Stability of Petrov-Galerkin discretizations: Application to the space-time weak formulation for parabolic evolution problems

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Abstract — This paper is concerned with the stability of Petrov-Galerkin discretizations with application to parabolic evolution problems in space-time weak form. We will prove that the discrete inf-sup condition for an a priori fixed Petrov-Galerkin discretization is satisfied uniformly under standard approximation and smoothness conditions without any further coupling between the discrete trial- and test spaces for sufficiently regular operators. It turns out that one needs to choose different discretization levels for the trial- and test spaces in order to obtain a positive lower bound for the discrete inf-sup condition which is independent of the discretization levels. In particular, we state the required number of extra layers in order to guarantee uniform boundedness of the discrete inf-sup constants explicitly. This general result will be applied to the space-time weak formulation of parabolic evolution problems as an important model example. In this regard, we consider suitable hierarchical families of discrete spaces. The results apply e.g. for finite element discretizations as well as for wavelet discretizations. Due to the Riesz basis property, wavelet discretizations allow for optimal preconditioning independently of the grid spacing. Moreover, our predictions on the stability, especially in view of the dependence on the refinement levels w.r.t. the test and trial spaces, are underlined by numerical results. Furthermore, it can be observed that choosing the same discretization levels would, indeed, lead to stability problems.

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1. Introduction

Suppose that we have a boundedly invertible operator $B \in \mathcal{L}(X,Y')$ mapping from a Hilbert space $X$ into a dual Hilbert space $Y'$. We denote by $\mathcal{L}(X,Y')$ the set of all linear and bounded mappings form $X$ into $Y'$. Consider the generic operator equation

$$ Bu = f, \quad u \in X, \quad f \in Y'. $$

(1)

It is well known (see e.g. [6, 12, 24]) that an operator $B \in \mathcal{L}(X,Y')$ is boundedly invertible if and only if the operator is bounded, surjective and if an inf-sup condition is
fulfilled, i.e.,

\[
C_B := \sup_{v \in X \setminus \{0\}} \sup_{q \in Y \setminus \{0\}} \frac{|\langle Bv, q \rangle|}{\|v\|_X \|q\|_Y} < \infty \quad \text{(boundedness)} \tag{2}
\]

\[
c_B := \inf_{v \in X \setminus \{0\}} \sup_{q \in Y \setminus \{0\}} \frac{|\langle Bv, q \rangle|}{\|v\|_X \|q\|_Y} > 0 \quad \text{(inf-sup condition)} \tag{3}
\]

\[
\sup_{v \in X \setminus \{0\}} |\langle Bv, q \rangle| > 0 \quad \forall q \in Y \setminus \{0\} \quad \text{(surjectivity)}, \tag{4}
\]

where \(Y'\) is the dual space of \(Y\) and \(\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{Y' \times Y}\) denotes the duality pairing on \(Y' \times Y\). For simplicity we will omit the indices and use the notation \(\langle \cdot, \cdot \rangle\) also for duality pairings on other spaces, if they are apparent from the context.

Next we want to consider the corresponding Petrov-Galerkin discretization with respect to discrete subspaces \(S_j \subset X\) and \(Q_\ell \subset Y\). Here the indices \(j\) and \(\ell\) refer to the refinement levels of the discrete spaces, which already indicate that we will deal with hierarchies of spaces \(S_j\) and \(Q_\ell\). The Petrov-Galerkin solution \(u_j \in S_j\) is given by the solution of the variational problem

\[
\langle Bu_j, q_\ell \rangle = \langle f, q_\ell \rangle \quad \text{for all } q_\ell \in Q_\ell. \tag{5}
\]

In order to arrive at our stability results, we allow discrete test spaces of higher dimension than that of the trial space. To this end, we introduce the minimal residual Petrov-Galerkin solution of \(Bu = f\) as the minimizer of the functional residual

\[
u_j := \arg \min_{v_j \in S_j} \sup_{q_\ell \in Q_\ell \setminus \{0\}} \frac{|\langle Bv_j - f, q_\ell \rangle|}{\|q_\ell\|_Y}, \tag{6}
\]

cf [3]. This leads to an overdetermined system of equations, i.e., a non-square system matrix, so that the numerical solution can obtained by solving the associated least squares problem respectively the normal equation.

Trivially, the operator is bounded on the discrete subspaces as well since

\[
\sup_{v_j \in S_j \setminus \{0\}} \sup_{q_\ell \in Q_\ell \setminus \{0\}} \frac{|\langle Bv_j, q_\ell \rangle|}{\|v_j\|_X \|q_\ell\|_Y} \leq \sup_{v \in X \setminus \{0\}} \sup_{q \in Y \setminus \{0\}} \frac{|\langle Bv, q \rangle|}{\|v\|_X \|q\|_Y} < \infty. \tag{7}
\]

But for general \(B\), the inf-sup condition (3) on the spaces \(X\) and \(Y\) does generally not imply its discrete counterpart

\[
\inf_{v_j \in S_j \setminus \{0\}} \sup_{q_\ell \in Q_\ell \setminus \{0\}} \frac{|\langle Bv_j, q_\ell \rangle|}{\|v_j\|_X \|q_\ell\|_Y} =: \beta_{j,\ell} > 0. \tag{8}
\]

The condition (8) is referred to as discrete inf-sup condition. The discrete inf-sup condition determines the stability of a Petrov-Galerkin approach. Especially, it is important for the uniform stability, that the constants \(\beta_{j,\ell}\) can be bounded uniformly from below by a constant \(\beta > 0\) which is independent of the discretization represented by \(j\) and \(\ell\). Otherwise, a stability problem appears for the limit \(S_j \to X\) and \(Q_\ell \to Y\), i.e. for \(j,\ell \to \infty\). The discrete inf-sup condition (8) plays an important role for quasi-optimality of Petrov-Galerkin solutions respectively minimal residual Petrov-Galerkin solutions since the quasi-optimality constant depends reciprocally on the discrete inf-sup constant. This follows from the following theorem, see [3, Th. 3.1].
Theorem 1.1. Let \( B \in \mathcal{L}(X,Y') \) and assume that the discrete inf-sup condition (8) with respect to \( S_j \subset X \) and \( Q_\ell \subset Y \) is satisfied. Then for any \( u \in X \) there exists a unique \( u_j \in S_j \) which satisfies
\[
\mathcal{R}_\ell(u_j) = \inf_{v_j \in S_j} \mathcal{R}_\ell(v_j), \quad \mathcal{R}_\ell(v_j) := \sup_{q_\ell \in Q_\ell \setminus \{0\}} |\langle Bv_j - Bu, q_\ell \rangle| / \|q_\ell\|_Y.
\] (9)
Moreover, there holds the quasi-optimality estimate
\[
\|u - u_j\|_X \leq \frac{C_B}{\beta_{j,\ell}} \inf_{v_j \in S_j} \|u - v_j\|_X,
\] (10)
with discrete inf-sup constant \( \beta_{j,\ell} \) given by (8) and continuity constant \( C_B \) given by (2).

Thus, if the discrete inf-sup condition (8) is satisfied uniformly, i.e., \( \inf_{j,\ell} \beta_{j,\ell} \geq \beta > 0 \) for some \( \beta \) independent of \( j \) and \( \ell \), then \( u_j \) is the quasi-optimal approximation of \( u \) with
\[
\|u - u_j\|_X \leq \frac{C_B}{\beta} \inf_{v_j \in S_j} \|u - v_j\|_X.
\]
Otherwise, if \( \beta_{j,\ell} \) tends to zero for \( j, \ell \to \infty \), the quasi-optimality constants \( \frac{C_B}{\beta_{j,\ell}} \) in (10) tend to infinity.

Moreover, the uniform discrete inf-sup condition is crucial for the optimal preconditioning in an adequate wavelet setting. This indeed was the main motivation for our work. The optimal preconditioning can only be guaranteed, if the discrete inf-sup condition holds with a lower bound which does not depend on the levels \( j \) and \( \ell \). The built-in optimal preconditioning effect is a well known property of suitable wavelet discretizations, see [18] and the survey [28] with practical improvements.

The idea of a space-time weak formulation is to obtain a full weak formulation in space and time. Classical approaches to solve parabolic evolution problems are the method of lines or Rothe’s method. Both methods can be applied to operator equations of the form
\[
\frac{du(t)}{dt} + A(t)u(t) = f(t), \quad u(0) = u_0,
\] (11)
with an elliptic operator \( A(t) \). The idea of the method of lines is to solve a coupled system of ordinary differential equations (ODEs) obtained by a spatial semidiscretization, see [27, 37], and Rothe’s method is based on a time semidiscretization, see [29, 34]. That is, these two methods as well as the discontinuous Galerkin method (see e.g. [24]), which has become more and more popular in the last years, are based on semidiscretizations and are time marching methods. The stability of explicit time marching methods generally requires a CFL condition, i.e., a restriction on the time step size. The aim of a full space and time weak formulation now is to eliminate the explicit time dependency in (11) and to discretize in space and time simultaneously. That means, one does not need any time stepping, but can calculate for instance the Petrov-Galerkin solution in space and time directly. It can be shown that one obtains a well-posed equation on Hilbert spaces defined on the whole space-time cylinder. That is, we arrive at a formulation of the form (1) such that the general stability results from section 2 can be applied to this class of problems. Moreover, in this formulation, only a comparatively low regularity of the solution is required and the minimal residual Petrov-Galerkin solution converges quasi-optimally to the exact solution, see Theorem 1.1.
Schwab and Stevenson respectively Chegini and Stevenson considered space-time adaptive wavelet methods in the recent works [13, 35]. In these methods the adaptive procedure stabilizes automatically by inheriting the stability properties of the underlying infinite-dimensional problem. Nevertheless, the inf-sup condition of the infinite-dimensional problem does not imply its discrete version on an \textit{a priori fixed} grid. In the present paper we will use the space-time formulation from [13]. A similar formulation was already used in [7, 8], where a stable space-time discretization of \(p\) type respectively \(h-p\) type in time is considered. However, the discrete inf-sup constants are not proven to be bounded uniformly in temporal direction. A space-time weak formulation also for stochastic PDEs is considered by Larson and Molteni, see [30]. It was already observed in [2, 13] that the sequence of stiffness matrices \(B_{j,j}\) which stems from a space-time weak formulation is generally not uniformly stable or consequently not optimally preconditioned when using the same discretization for the trial space as for the test space. In [13] the authors applied the discretization directly to the normal equation, i.e., they considered \((B^T B)_{j,j}\). Generally, this is only possible if the stiffness matrices are truly sparse, since otherwise it is not possible to calculate \((B^T B)_{j,j}\) exactly.

For parabolic partial differential equations (PDEs), an approach using space-time sparse grid multilevel methods was introduced in [25]. The authors interpreted space-time sparse grid discretization schemes as the sparse counterparts of the Crank-Nicolson scheme and the discontinuous Galerkin scheme with piecewise constant or linear functions in time. It was shown that the additional temporal dimension in the complexity estimates can be avoided with these constructions. Nevertheless, uniform stability was not proven and in [2] it was even shown that indeed the Crank-Nicolson method is generally not a stable space-time method. The interpretation of essentially time-stepping methods via the space-time formulation was also exploited for reduced basis approximations [39, 38] as well as for (quantized) tensor train (QTT) low rank tensor approximations [23] just to mention a few. In that regard, we would like to mention [39, 38] in the context of the Crank-Nicolson method and [31] in the context of the discontinuous Galerkin method for the heat equation. These works give stability results by introducing subspace-dependent norms but not directly with respect to the natural space-time norms. [40] gives an abstract guideline to establish stability with respect to natural norms, once it is proven with respect to other (e.g. mesh-dependent) norms. In [17], the authors first consider “ideal test spaces” which yield corresponding Petrov-Galerkin operators with condition number one. Since these ideal spaces are generally computationally infeasible, a perturbation will be introduced so that the perturbed spaces still give rise to uniformly stable Petrov-Galerkin discretizations. Different from our approach, these perturbed spaces are not given a priori, but are only constructed implicitly in a numerical solution scheme. Although formulated for more general linear operator equation, the authors focus on first order transport equations. Another idea in a different context of diffusion-convection problems in one dimension was introduced in [9, 10], where the authors choose test spaces which (approximately) symmetrize the associated bilinear form in order to obtain a symmetric, coercive bilinear form and so retain optimality.

In [2, 3], however, the author studied the stability of sparse space-time Petrov-Galerkin discretizations for parabolic PDEs in the weak formulation from [35] very detailed (see also [4]). The main result in [3, Th. 4.1] gives a stability proof under basically two general assumptions on the discrete subspaces. The main assumption can be interpreted as a stability relation between the discrete test spaces and the time derivative of the discrete trial spaces with respect to the duality pairing on the test space and its dual space. Such type of
relation is comparable to a reverse Cauchy-Schwarz inequality, cf. (17) and (18). A simplified criterion only assuming stability relations with respect to the temporal- and spatial spaces separately which ensures stability of the sparse space-time Petrov-Galerkin discretization [3, Prop. 4.2] and also specific examples of continuous, piecewise linear basis functions were introduced [3, Prop. 6.1, 6.3]. It turned out that for these type of bases the discrete temporal test space need to be enriched only by one level for this problem class of Petrov-Galerkin discretizations of parabolic PDEs in space-time weak form. Although directly applying our general statement from Theorem 2.1 to the space-time weak formulation of parabolic PDEs will yield a larger number of extra layers to guarantee uniform stability, our main result holds for more general operator equations and this problem class serves as an important model example. Moreover, we only need to assume standard Bernstein- and Jackson estimates on the temporal- and spatial discrete test- and trial spaces (cf. Table 1) which are known to hold for, e.g., finite elements of sufficiently high polynomial order and global smoothness, see Remark 3.1. That is, although it is likely that the assumptions made in [3, Th. 4.1] are also satisfied for spaces spanned by these functions, we have an easy recipe at hand in order to choose suitable bases for the discrete spaces in the case of parabolic PDEs in space-time weak form.

The main goal of this paper is to prove uniform stability for generic operator equations basically under standard approximation and smoothness assumptions on the discrete spaces. In this regard, we will prove the existence of a level independent lower bound $0 < \beta \leq \beta_{j,\ell}$ for the discrete inf-sup condition (8) for sufficiently regular boundedly invertible operators. We will explicitly state the value of $\beta$ as well as the number of required extra layers $L$ depending on the operator and on the discretization. We will use similar ideas as in [19], which deals with the LBB condition for second order elliptic boundary value problems with essential boundary conditions enforced with the aid of Lagrange multipliers leading to saddle point problems. It was shown that the discrete inf-sup condition is satisfied uniformly, if the discretization on the boundary is somewhat finer than the discretization on the domain. This statement holds under some general assumptions on the operator and the discrete spaces. For a brief overview of the results in the present paper we refer to [32]. Our work was also inspired by a presentation of Tantardini on the ECCOMAS workshop in Vienna ([36]).

After proving the main result for boundedly invertible operators under general assumptions on the discretization, the next step is to apply this result to parabolic evolution problems in a space-time weak form. To this end, we construct suitable families of approximation spaces and prove the assumptions made in Theorem 2.1. This problem class will turn out to be important in view of stability since we will see that choosing the same discretization for the discrete test- and solution spaces indeed would lead to instabilities.

The remainder of this paper is organized as follows. First, we will state the main result in section 2. This result requires some general assumptions which depend essentially on the particular operator and the involved spaces. After proving the main result, the goal is to verify these assumptions for a specific situation. This will be done in section 3. To this end, we first introduce the space-time weak formulation of a parabolic evolution problem. This formulation will serve as our (model) problem which will be considered for the rest of the paper. Afterward we will state suitable, still general, families of discretization spaces for this problem. For these discrete spaces we will prove the validity of the assumptions made in section 2. Finally, in section 4, we will underline the previous statements with some numerical results concerning the discrete inf-sup constants.
2. General Set-up and Main Result

Given Hilbert spaces $X$, $Y$ and $\mathcal{H}$ such that by identifying the pivot space $\mathcal{H}$ with its dual we obtain the Gelfand triple $X \hookrightarrow \mathcal{H} \hookrightarrow X'$. Moreover, let $X'_+ \hookrightarrow X'$ and $Y_+ \hookrightarrow Y$ be additional Hilbert spaces which are continuously and densely embedded in $X'$ and $Y$, respectively. Consider a boundedly invertible operator $B \in \mathcal{L}(X,Y')$ mapping from space $(X, \| \cdot \|_X)$ into the dual space of $(Y, \| \cdot \|_Y)$. Therefore, there are constants $0 < C_B < \infty$ and $0 < c_B < \infty$ with
\[
\|Bv\|_{Y'} \leq C_B \|v\|_X, \quad v \in X \quad \text{and} \quad \|B^{-1}\tilde{q}\|_X \leq c_B^{-1} \|	ilde{q}\|_{Y'}, \quad \tilde{q} \in Y'.
\] (12)

In order to prove a uniform discrete inf-sup condition, we need to consider the dual operator $B' : Y \to X'$ defined by
\[
\langle v, B'q \rangle_{X \times X'} := \langle Bv, q \rangle_{Y' \times Y} \quad \text{for all } v \in X, \ q \in Y.
\]

It is well known, that $B \in \mathcal{L}(X,Y')$ implies $B' \in \mathcal{L}(Y,X')$ with
\[
\|B'\|_{Y \to X'} = \|B\|_{X \to Y'} \quad \text{and} \quad \|(B')^{-1}\|_{X' \to Y} = \|B^{-1}\|_{Y' \to X},
\] (13)
see e.g. [5, Prop. 3.3.1]. That means, (12) also holds for the dual operator with the same constants.

Next, we need to make some general assumptions on the operator and on the discrete spaces.

The first assumption is a regularity assumption on the operator $B$. It is completely independent of the discretization.

**Assumption 1:** Assume that the dual operator $B'$ satisfies the regularity condition
\[
(B')^{-1} \in \mathcal{L}(X'_+, Y_+), \quad \text{with} \quad \|(B')^{-1}\|_{\mathcal{L}(X'_+, Y_+)} \leq C_+,
\] (14)
with a constant $0 < C_+ < \infty$.

Following (13), Assumption 1 could equivalently be stated for the primal operator $B$ instead of $B'$, but since we need to consider the dual operator in order to prove Lemma 2.1 and Theorem 2.1, we directly formulated Assumption 1 in this form. This assumption is very similar to the shift theorem in [40, (A1)].

The next assumptions are Jackson- and Bernstein estimates as well as an often called reverse Cauchy-Schwarz inequality with respect to sequences of subspaces of $X$ and $Y$ representing the discrete trial- and test spaces.

**Assumption 2:** Let $\{S_j\}_{j=j_0}^\infty \subset X$, $\{\tilde{S}_j\}_{j=j_0}^\infty \subset X'_+$ and $\{Q_{\ell}\}_{\ell=\ell_0}^\infty \subset Y_+$ be closed subspaces such that for some $\rho > 1$ the Bernstein estimate
\[
\|\tilde{v}_j\|_{X'_+} \leq C_{B,X'} \rho^j \|\tilde{v}_j\|_{X'}, \quad \tilde{v}_j \in \tilde{S}_j
\] (15)
as well as the Jackson estimate
\[
\inf_{q \in Q_{\ell}} \|q - q_{\ell}\|_Y \leq C_{J,Y} \rho^{-\ell} \|q\|_{Y_+}, \quad q \in Y_+
\] (16)
are satisfied with constants $C_{B,X'}, C_{J,Y} > 0$. Moreover, we assume that the reverse Cauchy-Schwarz inequality holds on $X$:

For every $v_j \in S_j$ there exists an element $\tilde{v}_j \in \tilde{S}_j$, depending on $v_j$, such that
\[
\|v_j\|_X \|\tilde{v}_j\|_{X'} \leq C_{CS}(v_j, \tilde{v}_j)_{X \times X'},
\] (17)
with a constant $C_{CS} > 0$, where $\langle \cdot, \cdot \rangle_{X \times X'}$ denotes the duality pairing between $X$ and $X'$ induced by the pivot space $\mathcal{H}$.

The reverse Cauchy-Schwarz inequality (17) can be formulated equivalently as

$$\inf_{v_j \in \tilde{S}_j \setminus \{0\}} \sup_{\tilde{v}_j \in \tilde{S}_j \setminus \{0\}} \frac{\langle v_j, \tilde{v}_j \rangle_{X \times X'}}{\|v_j\|_X \|\tilde{v}_j\|_{X'}} \geq (C_{CS})^{-1} > 0. \quad (18)$$

That is, the reverse Cauchy-Schwarz inequality can be seen as a stability property of $S_j$ and $\tilde{S}_j$ with respect to the duality pairing $\langle \cdot, \cdot \rangle_{X \times X'}$. The indices $j_0$ and $\ell_0$ refer to the lowest level of discretization. For simplicity one could start indexing with $j_0 = \ell_0 = 0$. Nevertheless, we have chosen this notation, since we will later deal with subspaces with, e.g., $\dim(S_j) \sim 2^{(d+1)j}$ with spatial dimension $d$, in accordance with the level indexing.

Of course the previous assumptions are somewhat abstract and maybe seem to be quite restrictive. But as we will see later in section 3 even for intersections of tensor product Hilbert spaces we can easily state families of spaces which satisfy Assumption 2. Moreover, Assumption 1 on the operator is a rather standard regularity result for many operators. Before we formulate the abstract main result, we need the following lemma.

**Lemma 2.1.** Assume that Assumption 1 and Assumption 2 are fulfilled. Then, for arbitrary $\tilde{v}_j \in \tilde{S}_j$ there exists an element $q_\ell \in Q_\ell$, depending on $\tilde{v}_j$, such that

$$\|\tilde{v}_j - B'q_\ell\|_{X'} \leq C_{J,X'} \rho^{-(\ell-j)} \|\tilde{v}_j\|_{X'}, \quad (19)$$

with a constant $C_{J,X'} := C_B C_+ C_{J,Y} C_{B,X'}$ and

$$\|q_\ell\|_{Y} \leq c_B^{-1}(C_{J,X'} \rho^{-(\ell-j)} + 1) \|\tilde{v}_j\|_{X'}. \quad (20)$$

**Proof.** Given $\tilde{v}_j \in \tilde{S}_j$ choose $q_\ell \in Q_\ell$ such that

$$\|\tilde{v}_j - q_\ell\|_{Y} = \min_{q_\ell \in Q_\ell \setminus \{0\}} \|\tilde{v}_j - q_\ell\|_{Y},$$

which exists since $Q_\ell$ is a closed subspace of a Hilbert space and so a Hilbert space itself. With this choice it follows

$$\|\tilde{v}_j - q_\ell\|_{Y} \geq C_B^{-1} \|\tilde{v}_j - B'q_\ell\|_{X'},$$

by using the boundedness (12). By the Jackson estimate (16), the regularity (14), and the Bernstein estimate (15) it follows

$$\|\tilde{v}_j - B'q_\ell\|_{X'} \leq C_B C_{J,Y} \rho^{-\ell} \|\tilde{v}_j\|_{Y} \leq C_B C_+ C_{J,Y} \rho^{-\ell} \|\tilde{v}_j\|_{X'} \leq C_B C_+ C_{J,Y} C_{B,X'} \rho^{-(\ell-j)} \|\tilde{v}_j\|_{X'},$$

proving (19). Similarly, we can prove (20) by

$$\|q_\ell\|_{Y} \leq \|q_\ell - (B')^{-1}\tilde{v}_j\|_{Y} + \|(B')^{-1}\tilde{v}_j\|_{Y} \leq C_B^{-1} \|B'q_\ell \tilde{v}_j\|_{X'} + C_B^{-1} \|\tilde{v}_j\|_{X'} \leq C_B^{-1}(C_{J,X'} \rho^{-(\ell-j)} + 1) \|\tilde{v}_j\|_{X'}. \quad \square$$

Finally, we can state our main result.
Theorem 2.1. Assume that Assumption 1 and Assumption 2 are fulfilled. Choose $L \in \mathbb{N}$ such that
\[ C_{CS}C_{J,X}\rho^{-L} < 1, \] (21)
with constants $C_{J,X}$ and $C_{CS}$ as in Lemma 2.1 and (17) and set
\[ \ell \geq j + L, \] (22)
for any refinement level $j$. Then the discrete inf-sup condition
\[ \inf_{v_j \in S_j \setminus \{0\}} \sup_{q_\ell \in Q_\ell \setminus \{0\}} |\langle Bv_j, q_\ell \rangle| \beta > 0 \]
is satisfied with a constant $\beta$ uniformly bounded away from 0 as $j \to \infty$. In particular, $\beta$ is given by
\[ \beta := \frac{C_{CS}^{-1} - C_{J,X}\rho^{-L}}{c_B(C_{J,X}\rho^{-L} + 1)}. \] (23)

Proof. Let $v_j \in S_j$ be arbitrary, then by Assumption 2 there exists an element $\tilde{v}_j \in \tilde{S}_j$ such that
\[ \|v_j\|_{X'} \leq CS\langle v_j, \tilde{v}_j \rangle_{X\times X'} = CS \langle v_j, \tilde{v}_j - B'q_\ell \rangle_{X\times X'} + \langle v_j, B'q_\ell \rangle_{X\times X'} \leq CS \langle v_j\|_{X} \|\tilde{v}_j - B'q_\ell\|_{X'} + \langle Bv_j, q_\ell \rangle_{Y'\times Y}. \]
Next, we choose $q_\ell \in Q_\ell$ according to Lemma 2.1 and obtain
\[ \|\tilde{v}_j - B'q_\ell\|_{X'} \leq C_{J,X}\rho^{-(\ell-j)}\|\tilde{v}_j\|_{X'}. \]
Then it follows
\[ (C_{CS}^{-1} - C_{J,X}\rho^{-(\ell-j)}) \|\tilde{v}_j\|_{X'} \leq \langle Bv_j, q_\ell \rangle_{Y'\times Y}. \]
Using (20) directly yields
\[ \frac{C_{CS}^{-1} - C_{J,X}\rho^{-(\ell-j)}}{c_B(C_{J,X}\rho^{-(\ell-j)} + 1)} \|q_\ell\|_{Y} \|v_j\|_{X} \leq \langle Bv_j, q_\ell \rangle_{Y'\times Y}, \]
where
\[ \frac{C_{CS}^{-1} - C_{J,X}\rho^{-(\ell-j)}}{c_B(C_{J,X}\rho^{-L} + 1)} =: \beta, \]
with the choice (22) of level $\ell$. Finally with the definition (21) of $L$ we have $\beta > 0$ which finishes the proof. \qed

This Theorem shows how to choose the level $\ell$ on the test space depending on the choice of level $j$ on the trial space in order to obtain a uniform discrete inf-sup condition. Obviously, (21) is satisfied when choosing
\[ L := \lceil \log_{\rho}(C_{CS}C_{J,X'}) \rceil. \] (24)

3. Application to Space-Time Weak Formulation

The aim of this section is to verify Assumption 1 and Assumption 2 of the previous section for a space-time weak formulation of parabolic evolution problems. To this end, we first introduce the concept of space-time weak formulations.
3.1. The Space-Time Weak Formulation

The space-time weak formulation in the present form can be found in [13]. For more general information and further properties concerning parabolic evolution equations, we refer to [22],[41].

Let \( V, H \) be separable (real) Hilbert spaces with dense embedding \( V \hookrightarrow H \). By identifying \( H \) with its dual, we obtain a Gelfand triple \( V \hookrightarrow H \hookrightarrow V' \). Let \( A(t) : V \to V' \) be a linear operator such that the mapping \( t \mapsto \langle Av, w \rangle \) is measurable on \([0,T]\) for any \( v,w \in V \) and given \( T \in \mathbb{R}^+ \). Moreover, we assume that \( A(t) \) is bounded and coercive for \( t \in [0,T] \) a.e., i.e., there exist constants \( M_a, \alpha > 0, \lambda \in \mathbb{R} \), such that for \( t \in [0,T] \) a.e.

\[
|\langle A(t)v, w \rangle| \leq M_a \|v\|_V \|w\|_V, \quad \text{for } v,w \in V, \\
\langle A(t)v, v \rangle + \lambda \|v\|_H^2 \geq \alpha \|v\|_V^2, \quad \text{for } v \in V. \tag{25}
\]

In order to state the parabolic problem, we briefly introduce the concept of Bochner spaces.

**Definition 3.1.** For a given Banach space \( V \), the **Bochner space** \( L_2(0;T;V) \) is the space of all strongly measurable functions \( u : [0,T] \to V \) such that the corresponding norm

\[
\|u\|_{L_2(0,T;V)}^2 := \int_0^T \|u(t)\|_V^2 \, dt < \infty \tag{26}
\]

is finite. Analogously, we define the vector valued Sobolev space \( H^m(0,T;V) \) with \( m \geq 0 \) as the space of all functions \( u \in L_2(0,T;V) \) with weak derivatives \( D^k u \in L_2(0,T;V) \) for all \( |k| \leq m \). A norm on \( H^m(0,T;V) \) is given by

\[
\|u\|_{H^m(0,T;V)} := \left( \sum_{|k| \leq m} \|D^k u\|_{L_2(0,T;V)}^2 \right)^{1/2}.
\]

Now, we can formulate the following parabolic problem.

**Problem 1.** For given right hand side \( f \in L_2(0,T;V') \) and initial condition \( u_0 \in H \), for a.e. \( t \in [0,T] \), find a solution \( u(t) \in V \) such that

\[
\frac{du}{dt}(t) + A(t)u(t) = f(t) \quad \text{in } V', \quad u(0) = u_0 \quad \text{in } H. \tag{27}
\]

This is the standard weak formulation with respect to space, formulated as a generic evolutionary equation with bounded and linear operator \( A(t) \in \mathcal{L}(V,V') \). In order to arrive at a space and time weak formulation, we test (27) with a test function \( q \in H^1(0,T;V) \) and integrate over \( t \in [0,T] \). Additionally, we perform integration by parts such that the initial condition is given as a natural boundary condition and not as an essential one. We end up with the following **space-time weak problem** of equation (27).

**Problem 2.** Find \( u \in X = L_2(0,T;V) \) such that

\[
\langle Bu, q \rangle = \ell(q) \quad \forall q \in Y = L_2(0,T;V) \cap H^1_{(T)}(0,T;V'), \tag{28}
\]

where

\[
\langle Bu, q \rangle := \int_0^T -\left( u(t), \frac{dq(t)}{dt} \right)_{V \times V'} \, dt + \int_0^T \langle A(t)u(t), q(t) \rangle_{V^* \times V} \, dt, \tag{29}
\]
\[
\ell(q) := \int_0^T \langle f(t), q(t) \rangle_{V' \times V} \, dt + (u_0, q(0))_H,
\]
where we have used \(\langle \cdot, \cdot \rangle_{V' \times V}\) and \(\langle \cdot, \cdot \rangle_{V \times V}\) in order to highlight that we are dealing with duality pairings on \(V\) and \(V'\) in space.

Here
\[
H_{\{T\}}^1(0, T; V') := \text{clos}_{H^1(0, T; V')} \{ v \in C^\infty(0, T; V') : v(T) = 0 \}
\]
denotes the Bochner space \(H^1(0, T; V')\) with zero final time condition and \((\cdot, \cdot)_H\) the inner product on the pivot space \(H\). Problem 2 is well defined since
\[
L_2(0, T; V) \cap H_{\{T\}}^1(0, T; V') \hookrightarrow C([0, T]; H) , \tag{30}
\]
where \(\hookrightarrow\) denotes continuous embedding, cf. [22, Ch. XVIII, §1, Th. 1].

We define norms on \(X\) and \(Y\) by
\[
\|u\|_X^2 := \|u\|^2_{L_2(0,T;V)}, \quad \|q\|_Y^2 := \|q\|^2_{L_2(0,T;V)} + \|q\|^2_{L_2(0,T;V')} + \|q'\|^2_{L_2(0,T;V')}, \tag{31}
\]
where we denote by \(q' := \frac{dq}{dt}\) the time derivative of \(q\).

These are the standard norms given on vector valued Sobolev spaces or intersections of them, respectively. Due to the isometry \(H^m(0, T; F) \cong H^m(0, T) \otimes F\), for separable Hilbert spaces \(F\), we can identify
\[
X \cong L_2(0, T) \otimes V, \quad Y \cong (L_2(0, T) \otimes V) \cap (H_{\{T\}}^1(0, T) \otimes V'),
\]
isometrically, see [5, Th. 12.7.1]. That means that the spaces \(X\) and \(Y\) are essentially tensor product spaces, respectively intersections of them. A detailed treatment of Bochner spaces can be found in [5, 22, 41] and of Sobolev spaces in [1]. Note that norm (31) on \(Y\) is equivalent to the norm
\[
|q|^2_Y := \|q\|^2_{L_2(0,T;V)} + \|q'\|^2_{L_2(0,T;V')},
\]
since \(V \hookrightarrow V'\), cf.[22, Ch. XVIII, §1, Prop. 6]. In [13, Th. 2.2] it was shown that the operator \(B \in \mathcal{L}(X, Y')\) defined by (29) is boundedly invertible. Therefore, property (12) holds. A similar approach is also valid for complex Hilbert spaces and sesquilinear forms. For explicit error bounds we refer to [35, Appendix A, Proof of Th. 5.1].

### 3.2. Verification of the Assumptions

First, we will state a criterion from [13, Th. 2.4.] which shows when Assumption 1 is satisfied for the operator defined by the bilinear form (29).

**Proposition 3.1.** Let \(W \hookrightarrow V\) with dense embedding. Assume that the dual elliptic operator \(A(\cdot)' \in C([0, T]; \mathcal{L}(W, H))\) and that \(A(t)' + \lambda I : W \to H\) is boundedly invertible for every \(t \in [0, T]\) with \(\lambda\) from (25). Then
\[
B \in \mathcal{L}(L_2(0, T; H), (L_2(0, T; W) \cap H_{\{T\}}^1(0, T; H))'),
\]
is boundedly invertible.

In order to verify Assumption 1, note that
\[
B^{-1} \in \mathcal{L}((L_2(0, T; W) \cap H_{\{T\}}^1(0, T; H))', L_2(0, T; H))
\]
is equivalent to
\[(B')^{-1} \in \mathcal{L}((L_2(0, T; H))', L_2(0, T; W) \cap H^1_{(T)}(0, T; H)).\]

It is well known, that by choosing \( W := H^{2m}(\Omega), V := H^m(\Omega) \) and \( H := L_2(\Omega) \) on a bounded domain \( \Omega \subset \mathbb{R}^d \), the assumptions in Proposition 3.1 are fulfilled e.g. for elliptic differential operators with smooth coefficients and smooth boundary \( \partial \Omega \), but also for many other relevant operators. See also the references in [13] for more details. Due to this fact, we choose
\[
X_+ := L_2(0, T; L_2(\Omega)), \quad Y_+ := L_2(0, T; H^{2m}(\Omega)) \cap H^1_{(T)}(0, T; L_2(\Omega))
\]
with bounded domain \( \Omega \subset \mathbb{R}^d \). That means, we consider the parabolic problem (28) with an elliptic operator \( A(t) \) of order \( 2m \) involved according to (29). Depending on the type of essential boundary conditions for the elliptic part, these need to be included into \( H^m(\Omega) \).

In order to verify Assumption 2 we need to specify the choice of discrete subspaces \( S_j \subset X \) and \( Q_j \subset Y_+ \). We choose \( S^x_j \subset H^m(\Omega), S^y_j \subset L_2(0, T), Q^y_j \subset H^m(\Omega) \) and \( Q^x_j \subset H^1_{(T)}(0, T) \) and define
\[
S_j := S^x_j \otimes S^y_j \subset L_2(0, T; H^m(\Omega)) = X,
\]
\[
Q_j := Q^y_j \otimes Q^x_j \subset L_2(0, T; H^m(\Omega)) \cap H^1_{(T)}(0, T; L_2(\Omega)) = Y_+.
\]

We arrange the spaces in such a way that they form a sequence of closed subspaces \( \{S^x_j\}_{j=j_0}^{\infty} \) satisfying
\[
S^x_j \subset S^x_{j+1} \subset \cdots \subset S^x_{j_0+1} \subset \cdots \subset H^m(\Omega), \quad \text{clos}_{H^m(\Omega)} \left( \bigcup_{j=j_0}^{\infty} S^x_j \right) = H^m(\Omega), \tag{33}
\]
and similarly for \( S^y_j, Q^y_j \) and \( Q^x_j \).

Next we need to make some general assumptions on the discrete subspaces. We consider another sequence \( \{\hat{S}^x_j\}_{j=j_0}^{\infty} \) of nested and closed subspaces of \( L_2(\Omega) \) with \( \dim \hat{S}^x_j = \dim S^x_j \) such that there exists a constant \( c_S^{(x)} > 0 \) so that
\[
\inf_{v \in \hat{S}^x_j \setminus \{0\}} \sup_{\tilde{v} \in \hat{S}^x_j \setminus \{0\}} \frac{|(v, \tilde{v})_{L_2(\Omega)}|}{\|v\|_{L_2(\Omega)} \|	ilde{v}\|_{L_2(\Omega)}} \geq c_S^{(x)}, \quad \text{for all } j \geq j_0. \tag{34}
\]
This condition is often referred to as \( L_2(\Omega) \)-stability relation. Obviously, this assumption is fulfilled when choosing \( \hat{S}^x_j = S^x_j \), but using (34) moreover allows us to use biorthogonal bases, for instance, cf. [20]. That is, we have more flexibility in the choice of discretization spaces. We consider sequences \( \{\hat{S}^y_j\}_{j=j_0}^{\infty}, \{\hat{Q}^y_j\}_{\ell=t_0}^{\infty} \) and \( \{Q^y_{\ell}\}_{\ell=t_0}^{\infty} \) in a similar way. It is important to note that the dual subspaces \( \hat{S}^y_j, \hat{S}^x_j, \hat{Q}^y_j \) and \( Q^x_j \) are only needed for the analysis of the discrete inf-sup condition but do not enter the implementation concerning the discrete operator discretization (52).

Since we deal with tensor product spaces, we need analog results for the space \( L_2(0, T) \otimes L_2(\Omega) \).

**Lemma 3.1.** The sequence \( \{\hat{S}_j\}_{j=j_0}^{\infty} := \{\hat{S}^x_j \otimes \hat{S}^y_j\}_{j=j_0}^{\infty} \) satisfies
\[
\inf_{v \in \hat{S}_j \setminus \{0\}} \sup_{\tilde{v} \in \hat{S}_j \setminus \{0\}} \frac{|(v, \tilde{v})_{L_2(0, T) \otimes L_2(\Omega)}|}{\|v\|_{L_2(0, T; L_2(\Omega))} \|	ilde{v}\|_{L_2(0, T; L_2(\Omega))}} \geq c_S, \quad \text{for all } j \geq j_0, \tag{35}
\]
where $c_S := c_S^{(t)}c_S^{(x)}$. The same holds for the sequence $\{\tilde{Q}_\ell\}_{\ell=t_0}^\infty := \{\tilde{Q}_\ell \otimes \tilde{Q}_\ell^x\}_{\ell=t_0}^\infty$ with $c_Q := c_Q^{(t)}c_Q^{(x)}$.

Proof. See [3, Cor. 5.3].

Furthermore, we can conclude the existence of certain projectors onto the discrete spaces.

**Proposition 3.2.** The stability (34) implies the existence of sequences $\{P_{S^j_i}\}_{j=0}^\infty$ of biorthogonal projectors $P_{S^j_i} : L_2(\Omega) \rightarrow S^j_i$ such that for $j \geq j_0$, range$(I - P_{S^j_i}) = (S^j_i)^{⊥L_2(\Omega)}$, while the adjoints $P_{S^j_i}' : L_2(\Omega) \rightarrow \tilde{S}^j_i$ satisfy range$(I - P_{S^j_i}') = (S^j_i)^{⊥L_2(\Omega)}$. The projectors are uniformly bounded with respect to $L_2(\Omega)$ with

$$\|P_{S^j_i}v\|_{L_2(\Omega)} \leq (c_S^{(x)})^{-1}\|v\|_{L_2(\Omega)}, \quad \text{for } j \geq j_0, \ v \in L_2(\Omega).$$

Furthermore, one has

$$P_{S^j_i}P_{S^j_i} = P_{S^j_i}, \quad P_{S^j_i}'P_{S^j_i}' = P_{S^j_i}'', \quad i \leq j.$$ (37)

The same holds for projectors $P_{Q^j_i}, P_{Q^\ell_i}, P_{Q^\ell_i}', P_{S^j_i}'P_{Q^\ell_i}'$ and $P_{Q^\ell_i}'$ defined in a similar fashion.

Proof. See [21, Th. 2.1].

Furthermore, we arrange the involved spaces such that they fulfill certain approximation and smoothness properties. In order to shorten the notation, let $F_k$ denote one of these spaces $\{S_j^i\}_{j=j_0}^\infty, \{S_j^\ell\}_{j=j_0}^\infty, \{Q_j^i\}_{\ell=t_0}^\infty$ and $\{Q_j^\ell\}_{\ell=t_0}^\infty$, respectively one of its dual spaces, with $k \in \{j, \ell\}$ as well as $\Omega' \in \{(0,T), \Omega\}$ in order to deal with time and space domain simultaneously.

For fixed $\omega > 1$ consider the Jackson inequality with respect to a sequence of spaces $\{F_k\}_{k=k_0}^\infty$:

$$\inf_{f_k \in F_k} \|f - f_k\|_{L_2(\Omega')} \lesssim \omega^{-sk}\|f\|_{H^s(\Omega')}, \quad f \in H^s(\Omega'), \quad 0 \leq s \leq d_F$$ (38)

and the Bernstein estimate with respect to a sequence of spaces $\{F_k\}_{k=k_0}^\infty$:

$$\|f_k\|_{H^s(\Omega')} \lesssim \omega^{sk}\|f_k\|_{L_2(\Omega')}, \quad f_k \in F_k, \quad 0 \leq s < \gamma_F.$$ (39)

The parameter $d_F$ characterizes the approximation order and the parameter $\gamma_F$ the smoothness of the space $F_k$. In view of the required Bernstein and Jackson estimate (15) and (16), we arrange the spaces $\{S_j^i\}_{j=j_0}^\infty, \{S_j^\ell\}_{j=j_0}^\infty, \{Q_j^i\}_{\ell=t_0}^\infty$ and $\{Q_j^\ell\}_{\ell=t_0}^\infty$ as well as its dual versions with $\sim$’, such that they fulfill the Bernstein and Jackson estimates with parameters listed in Table 1.

**Remark 3.1.** The approximation and smoothness properties given in Table 1 are known to hold for hierarchical spline spaces on uniform grids, i.e. for finite element as well as for spline-wavelet spaces. We also have in mind e.g. space-time sparse grid spaces [25] instead of uniform full grid spaces. Using spline discretizations, the smoothness is determined by the global smoothness of the elements and the approximation property by the piecewise polynomial degree of the elements. Of course, one could directly state one special discretization which fulfills all desired properties, but we would like to keep the construction more general. In a dyadic partitioning of the domain the parameter generally can be chosen as $\omega = 2$. For more details on splines, finite elements and wavelets we refer to [11], [12] and [16], respectively.
Table 1. Approximation and smoothness parameters.

<table>
<thead>
<tr>
<th>space</th>
<th>primal</th>
<th>dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_j^t$</td>
<td>$\gamma_{S^t}, d_{S^t} &gt; 0$</td>
<td>$\gamma_{S^t}, d_{S^t} &gt; 0$</td>
</tr>
<tr>
<td>$S_j^x$</td>
<td>$\gamma_{S^x}, d_{S^x} &gt; m$,</td>
<td>$\gamma_{S^x}, d_{S^x} &gt; 0$</td>
</tr>
<tr>
<td>$Q_t^i$</td>
<td>$\gamma_{Q^i}, d_{Q^i} &gt; 1$</td>
<td>$\gamma_{Q^i}, d_{Q^i} &gt; 0$</td>
</tr>
<tr>
<td>$Q_t^x$</td>
<td>$\gamma_{Q^x}, d_{Q^x} &gt; 2m$</td>
<td>$\gamma_{Q^x}, d_{Q^x} &gt; m$</td>
</tr>
</tbody>
</table>

The following theorem from [21, Th. 2.1] respectively [15, Th. 3.2] will be very helpful to prove Assumption 2.

**Theorem 3.1.** Assume stability property (34) for all involved spaces as well as the Jackson and Bernstein inequalities (38) and (39) with associated approximation and smoothness parameters from Table 1. Then, with $P_{S^t_{j_0}} := 0$, the following norm equivalence holds:

$$
\left( \sum_{j=j_0}^{\infty} \omega^{2s_j} \| (P_{S_j^t} - P_{S_{j-1}^t}) v \|_{L^2(\Omega)}^2 \right)^{1/2} \sim \| v \|_{H^s(\Omega)}, \quad v \in H^s(\Omega),
$$

for all $s \in (-\min\{\gamma_{S^t}, d_{S^t}\}, \min\{\gamma_{S^x}, d_{S^x}\})$. Analog equivalences hold for the sequence of spaces $\{S_j^t\}_{j=j_0}^{\infty}$, $\{Q_t^i\}_{t=t_0}^{\infty}$ and $\{Q_t^x\}_{t=t_0}^{\infty}$, as well as for $\{S_j^x\}_{j=j_0}^{\infty}$, $\{\tilde{S}_j^x\}_{j=j_0}^{\infty}$, $\{\tilde{Q}_t^x\}_{t=t_0}^{\infty}$ and $\{\tilde{Q}_t^x\}_{t=t_0}^{\infty}$ with interchanged roles of $\gamma_{S^x}, d_{S^x}$ and $(\gamma_{S^x}, d_{S^x})$.

Finally, we have collected all necessary ingredients to verify Assumption 2 by assuming the standard assumptions from Table 1. First, we will verify the Bernstein estimate (15).

**Proposition 3.3.** The Bernstein estimate

$$
\| \tilde{v}_j \|_{X_+^t} \leq C_{B,X'} \omega^{jm} \| \tilde{v}_j \|_{X'}, \quad \tilde{v}_j \in \tilde{S}_j
$$

holds for spaces $\tilde{S}_j$ constructed according to Table 1 and $X_+^t$ defined in (32).

**Proof.** For every $\tilde{v}_j \in \tilde{S}_j$ there holds

$$
\| \tilde{v}_j \|_{L_{2}(0,T;L_{2}(\Omega))}^2 = \int_0^T \| \tilde{v}_j(t) \|_{L_{2}(\Omega)}^2 \, dt = \int_0^T \sup_{\| u \|_{L_{2}(\Omega)} = 1} | \langle \tilde{v}_j(t), P_{S_j^x} u \rangle |^2 \, dt
$$

$$
\leq \int_0^T \sup_{\| u \|_{L_{2}(\Omega)} = 1} \| \tilde{v}_j(t) \|_{H^{-m}(\Omega)} \| P_{S_j^x} u \|_{H^m(\Omega)}^2 \, dt
$$

$$
\leq \int_0^T \sup_{\| u \|_{L_{2}(\Omega)} = 1} \| \tilde{v}_j(t) \|_{H^{-m}(\Omega)}^2 \omega^{2jm} \| P_{S_j^x} u \|_{L_{2}(\Omega)}^2 \, dt
$$

$$
\leq \omega^{2jm} \int_0^T \| \tilde{v}_j(t) \|_{H^{-m}(\Omega)}^2 = \omega^{2jm} \| \tilde{v}_j \|_{L_{2}(0,T;H^{-m}(\Omega))}^2,
$$

since $\gamma_{S^x} > m$ by the choice of parameters in Table 1, where we have used definition (26) and $L_{2}(\Omega)$-stability (36).
A more rigorous tracking of the involved constants would yield $C_{B,X'} \leq C_{B,S}^{(x)} (c_S^{(x)})^{-1}$, where $c_S^{(x)}$ denotes the $L_2$-stability constant from (34) and $C_{B,S}^{(x)}$ the constant in the Bernstein estimate (39) for $S^r_T$ as well as $\rho = \omega^m$.

The next proposition verifies the Jackson estimate (16).

**Proposition 3.4.** The Jackson estimate

$$
\inf_{q_t \in Q_t} \|q - q_t\|_{Y} \leq C_{d,Y} \omega^{-\ell m} \|q\|_{Y^+}, \quad q \in Y^+
$$

holds for spaces $Q_t$ constructed according to Table 1 and $Y^+$ defined in (32).

**Proof.** Due to Theorem 3.1 with Bernstein and Jackson parameters from Table 1 and by using [26, Prop. 1 and Prop. 2] the splittings

$$
\{Y; (\cdot, \cdot)_Y\} = \sum_{k=\ell_0}^{\infty} \sum_{i=\ell_0}^{\infty} \{\text{range}(D_{Q,k,i}); (\omega^{2mi} + \omega^{2k-2im})(\cdot, \cdot)_{L_2(0,T) \otimes (\cdot, \cdot)_{L_2(\Omega)}}\},
$$

and

$$
\{Y^+; (\cdot, \cdot)_{Y^+}\} = \sum_{k=\ell_0}^{\infty} \sum_{i=\ell_0}^{\infty} \{\text{range}(D_{Q,k,i}); (\omega^{2(2m)i} + \omega^{2k})(\cdot, \cdot)_{L_2(0,T) \otimes (\cdot, \cdot)_{L_2(\Omega)}}\},
$$

are stable. Here we have used the abbreviation $D_{Q,k,i} := (P_{Q_t}^t - P_{Q_{t-1}}^t) \otimes (P_{Q_t}^t - P_{Q_{t-1}}^t)$, with projectors defined in Proposition 3.2 and $P_{Q_{t-1}}^t = 0$ as well as $P_{Q_{t-1}}^0 = 0$. That means, we have the norm equivalences

$$
\left(\sum_{k=\ell_0}^{\infty} \sum_{i=\ell_0}^{\infty} (\omega^{2mi} + \omega^{2k-2im}) \|D_{Q,k,i}q\|_{L_2(0,T;L_2(\Omega))}^2\right)^{1/2} \sim \|q\|_{Y}, \quad q \in Y,
$$

as well as

$$
\left(\sum_{k=\ell_0}^{\infty} \sum_{i=\ell_0}^{\infty} (\omega^{2(2m)i} + \omega^{2k}) \|D_{Q,k,i}q\|_{L_2(0,T;L_2(\Omega))}^2\right)^{1/2} \sim \|q\|_{Y^+}, \quad q \in Y^+.
$$

For more details about tensor product subspace splittings we refer to [26]. Using these equivalences we can conclude

$$
\inf_{q_t \in Q_t} \|q - q_t\|_{Y} \leq \|q - \sum_{k=\ell_0}^{\ell} \sum_{i=\ell_0}^{\ell} D_{Q,k,i}q\|_{Y} = \|\sum_{k=\ell+1}^{\infty} \sum_{i=\ell+1}^{\infty} D_{Q,k,i}q\|_{Y}.
$$
for \( q \in X_+ \).

Again, we can bound the constant \( C_{J,Y} \) by tracking the involved constants. One obtains
\[
C_{J,Y} \leq \left( c_{Y_+} \right)^{-1} c_Y,
\]
where \( c_Y \) denotes the upper bound in (43) and \( c_{Y_+} \) the lower bound in (44) as well as \( \rho = \omega^m \).

Finally, we verify the reverse Cauchy-Schwarz inequality (17).

**Proposition 3.5.** Assume that the spaces \( S_j \) and \( \tilde{S}_j \) are constructed according to Table 1. Then for every \( v_j \in S_j \) there exists an element \( \tilde{v}_j \in \tilde{S}_j \), depending on \( v_j \), such that
\[
\| v_j \|_X \| \tilde{v}_j \|_{X'} \leq C_{CS}(v_j, \tilde{v}_j)_{L_2(0,T;L_2(\Omega))},
\]
with a constant \( 0 < C_{CS} < \infty \).

**Proof.** Let \( k \leq i \leq j \) and \( v \in S_j \), then \( D_{S,k,i} v \in S_i \), by the nestedness (33) with
\[
D_{S,k,i} := (P^i_{S_k} - P^i_{S_{k-1}}) \otimes (P^i_{S_i} - P^i_{S_{i-1}}),
\]
with \( P^i_{S_k} = 0 \) as well as \( P^i_{S_{i-1}} = 0 \), and due to the stability (35) there exists an element \( \tilde{v}_{k,i} \in \tilde{S}_i \) such that
\[
(D_{S,k,i} v, \tilde{v}_{k,i})_{L_2(0,T;L_2(\Omega))} \geq c_S \| D_{S,k,i} v \|_{L_2(0,T;L_2(\Omega))} \| \tilde{v}_{k,i} \|_{L_2(0,T;L_2(\Omega))}.
\]
Defining \( \tilde{v}_{k,i} := \omega^{2m_i} \frac{\| D_{S,k,i} v \|_{L_2(0,T;L_2(\Omega))}}{\| v \|_{L_2(0,T;L_2(\Omega))}} \tilde{v}_{k,i} \) yields
\[
(D_{S,k,i} v, \tilde{v}_{k,i})_{L_2(0,T;L_2(\Omega))} \geq c_S \omega^{2m_i} \| D_{S,k,i} v \|_{L_2(0,T;L_2(\Omega))}^2.
\]
Setting \( \tilde{v} := \sum_{k=j_0}^j \sum_{i=j_0}^j D'_{S,k,i} \tilde{v}_{k,i} \in \tilde{S}_j \) yields
\[
(v, \tilde{v})_{L_2(0,T;L_2(\Omega))} = \sum_{k=j_0}^j \sum_{i=j_0}^j (D_{S,k,i} v, \tilde{v}_{k,i})_{L_2(0,T;L_2(\Omega))} \geq c_S \sum_{k=j_0}^j \sum_{i=j_0}^j \omega^{2m_i} \| D_{S,k,i} v \|_{L_2(0,T;L_2(\Omega))}^2,
\]
by using \( D_{S,k',i'} D_{S,k,i} = D_{S,k',i} \delta_{i',i} \delta_{k',k} \) according to (37). Due to the construction of \( S_j \) and \( \tilde{S}_j \), we have the norm equivalence
\[
c_X \left( \sum_{k=j_0}^j \sum_{i=j_0}^j \omega^{2m_i} \| D_{S,k,i} v_j \|_{L_2(0,T;L_2(\Omega))}^2 \right)^{1/2} \leq \| v_j \|_X \leq C_X \left( \sum_{k=j_0}^j \sum_{i=j_0}^j \omega^{2m_i} \| D_{S,k,i} v_j \|_{L_2(0,T;L_2(\Omega))}^2 \right)^{1/2},
\]
for any \( v_j \in S_j \) and constants \( C_X \geq c_X > 0 \). Furthermore for the dual norm we obtain
\[
\| v_j \|_{X'} \leq C_X^{-1} \left( \sum_{k=j_0}^j \sum_{i=j_0}^j \omega^{-2m_i} \| D'_{S,k,i} v_j \|_{L_2(0,T;L_2(\Omega))}^2 \right)^{1/2}.
\]
Using norm equivalence (47) as well as (48), we obtain
\[
\| v \|_X \| \tilde{v} \|_{X'} \leq 4c_\delta^{-1} \kappa \sum_{k=j_0}^j \sum_{i=j_0}^j \omega^{2m_i} \| D_{S,k,i} v \|_{L_2(0,T;L_2(\Omega))}^2,
\]
with $\kappa := C_\gamma / C_X$, where we have used that
\[ \|D_{S,k,i}v\|_{L^2(0,T;L^2(\Omega))} = \omega^{2m_t} \|D_{S,k,i}v\|_{L^2(0,T;L^2(\Omega))} \|\tilde{v}_{k,i}\|_{L^2(0,T;L^2(\Omega))} \leq \omega^{2m_t} \|D_{S,k,i}v\|_{L^2(0,T;L^2(\Omega))} 4c_S^{-1}, \]
since $D_{S,k,i}$ is stable according to (36) with
\[
\| (P'_{S_{k}} - P'_{S_{k-1}}) \otimes (P'_{S_{l}} - P'_{S_{l-1}}) \| \\
\leq \left( \|P'_{S_{k}}\| + \|P'_{S_{l-1}}\| \right) \left( \|P'_{S_{l}}\| + \|P'_{S_{l-1}}\| \right) \\
\leq 4c_S^{-1}.
\]
Combining (46) and (49) finishes the proof.

That means, the reverse Cauchy-Schwarz inequality (17) holds with $C_{CS} \leq 4c_S^{-2}c_X^{-1}C_X$, with $L_2$-stability constant $c_S = C_S c_S' c_S''$ from (35), and the lower and upper bound $c_X$ and $C_X$, respectively, in (47).

Ultimately, by the previous propositions 3.1, 3.3, 3.4 and 3.5 we have shown that Assumption 1 and Assumption 2 of the previous section are satisfied for an operator $B$ defined by the bilinear form (29) and for the families of spaces introduced in this section. That means that Theorem 2.1 can be applied to this situation. So the discrete inf-sup condition holds with a constant which does not depend on the discretization when choosing the levels $j$ and $\ell$ according to Theorem 2.1. An overview of all relevant constants is given in Table 2.

To some extent, the following remark generalizes our previous results.

**Remark 3.2.** The choices of parameters we have made in this subsection (Table 1) as well as the corresponding results are only restricted due to the choices of subspaces $X'_+ \subset X'$ and $Y'_+ \subset Y$ according to (32). These particular choices in turn are induced by Proposition 3.1. Assuming generally that the regularity (14) holds for subspaces
\[ X'_+ := H^{d_1}(0, T; H^{d_x-m}(\Omega)), \quad Y'_+ := H^{d_t}(0, T; H^{d_x+m}(\Omega)) \cap H^{d_t+1}(0, T; H^{d_x-m}(\Omega)), \]
with $d_t \geq 0$, $d_x > 0$, one can prove in an analogous fashion, that Theorem 2.1 holds with
\[
L := \left[ \log_{C_{CS}C_{J,X}C} \right] \frac{\log_{C_{CS}C_{J,X}C}}{d_x + d_t}
\]
when choosing the parameters according to Table 3.

### 4. Numerical Results

In this section, we want to present some numerical calculations of the discrete inf-sup constants $\beta_{j,\ell}$ for different choices of levels $j$ and $\ell$. We consider a boundedly invertible operator $B \in L(X,Y')$ as in section 3, which stems from (29). The spaces $X$ and $Y$ are defined in Problem 2.

We choose a *Riesz basis* $\Psi^X := \{ \psi^X_\lambda : \lambda \in \nabla_X \}$ for $X$, that means, that each element in $X$ has a unique expansion in terms of $\Psi^X$ and that there exist Riesz-constants $0 < r_X \leq R_X < \infty$ such that
\[
r_X \|v\|_{\ell_2(\nabla_X)} \leq \|v\|_X \leq R_X \|v\|_{\ell_2(\nabla_X)},
\]
Table 2. Relevant constants

<table>
<thead>
<tr>
<th>constant</th>
<th>value</th>
<th>description and reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>$\left\lceil \log_{\omega}(C_{CS}C_{J,X'}) \right\rceil$ m</td>
<td>number of extra layers, (24)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$\frac{C_{CS}^{-1}-C_{J,X'}\omega^{-Lm}}{C_n(C_{J,X'}\omega^{-Lm+1})}$</td>
<td>stability constant, (23)</td>
</tr>
<tr>
<td>$C_{J,X'}$</td>
<td>$C_B C_J C_{J,Y} C_{B,X'}$</td>
<td>auxiliary constant from Lemma 2.1, (19)</td>
</tr>
<tr>
<td>$C_{CS}$</td>
<td>$4c_s^2 c_X^{-1} C_X$</td>
<td>constant in reverse Cauchy-Schwarz inequality, (45)</td>
</tr>
<tr>
<td>$C_{B,X'}$</td>
<td>$C_{B,S}(c^{(z)}_S)^{-1}$</td>
<td>constant in Bernstein estimate, (41)</td>
</tr>
<tr>
<td>$C_{J,Y}$</td>
<td>$(c_{Y+})^{-1} C_Y$</td>
<td>constant in Jackson estimate, (42)</td>
</tr>
<tr>
<td>$c_B$</td>
<td>cf. (A.6) in [35]</td>
<td>inf-sup constant, (3)</td>
</tr>
<tr>
<td>$C_B$</td>
<td>cf. (A.4) in [35]</td>
<td>boundedness constant, (2)</td>
</tr>
<tr>
<td>$C_+$</td>
<td>cf. [13]</td>
<td>constant for higher regularity, (14)</td>
</tr>
<tr>
<td>$c_s$</td>
<td>$c^{(t)}_S, c^{(x)}_S$</td>
<td>$L_2$-stability constant, (35)</td>
</tr>
<tr>
<td>$c_X, C_X$</td>
<td></td>
<td>lower and upper bound in norm equivalence, (47)</td>
</tr>
<tr>
<td>$C^{(x)}_{B,S}$</td>
<td>$-$</td>
<td>constant in Bernstein estimate for $S_j^x$, (39)</td>
</tr>
<tr>
<td>$c^{(t)}_S, c^{(x)}_S$</td>
<td>$-$</td>
<td>$L_2$-stability constant, (34)</td>
</tr>
<tr>
<td>$c_{Y+}$</td>
<td></td>
<td>lower bound in norm equivalence, (44)</td>
</tr>
<tr>
<td>$C_Y$</td>
<td></td>
<td>upper bound in norm equivalence, (43)</td>
</tr>
</tbody>
</table>

Table 3. Approximation and smoothness parameters (general setting).

<table>
<thead>
<tr>
<th>space</th>
<th>primal</th>
<th>dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_j^l$</td>
<td>$\gamma_s^l, d_s^l &gt; 0$</td>
<td>$\gamma_{s^l}, d_{s^l} &gt; d_l$</td>
</tr>
<tr>
<td>$S_j^x$</td>
<td>$\gamma_s^x, d_s^x &gt; m$</td>
<td>$\gamma_{s^x}, d_{s^x} &gt; \max{0, d_x - m}$</td>
</tr>
<tr>
<td>$Q_{l^x}$</td>
<td>$\gamma_q^x, d_q^x &gt; d_{l^x} + m$</td>
<td>$\gamma_{q^x}, d_{q^x} &gt; m$</td>
</tr>
</tbody>
</table>

for each function $v = \sum_{\lambda \in \nabla_X} v_\lambda \psi_\lambda^X$ and $v := \{v_\lambda\}_{\lambda \in \nabla_X}$. Analogously, we choose a Riesz basis $\Psi^Y := \{\psi_\lambda^Y : \lambda \in \nabla_Y\}$ for $Y$ with Riesz-constants $0 < r_Y \leq R_Y < \infty$. Moreover, let the discrete spaces $S_j := \text{span } \Psi_j^X$ with $\Psi_j^X := \{\psi_\lambda^X : |\lambda| \leq j\} \subset \Psi^X$. Analogously, we choose $Q_{l} := \text{span } \Psi_{l}^Y$ with $\Psi_{l}^Y := \{\psi_\lambda^Y : |\lambda| \leq \ell\} \subset \Psi^Y$ for the discrete test space. In order to shorten the notation we define $\nabla_j^X := \{\nabla_X : |\lambda| \leq j\}$ and $\nabla_{l}^Y := \{\nabla_Y : |\lambda| \leq \ell\}$. It is well known that the Riesz basis property (51) implies the existence of another $L_2$-stable biorthogonal Riesz basis according to (34) respectively (35), see [14, §3].
Using these bases we obtain the following matrix representation of the operator
\[ B_{j,\ell} := \{(B\psi^X_{\mu}; \psi^Y_{\lambda})\}_{\lambda \in \nabla^Y_{\mu}, \mu \in \nabla^X}. \] (52)
This matrix is usually called stiffness matrix or system matrix.

Due to the Riesz stability (51) and the definition of the system matrix (52), we obtain the equivalence
\[
\min_{v \in \ell^2(\nabla^X) \setminus \{0\}} \max_{w \in \ell^2(\nabla^Y) \setminus \{0\}} \frac{w^T B_{j,\ell} v}{\|w\|_{\ell^2(\nabla^Y)} \|v\|_{\ell^2(\nabla^X)}} \sim \inf_{v \in S^j_\Omega(0)} \sup_{w \in Q_\ell(0)} \langle Bv, w \rangle
\] (53)
of the discrete inf-sup condition (8) with constants which do not depend on the levels \( j \) and \( \ell \). That is, using Riesz bases, we can analyze the qualitative behavior of the discrete inf-sup condition (8) via the norm equivalence (53). Moreover, the equivalent representation has a simple algebraic interpretation. We rewrite the left hand side of (53) as
\[
\min_{v \in \ell^2(\nabla^X) \setminus \{0\}} \max_{w \in \ell^2(\nabla^Y) \setminus \{0\}} \frac{w^T B_{j,\ell} v}{\|w\|_{\ell^2(\nabla^Y)} \|v\|_{\ell^2(\nabla^X)}} = \min_{v \in \ell^2(\nabla^X) \setminus \{0\}} \frac{\|B_{j,\ell} v\|_{\ell^2(\nabla^Y)}}{\|v\|_{\ell^2(\nabla^X)}} = \lambda_{\min}(B^\top_{j,\ell} B_{j,\ell})^{1/2} =: \sigma_{\min}(B_{j,\ell}), \] (54)
where \( \lambda_{\min}(B^\top_{j,\ell} B_{j,\ell}) \) denotes the smallest eigenvalue of \( B^\top_{j,\ell} B_{j,\ell} \), i.e., \( \sigma_{\min}(B_{j,\ell}) \) is the smallest singular value of \( B_{j,\ell} \).

Next, we need to specify our choice of basis functions \( \psi^X_j \) of \( S_j \) and \( \psi^Y_\ell \) of \( Q_\ell \). We will use B-spline wavelets with \( d = 2 \) and \( \tilde{d} = 4 \) (2-4 DKU wavelets) in space for \( S^x_j \) as well as for \( Q^x_\ell \) and B-spline wavelets with \( d = 2 \) and \( \tilde{d} = 2 \) (2-2 DKU wavelets) in time for \( S^t_j \) as well as for \( Q^t_\ell \). The dimension of the spaces spanned by these wavelet bases is proportional to \( 2^j \) respectively \( 2^{\tilde{d}} \) and are known to meet the Jackson- and Bernstein estimates according to Table 1 with \( m = 1 \) for \( \omega = 2 \) except the condition for \( \gamma_{Q^x}, d_{Q^x} \), see [20, esp. Cor. 3.6 and Prop 3.7]. Nevertheless, since \( \gamma_{Q^x}, d_{Q^x} \gg \frac{3}{2} \) is truly larger than \( m = 1 \), we can expect that the statement of Theorem 2.1 stays true by Remark 3.2 with \( d_x := \frac{1}{2} \). Note that the regularity assumption according to Table 3 with \( d_t := 0 \) and \( \tilde{d}_t := \frac{1}{2} \) is actually weaker than the required regularity assumption for the choices in Table 1. To this end, according to our main Theorem 2.1, we have uniformly bounded discrete inf-sup constants when choosing levels \( \ell \gg j + L \) as in (22) with a constant \( L \) defined in (21) provided that the operator is sufficiently regular according to (14). For a detailed description of bioriented spline-wavelets we refer to [20]. Tensor products of these bases are Riesz bases of \( L_2(0, T) \otimes L_2(\Omega) \) which, normalized in \( X \) and \( Y \), are also Riesz basis for \( X \) and \( Y \), respectively, see [26, Prop.1 and Prop. 2] and [35, Ch. 6]. This in turn means, that the smallest singular value of the stiffness matrix \( B_{j,\ell} \) is uniformly bounded from below for \( \ell \gg j + L \) due to the equality (54) and the equivalence (53). Moreover, using these wavelet bases yields an optimally preconditioned sequence of matrices \( \{B^\top_{j,\ell} B_{j,\ell}\}_{\ell \geq j+L} \), since the largest singular values are uniformly bounded as well. To this regard, recall that the continuity of an operator is inherited to the subspaces so that the discrete operator is still bounded independently of the discretization, cf. (7).

The computations were performed using the adaptive wavelet C++ package written by Roland Pabel ([33]) and Matlab (R2012a). The assembling of system matrices was implemented with the aid of the adaptive wavelet C++ package and imported to Matlab, where the eigenvalues were computed by the standard Matlab routine \texttt{eigs}. For the computations, a computer with four Intel(R) Xeon(TM) CPU 2.00GHz processors and 16GB Ram on a 64-Bit Linux system was employed.
4.1. An ODE Example

First, we want to consider the easiest case that Problem 2 stems from the ODE

\[
\frac{du(t)}{dt} = f(t), \quad \text{a.e. } t \in [0, 1]
\]

\[
u(0) = 0,
\]

with \( f \in H^{-1}_1(0, 1) \). That means we consider the variational problem of finding a solution \( u \in X \) such that

\[
\langle Bu, q \rangle = \langle f, q \rangle \quad \forall q \in Y,
\]

with

\[
\langle Bu, q \rangle := -\int_0^1 u(t) \frac{dq(t)}{dt} dt,
\]

and the spaces \( X := L_2(0, 1); Y := H^1_1(0, 1) \), where

\[
\langle f, q \rangle = \int_0^1 f(t) q(t) dt
\]

for sufficiently smooth right hand side \( f \in L_2(0, 1) \). Since this example depends on time solely, we only have to ensure the approximation and smoothness properties according to Table 3 with respect to time and can ignore the spatial properties. This leads to a number of extra layers \( L \) according to (50) with \( d_x = 0 \). Due to (53) and (54) we have calculated the smallest singular values of \( B_{j,\ell} \) for different levels \( j \) and \( \ell \) as an indicator for the discrete inf-sup constants \( \beta_{j,\ell} \).

The results are presented in Figure 1 where we have plotted the slope of the smallest and largest singular values of \( B_{j,\ell} \) for a fixed level \( \ell = 12 \) and for the same levels \( \ell = j \).

\[
\sigma_{\min}(B_{j,j}) \text{ and therefore the discrete inf-sup constants } \beta_{j,j} \text{ decrease with increasing level } j. \quad \text{This behavior confirms that a uniformly bounded sequence of discrete}
\]

\[
\text{Figure 1. Plot of the smallest and largest singular values of } B_{j,\ell} \text{ for same levels } j = \ell \text{ as well as for fixed level } \ell = 12 \text{ w.r.t. (56). } \sigma_{\max}(B_{j,\ell}) \text{ for fixed level } \ell = 12 \text{ nearly coincides with } \sigma_{\max}(B_{j,j}).}
\]
inf-sup constants can only be guaranteed if we choose different levels \( j \) and \( \ell \), and that otherwise, indeed, stability problems occur. We can observe, that the values of \( \sigma_{\min}(B_{j,\ell}) \) and therefore the discrete inf-sup constants stay approximately constant, i.e., \( \sigma_{\min}(B_{j,\ell}) \sim 1 \), if \( \ell > j \) and is much smaller if \( \ell = j \). Such a behavior was suggested by Theorem 2.1. Recall that the main result of Theorem 2.1 says that the sequence of discrete inf-sup constants is uniformly bounded away from zero as long as \( \ell \geq j + L \) with \( L \) defined in (21). Moreover, we see that the largest singular values \( \sigma_{\max}(B_{j,\ell}) \) are asymptotically constant, so that the sequence of system matrices \( \{B_{j,\ell}^\top B_{j,\ell}\}_{\ell \geq j + L} \) is optimally preconditioned. It was proven in [3] that already one extra layer is sufficient to ensure uniform stability in the context of the slightly different formulation from [35]. Therefore, it is not surprising that already \( L = 1 \) yields a very satisfactory stable behavior of \( \beta_{j,\ell} \) bounded away from zero. This observation is also underlined by Figure 2. Here we have plotted the smallest and largest singular values for a fixed number of extra layers \( L = 1 \), i.e., for increasing \( j \) and \( \ell \) with \( \ell = j + 1 \). One can, indeed, observe that both, the smallest and largest singular values, stay asymptotically constant for increasing levels. For the sake of completeness, we also provide the error of the minimal Petrov-Galerkin solution in the natural norm \( \|\cdot\|_X = \|\cdot\|_{L_2(0,1)} \) on the solution space. To this end, we choose the right hand side \( f(t) := t^2 \), with exact solution \( u(t) = \frac{1}{3}t^3 \). We compute the approximate solution \( u_j \) respectively its expansion coefficients \( \mathbf{u}_j \) by using the mldivide operation of Matlab. Due to the Riesz basis property, the \( X \)-norm of the error can be estimated by the \( \ell_2 \)-norm of its expansion coefficients. As reference solution we took the best approximation of the exact solution on level \( j = 12 \). As expected due to quasi optimality (10), the optimal rate of convergence is attained.

4.2. A PDE Example – One-Dimensional Spatial Problem

In principle, all theoretical predictions were already confirmed by the ODE example in the previous subsection. Nevertheless, in view of tensor products and intersections of them, we would like to underline our numerical results also with an example of a PDE, where truly anisotropic multidimensional spaces are involved. As a model example we consider the
Figure 3. Estimated errors of the minimal residual Petrov-Galerkin solutions $u_j$ w.r.t. the norm $\|u_j - u_{\text{ref}}\|_{\ell_2(\nabla X)} \sim \|u_j - u_{\text{ref}}\|_X$ for (55) with $f(t) := t^2$ and $\ell = j + 1$.

parabolic evolution problem

$$\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = f(t,x), \quad \text{a.e. } t \in [0,1], \quad x \in \Omega := (0,1)$$

$$\left. \frac{\partial u(t,x)}{\partial x}\right|_{x=0,1} = 0, \quad \text{a.e. } t \in [0,1]$$

$$u(0,x) = 0, \quad x \in (0,1),$$

with $f \in Y'$. Searching for a solution $u \in X$, the corresponding space-time weak formulation reads as

$$\langle Bu, q \rangle = \langle f, q \rangle \quad \forall q \in Y,$$

with

$$\langle Bu, q \rangle := \int_0^1 \int_0^1 \left( -u(t,x) \frac{\partial q(t,x)}{\partial t} + \frac{\partial u(t,x)}{\partial x} \frac{\partial q(t,x)}{\partial x} + u(t,x)q(t,x) \right) \, dx \, dt,$$

and the spaces $X := L_2(0,1) \otimes H^1(0,1); \quad Y := (L_2(0,1) \otimes H^1(0,1)) \cap (H^1_{\{1\}}(0,1) \otimes \dot{H}^{-1}(0,1))$, where $\dot{H}^{-1}(0,1)$ denotes the dual space of $H^1(0,1)$ and

$$\langle f, q \rangle = \int_0^1 \int_0^1 f(t,x)q(t,x) \, dx \, dt,$$

for sufficiently smooth right hand side $f \in L_2(0,1) \otimes L_2(0,1)$. Therefore, the spatial operator $A(t) \in \mathcal{L}(H^1(0,1), \dot{H}^{-1}(0,1))$ for is given by

$$\langle A(t)v, w \rangle := \int_0^1 (v'(x)w'(x) + v(x)w(x)) \, dx, \quad v, w \in H^1(0,1)$$

for $t \in (0,T)$. We are only interested in the discrete inf-sup constants of the corresponding stiffness matrices, so we do not need to specify the right hand side $f \in Y'$. It can be easily
Figure 4. Plot of the smallest and largest singular values of $B_{j,\ell}$ for fixed level $j = 4$ w.r.t. (58).

seen that the assumptions of Proposition 3.1 are fulfilled with $A(\cdot)' = A(\cdot)$ and $\lambda = 0$ and spaces $W = H^2(\Omega) \hookrightarrow V = H^1(\Omega) \hookrightarrow H = L_2(\Omega)$. That is, property (14) is satisfied.

Again, we use the smallest singular values of $B_{j,\ell}$ as an indicator for the discrete inf-sup constants. The results are illustrated in Figure 4.

Different from Figure 1 we fixed the minimum level $j = 4$ and successively increase the level $\ell$ of the trial space in Figure 4. We have calculated these values instead of fixing a maximum level for $\ell$ and increasing level $j$ as in Figure 1, only to keep the necessary storage moderate. For this PDE example, we see qualitatively the same behavior as in the ODE example. We can observe, that for a level difference of one or more levels, the smallest singular values stay almost equal. This confirms also in this case that a level difference of one already suffices for $\beta_{j,\ell}$ being uniformly bounded away from zero. For the same level $j = \ell$ we can clearly see, that the value is much smaller than for different levels. As in Figure 1 the largest singular values are asymptotically constant.

5. Conclusion and Outlook

We have analyzed the stability of Petrov-Galerkin discretizations of boundedly invertible operators in a fairly general setting and applied the results to operators from a space-time weak formulation. In Theorem 2.1 we have given a stability result by assuming the validity of a regularity condition (Assumption 1) on the operator as well as standard smoothness, approximation and stability conditions (Assumption 2) on the discrete spaces. It was shown that we are able to explicitly state a lower bound for the discrete inf-sup constants (8) independently of the discretization by choosing discrete test spaces of higher dimension than the dimension of the trial spaces. The second task was to apply this abstract result to a more precise but still rather general operator stemming from a space-time weak formulation. First we have introduced the setting of such a space and time weak formulation following the lines of [13, 35] and gave a regularity condition on the involved operator $A(t)$, which is well-known to hold for a wide range of elliptic operators. The remainder of section 3 faced
the task of constructing suitable subspaces $S_j$ and $Q_\ell$ such that the conditions (15), (16) and (17) are fulfilled. To this end, we have introduced hierarchical tensor product spaces of temporal and spatial subspaces which satisfy standard smoothness and approximation properties according to Table 1, cf. Remark 3.1. We have shown that spaces constructed in this way satisfy the required assumptions, so that the abstract stability result from Theorem 2.1 applies to this particular situation. Finally, we have demonstrated the stability behavior of a minimal residual Petrov-Galerkin approach for an ODE example (55) as well as an PDE example (57). The numerical results underline our predictions on the stability, especially in view of the dependence of the refinement levels $\ell$ and $j$ with respect to the test and trial space, respectively.

The recipe for obtaining stable subspaces for the slightly different formulation in [35] may be developed along the same lines. Furthermore, the spaces $S_j := S_j^t \otimes S_j^x$ and $Q_\ell := Q_\ell^t \otimes Q_\ell^x$, respectively, are constructed such that the spatial and temporal resolution are equal. Obviously, one could also choose different resolutions and replace $S_j$ by $S_{(j_1,j_2)} := S_{j_1}^t \otimes S_{j_2}^x$ and $Q_\ell$ by $Q_{(\ell_1,\ell_2)} := Q_{\ell_1}^t \otimes Q_{\ell_2}^x$, respectively. It is most likely that considering such more general spaces would yield a sharper estimate (21) for the number of extra layers $L$ and therefore would improve our main result given by Theorem 2.1 for such particular situations. Moreover, one obtains a sharper estimate for $L$ the larger we can choose parameter $\rho$ in (21).

Due to Proposition 3.1 and by the choice of $W$, $V$ and $H$, $\rho$ is given as $\rho = \omega^m$, with $\omega = 2$ for dyadic partitioning. Assuming higher regularity of the operator $B$ allows larger values for $\rho$ if we choose discrete subspaces with corresponding stronger approximation and smoothness properties, see Remark 3.2. Furthermore, it could be very promising to apply the used techniques also to stochastic PDEs, as for instance to a space-time weak formulation of the stochastic heat equation [30].

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