(\hat{A})$ in terms of the thick subcategories of $\operatorname{mod}(A)$.

6. Classification of localizing subcategories

In this section we use the strategy of Theorem 5.1 to classify the localizing subcategories of the full derived category of a hereditary Grothendieck category. As an application we prove that the Smashing Conjecture is true for $\mathcal{D}(A)$ for a hereditary artin algebra A of finite representation type.

Recall that a full subcategory of a triangulated category with arbitrary direct sums is called *localizing* if it is thick and closed under arbitrary direct sums. These categories are the unbounded analogues of the thick subcategories.

Theorem 6.1. *Let $\mathcal{A}$ be a hereditary Grothendieck category. The assignments*

$$f\colon \mathcal{C} \mapsto \{H^0C \,|\, C \in \mathcal{C}\} \qquad \text{and} \qquad g\colon \mathcal{M} \mapsto \{C \in \mathcal{D}(\mathcal{A}) \,|\, H^nC \in \mathcal{M}\,\forall\, n \in \mathbb{Z}\}$$

induce mutually inverse bijections between

- *the class of localizing subcategories in $\mathcal{D}(\mathcal{A})$ and*
- *the class of thick subcategories in $\mathcal{A}$ that are closed under small coproducts.*

Proof. Adding the following comments the proof of Theorem 5.1 applies. Lemma 3.3 is not limited to the bounded derived category, and hence can be used here. The map g is well-defined, since the homology functor commutes with infinite direct sums. And finally if $\mathcal{C}$ is localizing, then $f(\mathcal{C})$ is closed under direct sums for the same reason. □

Theorem 6.2. [Aus74, RT74] *Let A be an artin algebra of finite representation type. Then every module is a direct sum of finitely generated indecomposable modules.*

Using this we can deduce:

Corollary 6.3. *Let A be a hereditary artin algebra of finite representation type.*

(i) *Every thick subcategory $\mathcal{M} \subset \mathrm{Mod}(A)$ that is closed under direct sums is the smallest thick subcategory that contains $\mathcal{M} \cap \mathrm{mod}(A)$ and is closed under direct sums.*
(ii) *Every localizing subcategory $\mathcal{C} \subset \mathcal{D}(A)$ is determined by its intersection with the perfect complexes: $\mathcal{C} = \langle \mathcal{C} \cap \mathcal{D}^{\mathrm{per}}(A)\rangle_{\mathrm{loc}}$.*

Proof. By Theorem 6.2, (i) is true. For the assertion (ii) let $\mathcal{C} \subset \mathcal{D}(A)$ be localizing and $C \in \mathcal{C}$ be an object. By Lemma 3.3 it suffices to show that H^0C is contained in $\langle \mathcal{C} \cap \mathcal{D}^{\mathrm{per}}(A)\rangle_{loc}$. Because of Theorem 6.2 there are a set I and finitely generated modules $\{M_i \,|\, i \in I\}$ such that $H^0C \cong \bigoplus_{i\in I} M_i$. Since $\mathcal{C}$ is thick, it follows that $M_i \in \mathcal{C}$. For every M_i choose a projective resolution

$$0 \to P_i^0 \to P_i^1 \to M_i \to 0$$

such that P_i^0, P_i^1 are finitely generated. The complex $P_i :\; 0 \to P_i^0 \to P_i^1 \to 0$ is perfect and hence in $\mathcal{D}^{\mathrm{perf}}(A)$. Since $P_i \to M_i$ is a quasi isomorphism and $M_i \in \mathcal{C}$ we can conclude that $P_i \in \mathcal{C} \cap \mathcal{D}^{\mathrm{per}}(A)$. Hence H^0C is a direct sum of perfect complexes in $\mathcal{C}$. □

Let $\mathcal{T}$ be a triangulated category with small coproducts, and let $\mathcal{T}^c$ denote the thick subcategory of compact objects. Recall that a localizing subcategory $\mathcal{C} \subset \mathcal{T}$ is called *smashing* if the canonical functor $\mathcal{T} \to \mathcal{T}/\mathcal{C}$ to the Verdier-quotient has a right adjoint that commutes with small coproducts. The *smashing conjecture* asserts that every smashing subcategory $\mathcal{C}$ is the smallest localizing subcategory containing $\mathcal{C} \cap \mathcal{T}^c$ [Nee92]. The smashing conjecture for the p-local stable homotopy category is a generalization of the telescope conjecture by Ravenel [Rav87] (see also [Brü07] for an overview). Since the perfect complexes form precisely the compact objects in $\mathcal{D}(A)$, Corollary 6.3(ii) shows:

Corollary 6.4. *The smashing conjecture is true for the derived category of a hereditary artin algebra of finite representation type.*

In fact, even all localizing subcategories are determined by the intersection with the compact objects.

If A is not of finite type the smashing conjecture is possibly also true since every module over A is a filtered colimit of finitely presented modules. Choosing a clever indexing category may lead to a proof of the smashing conjecture for arbitrary hereditary algebras.

References

[ASS06] I. Assem, D. Simson, and A. Skowroński. *Elements of the representation theory of associative algebras. Vol. 1*, volume 65 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2006. Techniques of representation theory.

[Aus74] M. Auslander. Representation theory of Artin algebras. II. *Comm. Algebra*, 1:269–310, 1974.

[BCR97] D. J. Benson, J. F. Carlson, and J. Rickard. Thick subcategories of the stable module category. *Fund. Math.*, 153(1):59–80, 1997.

[Beĭ78] A. A. Beĭlinson. Coherent sheaves on $\mathbf{P}^n$ and problems in linear algebra. *Funktsional. Anal. i Prilozhen.*, 12(3):68–69, 1978.

[Bek00] T. Beke. Sheafifiable homotopy model categories. *Math. Proc. Cambridge Philos. Soc.*, 129(3):447–475, 2000.

[Brü07] K. Brüning. Subcategories of triangulated categories and the smashing conjecture. PhD-thesis, 2007.

[GP07] G. Garkusha and M. Prest. Classifying Serre subcategories of finitely presented modules. `http://homotopy.nm.ru/papers/garkpr-ProcAMS.pdf` to appear in the Proceedings of the AMS, 2007.

[Gro60] A. Grothendieck. Éléments de géométrie algébrique. I. Le langage des schémas. *Inst. Hautes Études Sci. Publ. Math.*, (4):228, 1960.

[Hap88] D. Happel. *Triangulated categories in the representation theory of finite-dimensional algebras*, volume 119 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1988.

[Hop87] M. J. Hopkins. Global methods in homotopy theory. In *Homotopy theory (Durham, 1985)*, volume 117 of *London Math. Soc. Lecture Note Ser.*, pages 73–96. Cambridge Univ. Press, Cambridge, 1987.

[Hov01] M. Hovey. Classifying subcategories of modules. *Trans. Amer. Math. Soc.*, 353(8):3181–3191 (electronic), 2001.

[HS98] M. J. Hopkins and J. H. Smith. Nilpotence and stable homotopy theory. II. *Ann. of Math. (2)*, 148(1):1–49, 1998.

[Kra04] H. Krause. Derived categories, resolutions, and Brown representability. arXiv:math.KT/0511047, 2004. To appear in the proceedings of the summer school "Interactions between homotopy theory and algebra" (Chicago, 2004).

[Nee92] A. Neeman. The chromatic tower for $D(R)$. *Topology*, 31(3):519–532, 1992. With an appendix by Marcel Bökstedt.

[Rav87] D. C. Ravenel. Localization and periodicity in homotopy theory. In *Homotopy theory (Durham, 1985)*, volume 117 of *London Math. Soc. Lecture Note Ser.*, pages 175–194. Cambridge Univ. Press, Cambridge, 1987.

[RT74] C. M. Ringel and H. Tachikawa. QF-3 rings. *J. Reine Angew. Math.*, 272:49–72, 1974.

[Tho97] R. W. Thomason. The classification of triangulated subcategories. *Compositio Math.*, 105(1):1–27, 1997.

Kristian Brüning, Institut für Mathematik, Universität Paderborn, Warburger Strasse 100, 33098 Paderborn, Germany.

E-mail address: `bruening@math.upb.de`