# Upper Bounds on the Bisection Width of 3- and 4-regular Graphs

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**Abstract.** We derive new upper bounds on the bisection width of graphs which have a regular vertex degree. We show that the bisection width of large 3-regular graphs with $|V|$ vertices is at most $\frac{1}{6}|V|$. For the bisection width of large 4-regular graphs we show an upper bound of $\frac{2}{5}|V|$.

Keywords: *graph partitioning; bisection width; regular graphs; local improvement;*

## 1 Introduction

There are graph-partitioning problems in a wide range of applications. The task is to divide the set of vertices of a graph equally into a given number of parts while keeping the number of crossing edges between vertices belonging to different parts, called the *cut size* of the partition, as small as possible. The special case of a partition of the graph into 2 parts is called a *bisection*, and the minimal cut size of all balanced bisections of a graph is called its *bisection width*. Its calculation is **NP**-complete for arbitrary graphs [11] and remains **NP**-complete for regular graphs [5].

There are several results on bounds on the bisection width of regular graphs (discussed below). Results for 3- and 4-regular graphs are of special interest because these are the lowest non-trivial degrees. Some previous results for small degrees have been generalized to results for larger degrees.

It is a general theoretical interest to improve previous upper bounds on 3- and 4-regular graphs. Moreover, there are some direct applications of these results. As a motivating example, upper bounds on the bisection width of 4-regular graphs have successfully been applied to the configuration of transputer systems [13].

**Definitions and Previous Results** Let $G = (V, E)$ be a simple undirected graph with vertex set $V$ of cardinality $n := |V|$ and edge set $E$. A graph is $d$-regular if for all $v \in V$ it is $|\{w \in V; \{v, w\} \in E\}| = d$. Let $\pi : V \rightarrow \{0, 1\}$ be a **bisection** of $G$. It distributes the vertices among parts $V_0$ and $V_1$. We focus on **balanced** bisections, i. e. the number of vertices in the parts differ by at most 1. Let $cut(\pi) := |\{\{v, w\} \in E; \pi(v) \neq \pi(w)\}|$ be the **cut size** of $\pi$. The **bisection width** of a graph $G$ is $bw(G) := \min\{cut(\pi); \pi \text{ is a balanced bisection of G}\}$.

The bisection width is known for some graph classes with regular degree such as tori, cube-connected-cycles [18] or butterflies [4].

There are several results on bounds on the bisection width of arbitrary regular graphs. Clark and Entringer [7] present an upper bound of $\frac{n+138}{3}$ for the bisection width of 3-regular graphs. Kostochka and Melnikov improve this asymptotically and show an upper bound of $\frac{n}{4} + O(\sqrt{n} \log n)$ [15]. Recently, an upper bound of $0.198n + O(\log(n))$ has been proved in [24]. Hromkovic and Monien [13] proved an upper bound of $\frac{n}{2}+1$ for the bisection width of 4-regular graphs with $n \geq 350$. A general upper bound of $\frac{n}{2}+5$ for 4-regular graphs with any number of vertices is proven in [24]. The result of [15] for 3-regular graphs above is a corollary of an upper bound of $\frac{d-2}{4}n + O(d\sqrt{n}\log n)$ for the bisection width of $d$-regular graphs in the same paper. An upper bound of $\frac{d-2}{4}n+1$ for $n \geq n_0(d)$ with some function $n_0(d)$ is shown in [20, 21] by generalizing the techniques of [13]. Alon [1] uses probabilistic arguments to show that the bisection width is at most $(\frac{d}{2} - \frac{3\sqrt{d}}{16\sqrt{2}})\frac{n}{2}$ for $d$-regular graphs with $n > 40d^9$.

Bollobas [3] shows that for $d \to \infty$ the bisection width of almost every $d$-regular graph is at least $(\frac{d}{2} - \sqrt{\ln(2) \cdot d})\frac{n}{2}$. For $d = 4$ he shows that almost all 4-regular graphs have a bisection width of at least $\frac{11}{50}n = 0.22n$. Furthermore, Kostochka and Melnikov show that almost every 3-regular graph has a bisection width of at least $\frac{1}{9.9}n \approx 0.101n$ [16]. There are some (slightly weaker) results for explicitly constructible infinite graph classes with high bisection width. The *Ramanujan Graphs* (see e. g. [6, 17, 19, 22]) have a regular degree $d$ and a bisection width of at least $(\frac{d}{2} - \sqrt{d-1})\frac{n}{2}$. This value is derived by the use of the well-known spectral lower bound $\frac{\lambda_2 \cdot n}{4}$ with $\lambda_2$ being the second smallest eigenvalue of the Laplacian of the graph (cf. [10]). This implies lower bounds of $0.042n$ and $0.133n$ for the bisection widths of 3-regular and 4-regular Ramanujan graphs. The spectral lower bound has been improved in [2] to a lower bound of $0.082|V|$ for the bisection width of large 3-regular Ramanujan graphs and a lower bound of $0.176|V|$ for the bisection width of large 4-regular Ramanujan graphs.

There are many heuristics for graph partitioning which are successfully being used in applications. Furthermore, efficient software implementations of the most relevant methods are available by using software tools like e. g. CHACO [12], JOSTLE [25], METIS [14], SCOTCH [23] or PARTY [24]. These heuristics try to calculate a bisection with a small cut size. However, they do not guarantee an approximation of the bisection width. Recently, it has been shown that the bisection width can be approximated by a polynomial time algorithm within a factor of $O(\log^2(|V|))$ [9].

**New Results and Outline of the Paper** In this paper we improve previous upper bounds on the bisection width of large 3- and 4-regular graphs. In Section 2 we prove for any $\epsilon > 0$ an upper bound of $(\frac{1}{6} + \epsilon)n$ on the bisection width of large 3-regular graphs. We are able to prove an upper bound of $(\frac{2}{5} + \epsilon)n$ on the bisection width of large 4-regular graphs. This proof is omitted in this version of the paper due to space limitations and in favour of a detailed description of the 3-regular case. The proof will be published in the full version. As discussed above, there are large 3-regular graphs with a bisection width of at least $0.101n$ and large 4-regular graphs with a bisection width of at least $0.22n$. Thus, the results are optimal up to constant factors and our results improve these factors.

**Iterative Local Improvement with Helpful Sets** The proofs in this paper are constructive and follow an iterative local improvement scheme. It starts with an arbitrary balanced bisection. If the cut size of it does not fulfill the stated upper bound, it performs two steps to improve the bisection as illustrated in Fig. 1. In the first step, a small set $S_0 \subset V_0$ is moved to $V_1$. $S_0$ is chosen such that this move decreases the cut size. In the second step, a set $S_1 \subset V_1 \cup S_0$ with $|S_1| = |S_0|$ is moved to $V_0$. $S_1$ is chosen such that the cut size does not increase too much, i. e. such that the increase is less than the decrease in the first step. Thus, the resulting bisection is balanced and has a smaller cut size. These steps are repeated until the cut size drops below the upper bound. The proofs in this paper ensure that there are sets $S_0$ and $S_1$ with the desired property as long as the cut size is higher than the stated upper bound.

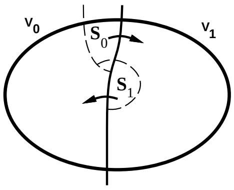

**Fig. 1.** One iteration of a local improvement.

This local improvement scheme has successfully been used to derive upper bounds on the bisection width of 4-regular graphs [13, 20]. Furthermore, it is the basis for the *Helpful-Set* heuristic which is able to calculate bisections with low cut sizes for very large graphs in a short time [8, 21]. An implementation of the Helpful-Set heuristic can be found in the software tool PARTY [24].

A move of a set of vertices from one part to the other changes the cut size of the bisection. The helpfulness of the set is the amount of this change.

**Definition 1.** *Let $\pi$ be a bisection of a graph $G = (V, E)$. For $S \subset V_p(\pi)$, $p \in \{0, 1\}$, let*

$$H(S) = |\{\{v, w\} \in E; v \in S, w \in V \backslash V_p(\pi)\}| - |\{\{v, w\} \in E; v \in S, w \in V_p(\pi) \backslash S\}|$$

*be the* **helpfulness** *of $S$. $S$ is called* **$H(S)$-helpful**.

## 2 Upper Bound on the Bisection Width of 3-regular Graphs

In this section we derive a new upper bound on the bisection width of 3-regular graphs. The proof is based on the iterative local improvement scheme described in the previous section. We will use Lemma 3 for the first step of the improvement scheme and Lemma 4 for the second step. These lemmas will be used to prove the theorem at the end of this section. Lemma 2 is the main lemma of this section and may be of interest on its own. It will be used to prove Lemma 3.

**Lemma 1.** *Let* $T = (V, E)$ *be a tree with weights* $w : V \to \mathbf{N}$*. Let* $A = (\sum_{v \in V} w(v))/(|V| - 1)$*. Let* $w(l) \geq A$ *for all leaves* $l$*. Then there are adjacent vertices* $u$ *and* $v$ *with* $w(u) + w(v) \leq 2 \cdot A$*.*

*Proof.* By induction on $|V|$. The case if $T$ is a path is trivial.

Otherwise consider an arbitrary leaf $v_0$. Let $v_1, \ldots, v_{k+1}$ be the path connecting $v_0$ to a vertex $v_{k+1}$ of degree at least 3, i. e. the degree of $v_i$, $1 \leq i \leq k$, is 2. Let $T'$ be the subtree (path) $v_0, \ldots, v_k$.

By considering the vertex pairs $\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{k-1}, v_k\}$ if $k$ is even or $\{v_0, v_1\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}$ if $k$ is odd we deduce that we either found the desired pair or it is $w(T') \geq A \cdot |V(T')|$. But in this case the tree $T \backslash T'$ satisfies $w(T \backslash T') \leq A \cdot |V(T \backslash T')|$ and we can apply induction on $T \backslash T'$. □

**Lemma 2.** *Let* $G = (V, E)$*,* $E = B \uplus R$*, be a 3-regular graph with black edges* $B$ *and red edges* $R$*. Let each vertex be adjacent to at least one black edge. Let* $|R| > (\frac{1}{2} + \epsilon)|V|$ *for an* $\epsilon > 0$*. Then there is a set* $S \subset V$ *of size* $O(\frac{1}{\epsilon^2})$ *such that the number of red edges between vertices of* $S$ *is larger than the number of black edges between* $S$ *and* $V \backslash S$*.*

*Proof.* Let $e$ be the number of vertices which are adjacent to 3 black edges, $d_1$ be the number of vertices which are adjacent to 2 black edges and $d_2$ be the number of vertices which are adjacent to 1 black edge, i. e. $|V| = e + d_1 + d_2$ and $2|R| = 2d_2 + d_1$. It is

$$\frac{|R|}{|V|} = \frac{d_1 + 2d_2}{2(d_1 + d_2 + e)} \ . \tag{1}$$

The fact $|V| < 2|R|$ leads us to

$$e = |V| - d_2 - d_1 < 2|R| - d_2 - d_1 = d_2 \ . \tag{2}$$

Call a set $S \subset V$ **positive** if it has more internal red edges than external black edges, **negative** if it has more external black edges than internal red edges and **neutral** if the numbers are equal.

Consider the graph consisting of black edges only and let $F$ be the family of its connected components. Clearly, the elements of $F$ are neutral or positive. As a simple example, a positive set $I \in F$ of size $O(\frac{1}{\epsilon^2})$ fulfills the lemma. In the following we manipulate the sets in $F$. However, the number of sets in $F$ remains constant and the sets in $F$ remain neutral or positive.

The number of vertices of degree 1 in a connected component exceeds the vertices of degree 3 by at most 2. Thus, for each $I \in F$ it is $e(I) \geq d_2(I) - 2$. More precisely, it is $e(I) = d_2(I) + 2r(I) - 2$ with $r(I)$ being the number of edges which can be removed from $I$ without splitting it into disconnected components. Let $r = \sum_{I \in F} r(I)$. It is

$$e = d_2 + 2r - 2|F| \ . \tag{3}$$

Let $\delta > 0$ be a constant. The value of $\delta$ will be assigned below. A set $I \in F$ is called **small** if $|I| \leq \frac{1}{\delta}$ and **large** otherwise. Denote with $\alpha(S)$, $S \subset V$, the number of red edges between vertices of $S$ and vertices of small sets. Call a black path $P$ a **thin path** if it has the following property. $P$ contains only vertices with black degree of 2, $P$ has a maximal path length, i. e. both end vertices are adjacent via a black edge to a vertex of black degree 1 or 3 (or, as described below, to a marked vertex of degree 2) and it is $|P| \leq \frac{\alpha(P)}{\delta}$. Call a black path $P$ a **thick path** if it has all these properties except the last one.

Let $B(I)$, $I \in F$, be the family of disconnected components of $I$ if all red edges and thin paths of $I$ are removed. Let $\beta(B) = |B(I)|$. Let $s(I)$ be the size of the union of $I$ and all small sets which are connected to $I$ via a red edge. A set $I \in F$ is called **thin** if $s(I) < \frac{\beta(I)-1}{\delta}$ and **thick** otherwise.

The outline of the proof is the following. We state an algorithm below which manipulates the elements of $F$. Especially, we will disconnect certain thin paths from the rest of their set. The disconnection of a thin path $P$ will shade it and mark both vertices previously connected to the ends of $P$ as well as mark the small sets connected to $P$ via a red edge. We disconnect thin paths in each set $I \in F$ until there is no thin path in any cycle of $I$. For such a set $I \in F$ we construct the tree $T(I) = (X, Y)$ with the vertex set $X$ being the disconnected components from $B(I)$ and with the edges connecting the components if there is an unshaded thin path between them. It is $\beta(I) = |X| = |Y| + 1$. Let the weight $w(v)$, $v \in X$, be the size of the subset represented by $v$.

The algorithm may terminate with a positive set of size $O(\frac{1}{\delta^2})$. Since we will set later $\delta = \frac{\epsilon}{6(1+2\epsilon)}$, we can say that any positive set of size at most $O(\frac{1}{\delta^2})$ fulfills the lemma. Otherwise, the algorithm terminates with a family $F$ consisting of sets of the following types only.

(i) $I \in F$ is large.
(ii) $I \in F$ is small and has at least one red edge to a thick set or to a thick path.
(iii) $I \in F$ is small and has at least 2 vertex marks or there is a cycle of black edges in $I$.

We will finish the proof by showing that if there are only sets of these types, we get a contradiction. Thus, the algorithm was successful in finding a set fulfilling the lemma. The following algorithm manipulates the elements of $F$.

**Step 1:** If there is a positive small set $I \in F$, then $I$ fulfills the lemma. If there is a red edge between two different small sets $I_a, I_b \in F$, then $I_a \cup I_b$ fulfills the lemma because $I_a$ and $I_b$ are neutral themselves and the union has an additional internal red edge. Both kind of fulfilling sets have a size of $O(\frac{1}{\delta})$.

**Step 2:** If there is a thin path $P$ with $\alpha(P) \geq 3$, we show that there is a set fulfilling the lemma. We will take 3 small sets which are connected to $P$ via a red edge and unify them with a subpath of $P$ such that the subpath connects the 3 small sets. This union is a positive set. We will show that there are 3 small sets such that the connecting subpath is not too long, i. e. such that the size of the union is at most $O(\frac{1}{\delta})$.

$P$ can be divided into subpaths by deleting the vertices in $P$ which are connected to a small set via a red edge. There are $\alpha(P)$ such vertices, i. e. $\alpha(P)+1$ such (possibly empty) subpaths $P_i$, $1 \leq i \leq \alpha(P)+1$. Let $x_i = |P_i|$. It is $|P| = \sum_{i=1}^{\alpha(P)+1} x_i + \alpha(P)$. In the following we do not consider the subpaths $P_1$ and $P_{\alpha(P)+1}$ on both ends of the path.

If $\alpha(P) = 3$, it is $x_2 + x_3 = 2\frac{1}{\alpha(P)-1}\sum_{j=2}^{\alpha(P)} x_j$. If $\alpha(P) > 3$, there is an $i$, $2 \leq i < \alpha(P)$ such that $x_i + x_{i+1} \leq 3\frac{1}{\alpha(P)-1}\sum_{j=2}^{\alpha(P)} x_j$. Otherwise we get

$2\sum_{j=2}^{\alpha(P)} x_j = \sum_{j=2}^{\alpha(P)-1}(x_j + x_{j+1}) + x_2 + x_{\alpha(P)} > 3\frac{\alpha(P)-2}{\alpha(P)-1}\sum_{j=2}^{\alpha(P)} x_j + x_2 + x_{\alpha(P)} \geq 2\sum_{j=2}^{\alpha(P)} x_j + x_2 + x_{\alpha(P)}$, which is a contradiction.

Thus, there is an $i$, $2 \leq i < \alpha(P)$, with $x_i + x_{i+1} \leq 3\frac{|P|}{\alpha(P)-1} \leq 3\frac{\alpha(P)}{(\alpha(P)-1)\delta} \leq 3\frac{3}{2\delta} = O(\frac{1}{\delta})$. The union of $P_i$, $P_{i+1}$, the three vertices connecting the paths $P_{i-1}$ with $P_i$, $P_i$ with $P_{i+1}$ and $P_{i+1}$ with $P_{i+2}$ and the three small sets connected to these vertices is a positive set with a size of $O(\frac{1}{\delta})$.

From now on it is $\alpha(P) \leq 2$ for each thin path $P$.

**Step 3:** If there is a thin path $P$ in a large set $I \in F$ and there is a cycle in $I$ which includes $P$, remove the edges connecting $P$ with $I \backslash P$ and shade $P$. The cycle in $I$ ensures that both vertices which are connected to the ends of $P$ had a black degree of 3 or (as we will see below) they had a degree of 2 and were marked once.

It is $\alpha(P) \leq 2$ due to step 2. If $\alpha(P) = 1$, let $S_1 \in F$ be the small set connected to $P$ via a red edge and assign 2 marks to $S_1$. If $\alpha(P) = 2$, let $S_1, S_2 \in F$ be the small sets connected to $P$ via a red edge and assign 1 mark to $S_1$ and 1 mark to $S_2$. The vertices which were connected to both ends of $P$ get a mark, too. Their black degree is reduced by one due to the removal of the connecting edges. Although a vertex with black degree of 2 may be generated, it is not to be taken as part of a thin or thick path.

The removal of the connecting edges changes the graph. However, only two black edges are removed which are internal to $I$, i. e. $I$ remains neutral or positive. Although $P$ is disconnected from the rest of $I$ and shaded, we still consider it to be a part of $I$. Therefore, neither the value $s(I)$ nor the number of elements in $F$ does change.

The changes of the graph in this step come into account again when a positive set $S$ of size $O(\frac{1}{\delta})$ is found in a future part of the algorithm. $S$ is only positive with respect to the graph with removed edges. As we will see in step 4, $S$ may include a subset of $I \backslash P$. Especially, $S$ may include a marked vertex $v$ which is connected to the end of $P$. However, this external edge can be compensated by enlarging $S$ with $P$, $S_1$ and, if existing, $S_2$. Thus, the black edge between $v$ and $P$ is now internal and the possible external black edge between $P$ and the vertex connected to the other end of $P$ is compensated by the internal edge between $P$ and $S_1$. Thus, the enlarged set $S$ is positive after adding the formerly removed edges. $S$ is getting enlarged for each vertex mark of a vertex in $S$. Each enlargement adds at most $|P| + |S_1| + |S_2| \leq \frac{4}{\delta}$ vertices. All enlargements lead to a positive set $S$ of size $O(\frac{1}{\delta^2})$ fulfilling the lemma. Repeat step 3 until there are no such thin paths.

**Step 4:** Let $I \in F$ be a thin set. Let $T(I) = (X, Y)$ be the graph of $I$ with edges $Y$ representing the unshaded thin paths in $I$ as described above. $T(I)$ is a tree due to step 3. For each leaf $l$ in $T(I)$ let $L(l)$ be the union of the vertices represented by $l$, all adjacent (possibly shaded) thin paths and all small sets which are adjacent to these vertices via a red edge. Clearly, such a set $L(l)$ is neutral or positive. We defined the weight $w(v)$ of a vertex $v \in T$ above to be the number of vertices represented by $v$. We redefine the weight $w(l)$ for each leaf $l$ in $T(I)$ to be $|L(l)|$. Call a leaf $l$ *small* if $w(l) < \frac{1}{\delta}$. If there is a shaded thin path between two small leaves $l_1$ and $l_2$, then $L(l_1) \cup L(l_2)$ is positive of size $O(\frac{1}{\delta})$ and the lemma is fulfilled. In the remainder we can assume that there is no such path and, therefore, for the new definition of the weights it holds $\sum_{v \in X} w(v) \leq s(I)$.

If there is a thin set $I \in F$ with $w(l) \geq \frac{1}{\delta}$ for each leaf $l$ in $T(I)$, we can fulfill the lemma. It is $w(l) \geq \frac{1}{\delta} \geq \frac{s(I)}{\beta(I)-1} \geq \frac{\sum_{v \in X} w(v)}{|X|-1}$. $T(I)$ fulfills the requirements of Lemma 1. Thus, there are adjacent vertices $u, v \in T$ with

$$w(u) + w(v) \leq 2 \cdot \left(\sum\nolimits_{v \in X} w(v)\right) / (|X| - 1) \leq 2 \cdot \frac{s(I)}{\beta(I) - 1} \leq \frac{2}{\delta} .$$

The union $S$ of the vertices represented by $u$, $v$ and the edge between $u$ and $v$ has a size of $O(\frac{1}{\delta})$. However, we have to insert the edges which were removed in step 3 and 4. For each removed edge of step 4 (see below) and for each vertex mark of step 3 we enlarge $S$ with the adjacent thin path and a small set connected to this path via a red edge. Each enlargement increases $S$ by at most $O(\frac{1}{\delta})$ and compensates an external black edge as discussed in step 3. All enlargements together result in a positive set $S$ of size $O(\frac{1}{\delta^2})$ and the lemma is fulfilled.

If we could not fulfill the lemma, consider all thin sets $I \in F$ which have a leaf $l$ in $T(I)$ with $w(l) < \frac{1}{\delta}$. We will manipulate two elements of $F$. Let $P$ be the thin path which connects $l$ with its neighbor $v$ in the tree and let $S$ be the small set connected to $P$. Remove $I$ and $S$ from $F$ and construct some new sets $I_1$ and $I_2$. Let $I_1$ be the union of the vertices represented by $l$, the path $P$ and the small set $S$. Let $I_2$ be the set $I$ reduced by $P$ and the vertices represented by $l$. If $|I_2| \leq \frac{1}{\delta}$, then the union $L(l) \cup I$ fulfills the lemma. Otherwise, remove the edge which connects $P$ with $v$. Now $I_1$ and $I_2$ are disconnected and both are neutral or positive with $I_1$ being a small set and $I_2$ being a large set. Unlike before, let the value $|L(l)|$ be the size of $I_1$. To keep our notations correctly we exchange the red color of the edge between $P$ and $S$ and the black color of the edge between $P$ and $v$. This does not change the neutral status of $I_1$.

We now add $I_1$ and $I_2$ to $F$. Thus, this step does not change the number of elements in $F$.

Note that, like in step 3, we will find a positive set of size $O(\frac{1}{\delta})$ in a future part of the algorithm. And, again, we have to insert the removed edges and we compensate the external edge with enlarging the set by the path $P$ and the small set $S$. Thus, each single enlargement will increase the size of the set by at most $O(\frac{1}{\delta})$ leading to a positive set of size $O(\frac{1}{\delta^2})$ fulfilling the lemma.

Go back to step 1 if any new small set was generated in step 4.

In the remaining we show the contradiction if only sets of types (i)-(iii) remain. Let $z$ be the number of sets of type (i) or (ii). Let $z_2$ be the number of sets of type (iii) with at least two vertex marks and let $z_1$ be the number of sets of type (iii) with an internal black cycle.

It is $e = d_2 + 2r - 2|F|$ due to equation (3). At most $r$ thin paths were shaded, leading to at most $2(r - z_2)$ vertex marks. Notice that only thin paths in large sets were shaded. It is $2(z_1 + z_2) \leq 2r$. It follows $e = d_2 + 2r - 2z - 2(z_1 + z_2) \geq d_2 - 2z$.

Each set of type (ii) is adjacent to a set of type (i) via a red edge. Let $I \in F$ be a set of type (i). We will show that $s(I)$ is large enough to reserve an average size of $\frac{1}{6\delta}$ for $I$ itself and for all small sets of type (ii) connected to thick paths and unshaded thin paths of $I$. Each thick path $P$ is connected to $\alpha(P)$ sets of type (ii) and has a size of at least $\frac{\alpha(P)}{\delta}$. We reserve a value of $\frac{1}{2\delta}$ for each set of type (ii) which is connected to a thick path. It remains a value of at least $\frac{s(I)}{2}$ for $I$ and all small sets connected to unshaded thin paths of $I$. There are at most $\beta(I) - 1$ unshaded thin paths in $I$, the others were shaded in step 3. Thus, there are at most $2\beta(I) - 2$ small sets of type (ii) which are adjacent to unshaded thin paths of $I$ via a red edge. Together with $I$ these are at most $2\beta(I) - 1$ sets of type (i) or (ii). If $\beta(I) = 1$ it is $\frac{s(I)}{2} \geq \frac{1}{2\delta}$. If $\beta(I) \geq 2$ it is $\frac{s(I)}{2} \geq \frac{\beta(I)-1}{2\delta} \geq \frac{2\beta(I)-1}{6\delta}$. Thus, we can reserve at least $\frac{1}{6\delta}$ for each set of type (i) and (ii) and it is $e + d_1 + d_2 \geq \frac{z}{6\delta}$. With $e < d_2$ (equation (2)) we get $z \leq 6\delta(d_1 + 2d_2)$.

With $e \geq d_2 - 2z$ and $z \leq 6\delta(d_1 + 2d_2)$ the equation (1) leads us to

$$\frac{|R|}{|V|} = \frac{d_1 + 2d_2}{2(d_1 + d_2 + e)} \leq \frac{d_1 + 2d_2}{2(d_1 + 2d_2 - 2z)} \leq \frac{d_1 + 2d_2}{2(d_1 + 2d_2 - 12\delta(d_1 + 2d_2))} = \frac{1}{2(1 - 12\delta)}.$$

However, it is $\frac{|R|}{|V|} > \frac{1}{2} + \epsilon$. This leads to contradiction for $\delta \leq \frac{\epsilon}{6(1+2\epsilon)}$. This shows that during the algorithm we have found a set fulfilling the lemma. □

In the following we state the lemmas 3 and 4 which are used for the two steps of the local improvement scheme. Before we do so, we classify the vertices.

**Definition 2.** *The vertices of $V_0$ (or $V_1$) are classified according to their distance to the cut. It is $V_0 = C \uplus D \uplus E$ with $C$ vertices being at a distance of 1 to the cut, i. e. they are incident to a cut edge. $D$ vertices are at a distance of 2 and $E$ vertices at a distance of at least 3. $D$ vertices are further classified with respect to the number of adjacent $C$ vertices. I. e. $D = D_3 \uplus D_2 \uplus D_1$ and each $D_x$ vertex is adjacent to $x$ vertices in $C$. Overall, it is $V_0 = C \uplus D_3 \uplus D_2 \uplus D_1 \uplus E$ .*

*The values $c$, $d_3$, $d_2$, $d_1$ and $e$ specify the number of vertices of each type and the values $c(X)$, $d_3(X)$, $d_2(X)$, $d_1(X)$ and $e(X)$ denote the number of the according vertices in a set $X \subset V$.*

**Lemma 3.** *Let $\pi$ be a bisection of a 3-regular graph $G = (V, E)$ with $V = V_0 \uplus V_1$. If $cut(\pi) > (\frac{1}{3} + 2\epsilon)|V_0|$, $\epsilon > 0$, then there is an at least 1-helpful set of size $O(\frac{1}{\epsilon^2})$ in $V_0$.*

*Proof.* We focus on part $V_0$ of the bisection only. Let $m := |V_0|$.

There are some structures which would directly lead to small 1-helpful sets.

(i) If a $C$ vertex is incident to two or three cut edges, it is an at least 1-helpful set by itself.

(ii) A set of three connected $C$ vertices is 1-helpful.

(iii) If there are two adjacent $C$ vertices and one of them is adjacent to a $D_3$ vertex, then the union of the adjacent $C$ vertices, the $D_3$ vertex and its two other adjacent $C$ vertices form a 1-helpful set of size 5.

(iv) Another 1-helpful set can be formed if two $D_3$ vertices are adjacent to a common $C$ vertices. Then, the union of both $D_3$ vertices and their adjacent $C$ vertices is a 1-helpful set of size 7.

(v) Let $v$ be a vertex which is adjacent to a $C$ vertex which itself is adjacent to another $C$ vertex or a $D_3$ vertex. If both other neighbors of $v$ are from $C \cup D_2 \cup D_3$, the union of the mentioned vertices and their adjacent $C$ vertices forms an at least 1-helpful set of size at most 11.

In the remainder we can assume that these types of structures do not exist. Especially, since (i) is excluded, it is $c = cut(\pi)$.

We manipulate the edges between vertices of $V_0$ such that $G$ is transformed into a 3-regular graph $\bar{G} = (V, \bar{E})$ with the following properties.

- $V_0$ of $\bar{G}$ does not contain $D_3$ vertices or adjacent $C$ vertices.
- If there is a 1-helpful set $\bar{H} \subset V_0$ of $\bar{G}$, then there is an at least 1-helpful set $H$ of $G$, $\bar{H} \subset H \subset V_0$, with $\frac{|H|}{|\bar{H}|}$ bounded from above by a constant value.

We omit the proof of the transformation of $G$ into $\bar{G}$ and the reverse transformation of $\bar{H}$ to $H$ due to space limitations.

$\bar{G} = (V, \bar{E})$ has no $D_3$ vertices and no adjacent $C$ vertices, i. e. it is now $d_3 = 0$ and $2c = 2d_2 + d_1$. Furthermore, because of $m < 3c$ it is (similar to equation (2))

$$e = m - c - d_2 - d_1 < 2c - d_2 - d_1 = 2d_2 + d_1 - d_2 - d_1 = d_2 \; . \tag{4}$$

Construct a new graph $K$ consisting of the $D$ and $E$ vertices of $\tilde{G}$, i. e. $K = (U, F)$ with $U = D \uplus E$. Let $F = B \uplus R$ with black edges $B$ and red edges $R$. The black edges are the edges between the $D$ and $E$ vertices as in $\tilde{G}$. Furthermore, there is a red edge between two vertices if they are adjacent to a common $C$ vertex in $\tilde{G}$, i. e. $|R| = c$. Thus, $K$ is 3-regular with a maximum red degree of 2, due to the fact that there are no $D_3$ vertices. It is

$$|R| = c > (\frac{1}{3} + 2\epsilon)(c + d_2 + d_1 + e) > (\frac{1}{3} + 2\epsilon)\frac{3}{2}(d_2 + d_1 + e) = (\frac{1}{2} + 3\epsilon)|U| \; .$$

Thus, $K$ fulfills the requirements of Lemma 2 for $\bar{\epsilon} = 3\epsilon$. We use Lemma 2 to derive a set $S$ of $D$ and $E$ vertices with size $O(\frac{1}{\bar{\epsilon}^2}) = O(\frac{1}{\epsilon^2})$.

The number $b_{ext}$ of external black edges of $S$ with respect to $K$ is equal to the number of edges between $S$ and other $D$ and $E$ vertices in $V_0$. The number $r_{int}$ of internal red edges of $S$ with respect to $K$ is equal to the number of $C$ vertices in $V_0$ which are connected to two vertices in $S$. Lemma 2 ensures $r_{int} > b_{ext}$.

Let $\bar{S}$ be the union of $S$ with all adjacent $C$-vertices. It is $|\bar{S}| = O(\frac{1}{\epsilon^2})$. Each external black edge connects $\bar{S}$ with $V_0 \backslash \bar{S}$. The external red edges are neutral, because they connect $S$ via a $C$ vertex to $V_0 \backslash \bar{S}$. Thus, such a $C$ vertex has one edge to $V_1$ and one edge to $V_0 \backslash \bar{S}$. Each internal red edge is a $C$ vertex which is connected to two vertices in $S$. Thus, such a vertex has one edge to $V_1$ and no edges to $V_0 \backslash \bar{S}$. Overall, there are $r_{int}$ edges between $\bar{S}$ and $V_1$ and $b_{ext}$ edges between $\bar{S}$ and $V_0 \backslash \bar{S}$. This leads to $H(\bar{S}) = r_{int} - b_{ext} > 0$ and $\bar{S}$ fulfills the lemma. □

In the following, 'log($x$)' denotes the logarithm of $x$ to the basis 2.

**Lemma 4.** *Let $G = (V, E)$ be a connected 3-regular graph and let $\pi$ be a bisection of $G$. If $|V_1(\pi)| < 3 \cdot cut(\pi)$ and $0 < x < |V_1(\pi)|$, then there is a set $S \subset V_1(\pi)$ with $|S| = x$ and $H(S) \geq -1 - \log(|S|)$.*

*Proof.* We first discuss the following cases.

(i) If $x \leq 2$, any set $S$ of $x$ vertices which are incident to a cut edge has the desired property $H(S) \geq -1 - \log(|S|)$.

(ii) If we find a set $Z \subset V_1(\pi)$ with $|Z| \leq x$ and $H(Z) \geq 0$, we can move $Z$ from $V_1$ to $V_0$ without increasing the cut size. It remains to apply the lemma again with $\bar{x} = x - |Z|$. Notice that in the case $H(Z) > 0$ the move may result in $|V_1| \geq 3 \cdot cut$. In this case vertices which are incident to a cut edge can be moved from $V_1$ to $V_0$ until we either moved a total of $x$ vertices or until it holds $|V_1| < 3 \cdot cut$. In the latter case we apply the lemma again.

(iii) If we find a set $Z \subset V_1(\pi)$ with $\frac{x}{2} \leq |Z| \leq x$ and $H(Z) \geq -1$, we can move $Z$ from $V_1$ to $V_0$ with increasing the cut size by at most 1. It remains to apply the lemma again with $\bar{x} = x - |Z| < \frac{x}{2}$. This will construct a set $\bar{S}$ with $|\bar{S}| = x - |Z|$ and $H(\bar{S}) \geq -1 - \log(|\bar{S}|)$, and a unified set $S = Z \cup \bar{S}$ with $|S| = x$ and $H(S) \geq -1 - 1 - \log(|\bar{S}|) \geq -1 - \log(|S|)$.

In the following we can exclude the existence of certain small 0-helpful sets. One example are $C$ vertices incident to two or three cut edges and any set of two adjacent $C$ vertices. A $D_3$ vertex, together with its adjacent $C$ vertices, also forms a 0-helpful set. In the remainder there are no such sets, i. e. it is $cut = c$, $d_3 = 0$ and $2c = 2d_2 + d_1$. Because of $|V_1| < 3c$ it holds equation (4).

Consider the graph induced by the vertex set $D \cup E$ and its connected components. Let $F$ be the family of these components. For a set $I \subset V_1(\pi)$ define the enlarged set $Z(I) = I \cup \{v \in C; \exists w \in I \text{ with } \{v, w\} \in E\}$ which includes the adjacent $C$-vertices. Clearly, each set $Z(I)$ for an $I \in F$ is at least 0-helpful. If there is a set $Z(I)$, $I \in F$, with $|Z(I)| \leq x$, we proceed as discussed in case (ii).

Consider a connected component $I \in F$ and let $K = (I, J)$ be the subgraph of $G$ induced by $I$. The $E$ vertices in $K$ have degree 3, $D_1$ vertices have degree 2 and $D_2$ vertices have degree 1. It is easy to see that $e(I) \geq d_2(I)$ iff $K$ contains a cycle and $e(I) = d_2(I) - 2$ otherwise. Because of equation (4) there is an $I \in F$ for which the induced subgraph is a tree.

Let $I \in F$ be a connected component with the induced subgraph $T = (I, J)$ being a tree. Assign a weight $w(v)$ to each vertex $v$ in the tree with $w(v) = |Z(\{v\})|$. For each vertex $v$ this is one higher than the number of $C$ vertices adjacent to $v$. Thus, each leaf has a weight of 3, each vertex of degree 2 has a weight of 2 and each vertex with a degree of 3 has a weight of 1. It is $\sum_{v \in L} w(v) = |Z(L)|$ for an $L \subset I$ if there are no $C$ vertices which are connected to two vertices of $L$. It is $\sum_{v \in L} w(v) > |Z(L)|$ for an $L \subset I$ if there is at least one such vertex.

With $|Z(I)| > x$ it is $\sum_{v \in T} w(v) > x$. Clearly, for this type of weight distribution there is an edge in $T$ which separates $T$ into $T_1$ and $T_2$ with $\frac{x}{2} \leq \sum_{v \in T_1} w(v) \leq x$. If $|Z(T_1)| < \sum_{v \in T_1} w(v)$, it is $|Z(T_1)| < x$ and $H(Z(T_1)) \geq 0$ and we proceed with case (ii) above. If $|Z(T_1)| = \sum_{v \in T_1} w(v)$, it is $\frac{x}{2} \leq |Z(T_1)| \leq x$ and $H(Z(T_1)) \geq -1$. We proceed with case (iii) above. ☐

**Theorem 1.** *For any* $\epsilon > 0$ *there is a value* $n(\epsilon)$ *such that the bisection width of any 3-regular graph* $G = (V, E)$ *with* $|V| > n(\epsilon)$ *is at most* $(\frac{1}{6} + \epsilon)|V|$.

*Proof.* We start with an arbitrary bisection and follow the iterative local improvement scheme described in Section 1. As long as the cut is above the bound, we repeatedly use Lemma 3 and 4 to calculate a new bisection with a lower cut. Thus, we can limit our focus on one iteration of the two lemmas. Let $\pi_0$ be a balanced bisection at the start of the iteration with $cut(\pi_0) > (\frac{1}{6} + \epsilon)|V|$.

**Step 1:** We construct a small helpful set $S \subset V_0$. Set $k = 4 \cdot \log(\frac{1}{\epsilon})$. The value of $k$ is discussed below. We apply Lemma 3 several times. Each time we find an at least 1-helpful set. We proceed until we reach a total helpfulness of at least $k$, i. e. we apply the lemma $k'$ times with $k' \leq k$. Let $S_i \subset V_0$, $1 \leq i \leq k'$, with $|S_i| = O(\frac{1}{\epsilon^2})$ be the sets constructed with Lemma 3. After a 1-helpful set $S_i$ is constructed, it is moved from $V_0$ to $V_1$ and the next set $S_{i+1}$ is constructed. Let $S = \uplus_{1 \leq i \leq k'} S_i$. It is $|S| = k' \cdot O(\frac{1}{\epsilon^2}) = k \cdot O(\frac{1}{\epsilon^2})$ and $H(S) \geq k$.

It remains to show that the requirement of Lemma 3 is fulfilled before each construction of a helpful set. Let $\bar{\epsilon} = \frac{\epsilon}{2}$. It is $|V| \geq 2|V_0| - 1$ and $cut(\pi_0) > (\frac{1}{3} + 2\bar{\epsilon})|V_0(\pi_0)| - (\frac{1}{6} + \bar{\epsilon}) + \bar{\epsilon}|V|$ at the beginning. Let $n(\epsilon)$ be large enough such that $\bar{\epsilon}|V| \geq k + (\frac{1}{6} + \bar{\epsilon})$ for all $|V| > n(\epsilon)$. Thus, it is $cut(\pi_0) > (\frac{1}{3} + 2\bar{\epsilon})|V_0(\pi_0)| + k$. Each application of Lemma 3 decreases the size of the cut. We perform the lemma as long as $cut(\pi) > cut(\pi_0) - k > (\frac{1}{3} + 2\bar{\epsilon})|V_0(\pi_0)| \geq (\frac{1}{3} + 2\bar{\epsilon})|V_0(\pi)|$ with $\pi$ being the current bisection. Thus, the condition $cut(\pi) > (\frac{1}{3} + 2\bar{\epsilon})|V_0(\pi)|$ is true before each application.

Let $\pi_1$ be the new bisection with $cut(\pi_1) = cut(\pi_0) - H(S)$.

**Step 2:** If $H(S) = k$, it is $cut(\pi_1) = cut(\pi_0) - k$. If $H(S) > k$, it is $cut(\pi_1) < cut(\pi_0) - k$ and we change $\pi_1$ by iteratively moving border vertices from $V_1$ to $V_0$ until we either get to $cut(\pi_1) = cut(\pi_0) - k$ or to a balanced bisection (in this case we are already finished). Each move of a border vertex decreases the imbalance of the bisection and increases the cut by at most one.

Let $i := |V_1(\pi_1)| - \frac{n}{2}$ be the imbalance of $\pi_1$. It is $i \leq k \cdot O(\frac{1}{\epsilon^2})$. We use Lemma 4 to find a balancing set $\bar{S} \subset V_1(\pi_1)$ with $|\bar{S}| = i$.

Lemma 4 can only be applied if $|V_1(\pi_1)| < 3 \cdot cut(\pi_1)$. The fact $cut(\pi_0) > (\frac{1}{6} + \epsilon)n$ implies $|V_1(\pi_1)| = \frac{n}{2} + i < 3cut(\pi_0) - 3\epsilon \cdot n + i = 3cut(\pi_1) + 3k - 3\epsilon \cdot n + i \leq 3cut(\pi_1)$ if $3k + i \leq 3\epsilon \cdot n$. Clearly, there is a value $n(\epsilon)$ such that this equation holds for all graphs with $n > n(\epsilon)$.

We use Lemma 4 to get a set $\bar{S} \subset V_1(\pi_1)$ with $|\bar{S}| = i$ and $H(\bar{S}) \geq -1 - \log(i)$. The move of $\bar{S}$ from $V_1$ to $V_0$ results in a balanced bisection $\pi_2$ with $cut(\pi_2) \leq cut(\pi_1) + 1 + \log(i)$.

We need to ensure $cut(\pi_2) < cut(\pi_0)$ in order to show a decrease of the cut size. It is $cut(\pi_2) \leq cut(\pi_0) - k + 1 + \log(i)$ and $i \leq k \cdot x\frac{1}{\epsilon^2}$. for some constant $x$. Choosing $k = 4 \cdot \log(\frac{1}{\epsilon})$ fulfills $k > 1 + \log(k \cdot x\frac{1}{\epsilon^2})$ for $\frac{1}{\epsilon} \geq 2^8$ and $\frac{1}{\epsilon} \geq x$. ☐

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