The constant term of tempered functions on a real spherical space of wavefront type

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March 18, 2018

Abstract

Let $Z$ be a unimodular real spherical space which is assumed to be of wavefront type. We develop a theory of constant terms for tempered functions on $Z$ which parallels the work of Harish-Chandra. The constant terms $f_I$ of an eigenfunction $f$ are parametrized by subsets $I$ of the set $S$ of spherical roots which determine the fine geometry of $Z$ at infinity. Constant terms are transitive i.e. $(f_J)_I = f_I$ for $I \subset J$, and our main result is a quantitative bound of the difference $f - f_I$, which is uniform in the parameter of the eigenfunction.

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*The first author was supported by a grant of Agence Nationale de la Recherche with reference ANR-13-BS01-0012 FERPLAY.
Introduction

Let \( Z = G/H \) be a real unimodular wavefront real spherical space. In this introduction \( G \) is the group of real points of a connected reductive algebraic group \( G \) defined over \( \mathbb{R} \), and \( H = G \cap H(\mathbb{R}) \) for an algebraic subgroup of \( H \) of \( G \) defined over \( \mathbb{R} \). Real spherical then means that there exists a minimal parabolic subgroup \( P \subset G \) with \( PH \) open in \( G \) and the term wavefront will be characterized later in the introduction. At this point we only remark that all symmetric spaces are wavefront together with the interesting classes of Gross-Prasad spaces such as \( \text{GL}(n+1, \mathbb{R}) \times \text{GL}(n, \mathbb{R})/\text{diag GL}(n, \mathbb{R}) \) or \( \text{O}(p+1, q) \times \text{O}(p, q)/\text{diag O}(p, q) \).

This paper is motivated in part by the work of Sakellaridis and Venkatesh for \( p \)-adic spherical varieties in [20] of wavefront-type, in particular their theory of asymptotics for smooth functions.

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1 All results mentioned in this intro hold true in the analytically more general setup with \( G \) an open subgroup of \( G(\mathbb{R}) \) and \( H \) replaced by its identity component.
In this paper we supply asymptotics for a more limited class of functions on a wavefront real spherical space $Z$. They are $C^\infty$, tempered and generalized eigenfunctions under the center of the enveloping algebra. We develop a theory of constant terms for tempered functions on $Z$ which parallels the work of Harish-Chandra [8, 9]. The constant terms $f_I$ of an eigenfunction $f$ are parametrized by subsets $I$ of the set $S$ of spherical roots which determine the fine geometry of $Z$ at infinity (cf. [11]). Constant terms are transitive i.e. $(f_J)_I = f_I$ for $I \subset J$, and our main result is a quantitative bound of the difference $f - f_I$, which is uniform in the parameter of the eigenfunction.

Let us describe the results more precisely. The local structure theorem (cf. [14, Theorem 2.3] and Subsection 1.1) associates a parabolic subgroup $Q$, said $Z$-adapted to $P$, with Levi decomposition $Q = LU$ (one has $P \subset Q$).

We will say that $A$ is a split torus of $G$ if $A = A(\mathbb{R})$, where $A$ is a split $\mathbb{R}$-torus of $G$. Let now $\Lambda$ be a maximal split torus of $L$ and set $A_H := A \cap H$. We define $A_Z$ to be the identity component of $A/A_H$ and recall the spherical roots $S$ as defined in e.g. [15, Section 3.2]. The spherical roots are linear forms on $a_Z = \text{Lie } A_Z$ and give rise to the co-simplicial compression cone $a_Z := \{X \in a_Z \mid (\forall \alpha \in S) \alpha(X) \leq 0\}$. Set $A_Z := \exp(a_Z) \subset A_Z$.

Denote by $z_0 = H \in Z$ the standard base point of $Z = G/H$. The polar decomposition asserts that there is a compact subset $\Omega \subset G$ and a finite subset $W$ of $G$ such

$$Z = \Omega A_Z W \cdot z_0.$$  \hspace{1cm} (0.1)

More precisely, the set $W$ parametrizes the open $P \times H$-double cosets in $G$ and we refer to Lemma 1.5 and Remark 1.8 below for more explicit expressions of its elements.

Let $\rho_Q$ be the half sum of the roots of $a$ in Lie $U$. Actually, as $Z$ is unimodular, $\rho_Q \in a_Z^*$. For $f \in C^\infty(Z)$ we set

$$q_N(f) = \sup_{\omega \in \Omega, w \in W, a \in A_Z} a^{-\rho_Q}(1 + \|a\|)^{-N}|f(\omega aw \cdot z_0)|$$

and define $C^\infty_{\text{temp},N}(Z)$ as the space of all $f \in C^\infty(Z)$ such that, for all $u$ in the enveloping algebra $\mathcal{U}(g)$ of the complexification $g_C$ of $g = \text{Lie } G$,

$$q_N,\omega(f) := q_N(L_u f)$$

is finite. The semi-norms $q_{\omega,u}$ induce a Fréchet structure on $C^\infty_{\text{temp},N}(Z)$ for which the $G$-action is smooth and of moderate growth (in [2] these are called $SF$-representations). We define the space of tempered functions $C^\infty_{\text{temp}}(Z) = \bigcup_{N \in \mathbb{N}} C^\infty_{\text{temp},N}(Z)$ and endow it with the inductive limit topology.

We denote by $Z(g)$ the center of $\mathcal{U}(g)$ and define $\mathcal{A}_{\text{temp}}(Z)$ as the subspace of $C^\infty_{\text{temp}}(Z)$ consisting of $Z(g)$-finite functions. It is then a variant of Frobenius-reciprocity that elements $f \in \mathcal{A}_{\text{temp}}(Z)$ can be expressed as generalized matrix coefficients

$$f(gH) = m_{\eta,v}(gH) := \eta(\pi(g)^{-1}v)$$

for $v \in V^\infty$ where $(\pi, V^\infty)$ is a $Z(g)$-finite $SF$-representation of $G$ and $\eta : V^\infty \to \mathbb{C}$ a $H$-invariant continuous functional. We denote by $V^{-\infty}$ the continuous dual of $V^\infty$. An element $\eta \in (V^{-\infty})^H$ is called $Z$-tempered provided $m_{\eta,v} \in \mathcal{A}_{\text{temp}}(Z)$ for all $v \in V^\infty$. 


We move on to boundary degenerations \( h_I \) of \( h \) which are parametrized by subsets \( I \subset S \). These geometric objects show up naturally in the compactification theory of \( Z \) (see [12], [11] and Section 2 below) which in turn is closely related to the polar decomposition (0.1). In more detail let \( \mathfrak{a}_I = \bigcap_{\alpha \in I} \text{Ker} \alpha \) and pick \( X \in \mathfrak{a}_I^{-} = \{ X \in \mathfrak{a}_I : \alpha(X) < 0, \alpha \in S\backslash I \} \). Let \( H_I \) be the analytic subgroup of \( G \) with Lie algebra

\[
\text{Lie } H_I = \lim_{t \to +\infty} e^{t \text{ad} X} \text{Lie } H,
\]

where the limit is taken in the Grassmanian \( \text{Gr}(g) \) of \( g \) and does not depend on \( X \). Then (see [15, Proposition 3.2]) \( Z_I = G/H_I \) is a real spherical space, \( PH_I \) is open in \( G \) and \( Q \) is \( Z_I \)-adapted to \( P \). We denote \( H_I \) by \( z_{0,I} \) in the quotient space \( Z_I \). We choose a set \( \mathcal{W}_I \) corresponding to \( W \) for \( Z_I \) as in Lemma 1.5 and Remark 2.2. One can define similarly \( C_{\text{temp},N}(Z_I) \) and \( A_{\text{temp},N}(Z_I) \).

The main result of this paper is the following (cf. Proposition 5.8 and Theorem 5.15 for (i)-(iii) and Theorem 7.6 for (iv)).

A Theorem. Let \( I \) be a finite codimensional ideal of the center \( Z(g) \) of \( U(g) \) and let \( A_{\text{temp},N}(Z : I) \) be the space of elements of \( A_{\text{temp},N}(Z) \) annihilated by \( I \). There exists \( N_Z \in \mathbb{N} \) such that, for all \( N \in \mathbb{N} \), for each \( f \in A_{\text{temp},N}(Z : I) \), there exists a unique \( f_I \in A_{\text{temp},N+I}(Z_I : I) \) such that, for all \( g \in G \), \( X \in \mathfrak{a}_I^{-} \):

(i) \( \lim_{T \to +\infty} e^{-T \rho_q(X)} (f(g \exp(TX)) - f_I(g \exp(TX))) = 0 \).

(ii) \( T \mapsto e^{-T \rho_q(X)} f_I(g \exp(TX)) \) is an exponential polynomial with unitary characters, i.e. of the form \( \sum_{j=1}^{n} p_j(T) e^{i\nu_j T} \), where the \( p_j \)'s are polynomials and the \( \nu_j \)'s are real numbers.

(iii) The linear map \( f \mapsto f_I \) is a continuous \( G \)-morphism. Moreover, for each \( w_I \in \mathcal{W}_I \) there exist \( w \in W \) such that for any compact subset \( C \) in \( \mathfrak{a}_I^{-} \) and any compact subset \( \Omega \) of \( G \) there exists \( \varepsilon > 0 \) and a continuous semi-norm \( p \) on \( A_{\text{temp},N}(Z) \) with:

\[
\left| (a \exp TX)^{-\rho_q} (f(\omega \exp(TX)w \cdot z_0) - f_I(\omega \exp(TX)w_I \cdot z_0, I)) \right| \leq e^{-\varepsilon T} P(f)(1 + \| \log a \|)^N, \quad a \in A_Z, X \in C, \omega \in \Omega, T \geq 0.
\]

(iv) The bound in (iii) is uniform for all \( I \) is of co-dimension 1, i.e. \( I = \ker \chi \) for a character \( \chi \) of \( Z(g) \).

Part (i) - (iii) generalizes the work of Harish-Chandra in the group case (see [8, Sections 21 to 25], also the work of Wallach [22, Chapter 12]) and the one of Carmona for symmetric spaces (see [4]). Part (iv) generalizes the uniform results of Harish-Chandra in the group case (cf. [9, Section 10]) and is already new for symmetric spaces. This uniform bound relies on the recently established fact that infinitesimal characters of tempered representations have integral real parts (see [16] and Lemma 7.4 below).

While the work of Harish-Chandra is for \( K \)-finite tempered functions, where \( K \) is a maximal compact subgroup of \( G \), we deal with smooth tempered functions, but without using asymptotic expansions as it is done in [22, Chapter 12].
For a $\mathbb{Z}$-tempered continuous linear form $\eta$ on a $\mathbb{Z}(\mathfrak{g})$-finite $SF$-representation $(\pi, V^\infty)$, one can define a constant term $\eta_I$ which is a $\mathbb{Z}_I$-tempered continuous linear form on $V^\infty$ in such a way that, for all $v \in V^\infty$,

$$m_{\eta_I,v}(z_I) = (m_{\eta,v})_I(z_I), \quad z_I \in \mathbb{Z}_I$$

(cf. Proposition 5.16). Moreover we show that, if $(\pi, V^\infty)$ is irreducible with unitary central character, then $(\pi, V^\infty, \eta)$ is a discrete series modulo the center of $Z$ if and only if for all $I \not\subset S$, $\eta_I = 0$ (see Theorem 5.17). Again it is analogous to a result of Harish-Chandra. For this we use in a crucial manner some results on discrete series from [15, Section 8].

The proof of Theorem A is fairly parallel to [8, 9] by studying certain systems of linear differential equations applied to $K$-finite functions. Using the $SF$-completion theory of Harish-Chandra modules (see e.g. [5], [22, Chapter 11] or [2]) we can formulate results in the smooth category, i.e. we are no more limited to $K$-finite functions.

The method used in this paper is limited to wavefront spaces. These spaces are defined by the property that the compression cone $A_Z^-$ is the homomorphic image of the closure of the negative Weyl-chamber $A_+^-$ for $P$ under the natural quotient map $A_0 \to A_Z$ where $A_0$ denotes the identity component of $A$. In particular, we find a section $s : A_Z^- \to A_-$ such that $s(A_Z^-)$ acts by contractions on the nilradical Lie $U$ of the $Z$-adapted parabolic $Q$. Moreover, for all subsets $I \subset S$ we find a parabolic $P_I \supset P_{\text{opp}}$, where $P_{\text{opp}}$ is the parabolic opposite to $P$ containing $A$, with Levi decomposition $P_I = L_I U_I^-$ such that $U_I^- \subset H_I \subset P_I$. These two facts are heavily used in the present account (beginning with Lemma 4.2 and then essentially via (4.14)) and are not true for general real spherical spaces.

**Acknowledgement:** We thank Raphael Beuzart-Plessis and Friedrich Knop for their useful comments and help. It is our pleasure to acknowledge the input of the referee whose substantial report and requests led to a strongly improved paper.

1 **Notation**

In this paper, we will denote (real) Lie groups by upper case Latin letters and their Lie algebras by lower case German letters. If $R$ is a real Lie group, then $R_0$ will denote its identity component. Further, if $Z$ is an algebraic variety defined over $\mathbb{R}$ and $k$ is any field containing $\mathbb{R}$, then we denote by $Z(k)$ the $k$-points of $Z$.

Let $G$ be a connected reductive algebraic group defined over $\mathbb{R}$ and let $G := \underline{G}(\mathbb{R})$ be its group of real points.

1.1 **Remark.** More generally we could define $G$ as an open subgroup of the real Lie group $\underline{G}(\mathbb{R})$. The main analytic result of this paper (i.e. Theorem A) is not affected by this more general assumption.

For an $\mathbb{R}$-algebraic subgroup $R$ of $G$ we set $R := \underline{R}(\mathbb{R})$ and note that $R \subset G$ is a closed subgroup.
Let now $H \subset G$ be a connected $\mathbb{R}$-algebraic subgroup. Having $G$ and $H$ we form the homogeneous variety $Z = G/H$. We note that $Z(\mathbb{C}) = G(\mathbb{C})/H(\mathbb{C})$ and denote by $z_0 = H(\mathbb{C})$ the standard base point of $Z(\mathbb{C})$. Set $Z = G/H$ and record the $G$-equivariant embedding

$$Z \to Z(\mathbb{C}), \quad gH \mapsto g \cdot z_0.$$ 

In the sequel we consider $Z$ as a submanifold of $Z(\mathbb{C})$ and in particular identify $z_0$ with the standard base point $H$ of $Z$ as well.

1.2 Remark. Note that $Z$ is typically strictly smaller than $Z(\mathbb{R})$ which is a finite union of $G(\mathbb{R})$-orbits. The first instance where this occurs is for $Z = G$ and $G \not= G(\mathbb{R})$. More illuminating is the space $Z = GL_n / O(n)$ of invertible symmetric matrices which features $Z(\mathbb{R}) = \bigcup_{p+q=n} GL(n, \mathbb{R}) / O(p,q)$. In particular, $Z(\mathbb{R}) \cong G(\mathbb{R}) / H(\mathbb{R}) = GL(n, \mathbb{R}) / O(n)$.

As a further piece of notation we use for an algebraic subgroup $R \subset G$ defined over $\mathbb{R}$ the notation $R_H := R \cap H$ and likewise $R_H := R \cap H$. In the sequel we use the letter $P$ to denote a minimal $\mathbb{R}$-parabolic subgroup of $G$. The unipotent radical of $P$ is denoted by $N$.

1.1 The local structure theorem

From now on we assume that $Z$ is real spherical, that is there is a choice of $P$ such that $P \cdot z_0$ is open in $Z$.

1.3 Remark. Notice that $P(\mathbb{C})H(\mathbb{C})$ is Zariski open and hence dense in $G(\mathbb{C})$ as $G$ was assumed to be connected. Thus any other choice $P'$ of $P$ with $P' \cdot z_0$ open is conjugate to $P$ under $H$.

We now recall the local structure theorem for real spherical varieties (cf. [14, Theorem 2.3] or [11, Corollary 4.12]; see also [3, 10] for preceding versions in the complex case), which asserts that there is a unique parabolic subgroup $Q \supset P$ endowed with a Levi-decomposition $Q = L \ltimes U$, defined over $\mathbb{R}$, such that

$$QH = PH, \quad Q_H = L_H, \quad L_H \supset L_n,$$

where $L_n$ denotes the connected normal subgroup of $L$ generated by all unipotent elements defined over $\mathbb{R}$.

1.4 Remark. (a) The Lie algebra $l_n$ is the sum of all non-compact simple ideals of $l$.

(b) As mentioned above, $Q$ is the unique parabolic subgroup above $P$ with properties (1.1) - (1.3). Slightly differently we could have defined $Q$ via [12, Lemma 3.7] which asserts

$$Q(\mathbb{C}) = \{ g \in G(\mathbb{C}) \mid gP(\mathbb{C}) \cdot z_0 = P(\mathbb{C}) \cdot z_0 \}.$$ 

The group $L_H$ is uniquely determined by $Q$ and we recall from [11, Lemma 13.5] that $L_H$ is an invariant of $Z$, i.e. its $H$-conjugacy class is defined over $\mathbb{R}$. In contrast to $L_H$ the Levi
$L$ is only unique up to conjugation with elements from $U$ which stabilize $L_H$. In this regard we note that it is quite frequent that $L_H$ is trivial and then $L$ could be an arbitrary Levi of $Q$. For later purposes of compactifications we will only use those choices of $L$ which are obtained from the constructive proof of the local structure theorem (cf. [14, Subsection 2.1]).

In a nutshell this means that $I$ is defined to be as the centralizer of a generic hyperbolic element of $g$ which is contained in $(\mathfrak{h} + \mathfrak{n})^\perp$ where $\perp$ refers to the orthogonal complement of a standard non-degenerate invariant extension of the Cartan-Killing form of $[\mathfrak{g}, \mathfrak{g}]$ to $g$.

Such a parabolic subgroup $Q$ as above will be called $Z$-adapted to $P$.

Let $A_L$ be a maximal split torus of the center of $L$ and $\mathcal{A}$ be a maximal split torus of $P \cap L$. Note that $A_L \subset \mathcal{A}$. Define $A_Z := A/A_H$ and let $A_Z := (A/A_H)_0 \simeq A_0/(A_H)_0$. From the fact that $L_0 \subset L_H$ and $\mathcal{A} = A_0(A \cap \mathcal{L}_0)$ we obtain $A_Z \simeq (A_L)_0/(A_L)_0 \cap H$ with $\mathfrak{a}_Z \simeq \mathfrak{a}_L/\mathfrak{a}_L \cap \mathfrak{h}$.

We choose a section $s : A_Z \to (A_L)_0$ of the projection $(A_L)_0 \to A_Z$ which is a morphism of Lie groups. We will often use $\tilde{s}$ instead of $s(a)$.

Note that $Z_G(A) = MA$ for some anisotropic group $M \subset P$. Moreover, $MA$ as a Levi of $P$ is connected (recall that Levi-subgroups of connected algebraic groups are connected). Notice that $M$ commutes with $A$ and $P = MAN$. Observe that $M \cap A$ equals the 2-torsion points $A_2$ of $A$.

From (1.1)-(1.2) we obtain $P\mathcal{L}/H = Q\mathcal{L}/H \simeq U \times L/L_H$ and taking real points we get

$$[P \cdot z_0](\mathbb{R}) \simeq U \times (L/L_H)(\mathbb{R}).$$

Next we collect some elementary facts about $(L/L_H)(\mathbb{R})$. To begin with we define

$$\widehat{M}_H := \{ m \in M \mid m \cdot z_0 \subset A_Z(\mathbb{R}) \}$$

and note that $M_H$ is a cofinite normal 2-subgroup of $\widehat{M}_H$, see Proposition B.2. We denote by $F_M := \widehat{M}_H/M_H$ this finite 2-group. Since $L/L_H$ is transitive for the $P$-Levi $MA \subset L$ we obtain for the real points by Proposition B.2

$$(L/L_H)(\mathbb{R}) = [M/M_H] \times_{F_M} A_Z(\mathbb{R}).$$

From that we derive the local structure theorem in the form

$$[P \cdot z_0](\mathbb{R}) = U \times [[M/M_H] \times_{F_M} A_Z(\mathbb{R})]$$

which we will use later. Let us define by $T_2 \subset A_Z(\mathbb{R})$ the group of 2-torsion elements and note that (1.5) shows that the open $P$-orbits on $Z(\mathbb{R})$ are parametrized by $T_2/F_M$. If we intersect (1.5) with $Z$ we obtain

$$[P \cdot z_0](\mathbb{R}) \cap Z = U \times [[M/M_H] \times_{F_M} A_{Z,\mathbb{R}}]$$

with $A_{Z,\mathbb{R}} := A_Z(\mathbb{R}) \cap Z$.

In particular we obtain a set of representatives $\mathcal{W} \subset G$ of the open $P \times H$-orbits in $G$ of a special shape:
1.5 Lemma. The set \( \mathcal{W} \) can be chosen such that any \( w \in \mathcal{W} \) can be written as:

\[
w = \tilde{t}h, \text{ where } \tilde{t} \in \exp(i\tilde{a}_Z) \text{ and } h \in H(\mathbb{C}).
\]

In particular, if \( a \in A_H \), \( aw \cdot z_0 = w \cdot z_0 \).

Proof. It follows from (1.6) that every open \( P \)-orbit on \( Z \subset \mathbb{Z}(\mathbb{C}) \) has a representative of the form \( t \in T_2 \cap Z \). Now \( t = w' \cdot z_0 \) for some \( w \in G \) by definition. In particular, \( w = \tilde{t}h \) for some \( \tilde{t} \in \exp(i\tilde{a}_Z) \), \( h \in H(\mathbb{C}) \) with \( \tilde{t} \cdot z_0 = t \).

\[\square\]

1.2 Spherical roots and polar decomposition

Let \( K \subset G \) be a maximal compact subgroup associated to a Cartan-involution \( \theta \) of \( g \) with \( \theta(X) = -X \) for all \( X \in a \). Further let \( B \) be an \( \text{Ad} \)G and \( \theta \)-invariant bilinear form on \( g \) such that the quadratic form \( X \mapsto \|X\|^2 = -B(X, \theta X) \) is positive definite. We will denote by \( (\cdot, \cdot) \) the corresponding scalar product on \( g \). It defines a quotient scalar product and a quotient norm on \( a_Z \) that we still denote by \( \| \cdot \| \).

For later reference we record that \( K \) is algebraic, i.e. \( K = K(\mathbb{R}) \), and further \( M \subset K \) as we requested \( \theta|_a = -\text{id}_a \).

Let \( \Sigma \) be the set of roots of \( a \) in \( g \). If \( \alpha \in \Sigma \), let \( g^\alpha \) be the corresponding weight space for \( a \). We write \( \Sigma_u \) (resp. \( \Sigma_n \)) for the set of \( a \)-roots in \( u \) (resp. \( n \)) and set \( u^- = \sum_{\alpha \in \Sigma_u} g^{-\alpha}, \) i.e. the nilradical of the parabolic subalgebra \( q^- \) opposite to \( q \) with respect to \( a \).

Let \((l \cap h)^{\perp_1}\) be the orthogonal of \( l \cap h \) in \( l \) with respect to the scalar product \( (\cdot, \cdot) \). One has:

\[
g = h \oplus (l \cap h)^{\perp_1} \oplus u. \tag{1.7}
\]

Let \( T \) be the restriction to \( u^- \) of minus the projection from \( g \) onto \((l \cap h)^{\perp_1} \oplus u \) parallel to \( h \). Let \( \alpha \in \Sigma_u \) and \( X_{-\alpha} \in g^{-\alpha}. \) Then (cf. [15, equation (3.3)])

\[
T(X_{-\alpha}) = \sum_{\beta \in \Sigma_u \cup \{0\}} X_{\alpha, \beta}, \tag{1.8}
\]

with \( X_{\alpha, \beta} \in g^\beta \subset u \) if \( \beta \in \Sigma_u \) and \( X_{\alpha, 0} \in (l \cap h)^{\perp_1}. \)

Let \( \mathcal{M} \subset N_0[\Sigma_u] \) be the monoid generated by:

\[
\{\alpha + \beta : \alpha \in \Sigma_u, \beta \in \Sigma_u \cup \{0\} \text{ such that there exists } X_{-\alpha} \in g^{-\alpha} \text{ with } X_{\alpha, \beta} \neq 0\}. \tag{1.9}
\]

The elements of \( \mathcal{M} \) vanish on \( a_H \) so \( \mathcal{M} \) identifies to a subset of \( a_Z^* \). We define

\[
a_Z^- = \{X \in a_Z : \alpha(X) < 0, \alpha \in \mathcal{M}\}
\]

and \( a_Z = \{X \in a_Z : \alpha(X) \leq 0, \alpha \in \mathcal{M}\}. \)

Following e.g. [15, Section 3.2] we recall that \( a_Z^- \) is a co-simplicial cone and our choice of spherical roots \( S \) consists of the irreducible elements of \( \mathcal{M} \) which are extremal in \( \mathbb{R}_{\geq 0}\mathcal{M}. \)
Here an element of \( M \) is called irreducible if it cannot be expressed as a sum of two non-zero elements in \( M \). Later we will also need the edge of \( a_Z \)

\[
a_{Z,E} := a_Z^- \cap (-a_Z^-) = \{ X \in a_Z : \alpha(X) = 0, \alpha \in S \}.
\]

Note that \( a_{Z,E} \) (more precisely \( s(a_{Z,E}) \)) normalizes \( \mathfrak{h} \) and likewise \( A_{Z,E} := \exp(a_{Z,E}) \subset A_Z \).

We turn to the polar decomposition for \( Z \). Set \( A^{-}_Z := \exp(a_Z^-) \).

**1.6 Lemma** (Polar decomposition). *There exists a compact subset \( \Omega \subset G \) such that \( Z = \Omega A^{-}_Z W \cdot z_0 \).*

**1.7 Remark.** Recall that \( W \) was such that \( PwH \) is open and \( A_Hw \subset wH \) for each \( w \in W' \). In particular, the notion \( A^{-}_Z W \cdot z_0 \) is defined.

**Proof.** Recall \( T_2 \) the 2-torsion subgroup of \( A_Z(\mathbb{R}) \) and \( F_M = \hat{M}_H/M_H \) the the finite 2-group defined before (1.5). Then, according to (1.5), the 2-group \( T_2/F_M \) parametrizes the open \( P \)-orbits in \( Z(\mathbb{R}) \). Now, according to [11, Theorem 13.2 with Remark 13.3(ii)] (building up on the earlier work [12, Theorem 5.13]), we have \( Z(\mathbb{R}) = \Omega \cdot A^{-}_Z T_2 \) for some compact subset \( \Omega \) of \( G \). Note that \( A^{-}_Z T_2 \cap Z = F_M(A^{-}_Z W \cdot z_0) \) by the definition of \( W \) and \( F_M \). Hence it follows that \( Z = \Omega A^{-}_Z W \cdot z_0 := \Omega \cdot (A^{-}_Z W \cdot z_0) \) for some some compact subset \( \Omega \) of \( G \).

**1.8 Remark.** (Passage to \( H \) connected) An analytically more general setup would be to work with connected \( H \), i.e. with \( Z_0 = G/H_0 \) instead of \( Z = G/H \). For that only some adjustments are needed. In detail, by right-enlarging \( W \) with a set \( F_H \) of representatives for \( H/H_0(H \cap M) \) we obtain with \( W_0 := WF_H \) a set which is in bijection with the set of the open \( P \times H_0 \)-double cosets in \( G \). Similar one obtains a polar decomposition for \( Z_0 \) as \( Z_0 = \Omega A^{-}_Z W_0 \cdot z_0 \) with \( z_0 = H_0 \) now denoting the base point of \( Z_0 \). This is all what is needed for this paper. However, working with \( H_0 \) requires additional notational adjustments and the statements of some results (e.g. elements \( m_w \) from Remark 2.2 below cannot chosen to be 1 etc.) become less clean. For the sake of readability we decided to stay in the algebraic framework.

The polar decomposition is closely related to compactification theory of \( Z \) which we summarize in the next section.

## 2 Boundary degenerations and quantitative geometry at infinity

### 2.1 Boundary degenerations of \( Z \)

Let \( I \) be a subset of \( S \) and set:

\[
a_I = \{ X \in a_Z : \alpha(X) = 0, \alpha \in I \}, \quad A_I = \exp(a_I) \subset A_Z,
\]

\[
a_I^- = \{ X \in a_I : \alpha(X) < 0, \alpha \in S \setminus \{I\} \}, \quad A_I^- = \exp(a_I^-).
\]
Then there exists an algebraic Lie subalgebra \( h_I \) of \( g \) such that, for all \( X \in a_I^- \), one has:

\[
h_I = \lim_{t \to +\infty} e^{ad tX} h
\]

in the Grassmanian \( Gr_d(g) \) of \( g \), where \( d := \dim(h) \) (cf. [15, equation (3.9)]).

Note that \( h_I \) is algebraic and let \( H_I \subset G \) be the corresponding connected \( \mathbb{R} \)-algebraic subgroup. By slight abuse of notation we set \( H_I := (H_I(\mathbb{R}))_0 \). Let \( Z_I = G/H_I \). Then \( Z_I \) is a real spherical space for which:

(i) \( PH_I \) is open,

(ii) \( Q \) is \( Z_I \)-adapted to \( P \),

(iii) \( a_{Z_I} = a_Z \) and \( a_{Z_I} = \{ X \in a_Z : \alpha(X) \leq 0, \alpha \in I \} \) contains \( a_Z^- \).

(cf. [15, Proposition 3.2]). Similarly to (1.7), one has

\[
g = h_I \oplus (1 \cap h)^{1-I} \oplus u_I.
\]

Let \( T_I \) be the restriction to \( u^- \) of minus the projection onto \( (1 \cap h)^{1-I} \oplus u_I \) parallel to \( h_I \) and \( \langle I \rangle \subset \mathbb{N}_0[S] \) be the monoid generated by \( I \). Let \( X^I_{\alpha,\beta} = X_{\alpha,\beta} \) if \( \alpha + \beta \in \langle I \rangle \) and zero otherwise. It follows from [15, equation (3.12)] that \( X_{-\alpha} + \sum_{\beta \in \Sigma_u \cup \{0\}} X^I_{\alpha,\beta} \in h_I \). This implies that, for \( \alpha \in \Sigma_u \),

\[
T_I(X_{-\alpha}) = \sum_{\beta \in \Sigma_u \cup \{0\}} X^I_{\alpha,\beta}
\]

Let \( A_{Z_I} = \exp a_{Z_I} \). Similarly to \( Z \), the real spherical space \( Z_I \) has a polar decomposition:

\[
Z_I = \Omega_I A_{Z_I} W_I \cdot z_{0,I},
\]

where \( z_{0,I} = H_I \), \( \Omega_I \subset G \) compact and \( W_I \subset G \) finite (cf. Lemma 1.6 and Remark 1.8 for the choice of \( W_I \) as \( H_I \) is defined to be connected). In more detail, using Lemma 1.5 applied to the real spherical space \( Z_I \), we can make the same kind of choice for \( W_I \) as for \( W \), i.e. elements \( w_I \in W_I \) are of the form

\[
w_I = \tilde{t}_I h_I, \quad \text{for some } \tilde{t}_I \in \exp(i\tilde{a}_Z) \text{ and } h_I \in \overline{H_I(\mathbb{C})}.
\]

### 2.2 Quantitative escape to infinity

Let \( I \subset S \). Let us pick \( X_I \in a_I^- \), i.e. \( X_I \in a_I \) and \( \alpha(X_I) < 0 \) for all \( \alpha \in S \backslash I \). For \( s \in \mathbb{R} \), let

\[
a_s := \exp(sX_I).
\]

Fix \( w_I = \tilde{t}_I h_I \in W_I \). According to [15, Lemma 3.9] there exists \( w \in W \) and \( s_0 > 0 \) with

\[
Pw_I a_s H = Pw H, \quad s \geq s_0.
\]

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Note that (cf. Lemma 1.5):

\[ w = \tilde{t}h \quad \text{for some } \tilde{t} \in \exp(i\tilde{a}_Z) \text{ and } h \in H(C). \quad (2.6) \]

According to [15, Lemma 3.10] there exist elements \( u_s \in U, b_s \in A_Z, \) and \( m_s \in M \) such that

\[
\begin{align*}
    w_I \tilde{a}_s \cdot z_0 &= u_s m_s \tilde{b}_s w \cdot z_0 \quad s \geq s_0, \\
    \lim_{s \to +\infty} (a_s b_s^{-1}) &= 1, \\
    \lim_{s \to +\infty} u_s &= 1, \\
    \lim_{s \to +\infty} m_s &= m_{w_I} \quad \text{for some } m_{w_I} \in M.
\end{align*}
\]

Notice that in case \( w_I = 1 \), one can take \( w = 1 \) and then one has \( m_{w_I} = 1 \). The goal of this section is to give a quantitative version of the convergence in (2.7). For that we first refer to Appendix A for the definition and basic properties of rapid convergence.

Recall the finite 2-group \( F_M = \hat{M}_H/\hat{M} \) defined before (1.5) and fix with \( \hat{F}_M \subset M \) a set of its representatives containing 1. Then we have the following result:

2.1 Proposition. The families \( (a_s b_s^{-1}) \) and \( (u_s) \) converge rapidly to 1 and one can choose the family \( (m_s) \) such that \( (m_s) \) converges rapidly to \( m_{w_I} \in \hat{F}_M \).

2.2 Remark. Proposition 2.1 allows us to change the representatives \( w_I \) to \( m_{w_I}^{-1} w_I \) without loosing the special form \( w_I = \tilde{t}_I h_I \). This is because of \( F_M A_Z \subset A_Z \subset \exp(i\tilde{a}_Z) A \cdot z_0 \). Hence we may assume in the sequel that \( m_{w_I} = 1 \) for all \( w_I \in \mathcal{W}_I \).

The key to understand this Proposition lies in the interpretation of \( \lim_{s \to +\infty} \tilde{a}_s \cdot z_0 \) as an appropriate rapid limit in a suitable smooth compactification of \( Z \).

2.3 Smooth equivariant compactifications

By an equivariant compactification of \( Z(\mathbb{R}) \) we understand here a \( G \)-variety \( \hat{Z} \) defined over \( \mathbb{R} \) such that \( \hat{Z}(\mathbb{R}) \) is compact and contains \( Z(\mathbb{R}) \) as an open dense subset. In this context we denote by \( \partial Z \) the boundary of \( Z \) in \( \hat{Z}(\mathbb{R}) \).

Suitable (i.e. smooth and equivariant) compactifications exist:

2.3 Proposition. Let \( Z = G/H \) be an algebraic real spherical space. Then there exists a smooth equivariant compactification \( \hat{Z}(\mathbb{R}) \) of \( Z(\mathbb{R}) \) with the following property: for all \( I \subset S \) and \( X \in a_I^- \) the limit \( z_X := \lim_{s \to +\infty} (\exp(sX) \cdot z_0) \) exists in \( \partial Z \) and the convergence is rapid. If \( h_X \) is the stabilizer Lie-subalgebra of \( z_X \) in \( g \), then \( h_I \subset h_X \subset h_I + a_I \).

This result is part of the forthcoming work [6]. Since the constructive proof is of relevance for us we allow ourselves to repeat the fairly short proof.

Proof. The starting point is the local structure theorem for the open \( P \)-orbit on \( Z \) as in (1.5)

\[
(P \cdot z_0)(\mathbb{R}) = U \times \left( [M/M_H] \times_{F_M} A_Z(\mathbb{R}) \right). \quad (2.7)
\]
One of the main results in [11], see Theorem 7.1, was that the compactification theory of \( \mathcal{Z} \) can, via the local structure theorem, be reduced to the partial toric compactification theory of \( \hat{A}_\mathcal{Z} \). Let us be more precise and denote by \( \Xi \) the character group of \( \hat{A}_\mathcal{Z} \). Note that \( \Xi \cong \mathbb{Z}^n \) with \( n = \dim \hat{A}_\mathcal{Z} \). If we denote by \( \mathcal{N} \) the co-character group of \( \hat{A}_\mathcal{Z} \), then there is a natural identification of \( \mathfrak{a}_\mathcal{Z} \) with \( \mathcal{N} \otimes \mathbb{Z} \mathbb{R} \). Further the compression cone \( \mathfrak{a}_\mathcal{Z}^{-} \) identifies as a co-simplicial cone (in [11] one uses the rational valuation cone, denoted \( \mathcal{Z}_k(X) \): take \( k = \mathbb{R} \) and \( X = \mathbb{Z} \). Then \( \mathfrak{a}_\mathcal{Z}^{-} = \mathbb{R} \otimes \mathbb{Q} \mathcal{Z}_k(X) \)). The set of spherical roots \( S \subset \Xi \) are then the primitive extremal elements co-spanning \( \mathfrak{a}_\mathcal{Z}^{-} \). Best possible compactifications (a.k.a. wonderful compactifications) exists when \( \# S = \dim \mathfrak{a}_\mathcal{Z} \) and \( S \) is a basis of the lattice \( \Xi \). In general this is not satisfied and we proceed as follows: We choose a complete fan \( \mathcal{F} \subset \mathfrak{a}_\mathcal{Z} \) supported in \( \mathfrak{a}_\mathcal{Z}^{-} \) which is generated by simple simplicial cones \( C_1, \ldots, C_N \), i.e.

- \( \bigcup C_i = \mathfrak{a}_\mathcal{Z}^{-} \),
- \( C_i \cap C_j \) is a face of both \( C_i \) and \( C_j \) for all \( 1 \leq i, j \leq N \),
- \( C_i = \{ X \in \mathfrak{a}_\mathcal{Z} | d\psi_{ij}(X) \leq 0, 0 \leq j \leq n \} \) for \( (\psi_{ij})_{1 \leq i, j \leq n} \) a basis of the lattice \( \Xi \).

For the existence of such a subdivision we refer to [7, Chapter III]. Now attached to the fan \( \mathcal{F} \) we construct the toric variety \( \hat{A}_\mathcal{Z}(\mathcal{F}) \) expanding \( \hat{A}_\mathcal{Z} \) along \( \mathcal{F} \). Note that the toric variety \( \hat{A}_\mathcal{Z}(\mathcal{F}) \) is smooth as the fan consists of simple cones (third bulleted property). Thus we obtain a smooth variety

\[
\mathcal{Z}_0(\mathcal{F}) := U \times \left[ \left[ \mathbb{M}/\mathbb{M}_H \right] \times_{\mathbb{M}} \hat{A}_\mathcal{Z}(\mathcal{F}) \right]
\]

which inflates to a smooth \( G \)-variety \( \mathcal{Z}(\mathcal{F}) := G \cdot \mathcal{Z}_0(\mathcal{F}) \) containing \( \mathcal{Z}_0(\mathcal{F}) \) as an open subset. This is content of [11, Theorem 7.1]. Now set \( \hat{\mathcal{Z}}(\mathbb{R}) := \mathcal{Z}(\mathcal{F})(\mathbb{R}) \) and note that \( \hat{\mathcal{Z}}(\mathbb{R}) \) is compact by [11, Corollary 7.12] as \( \mathcal{F} \) was assumed to be complete.

We now claim that the limit \( \lim_{s \to \infty} (\exp(sX) \cdot z_0) \) exists in \( \hat{A}_\mathcal{Z}(\mathcal{F})(\mathbb{R}) \) and that the convergence is rapid. For that we pick a cone \( C_i \) which contains \( X \) and let \( \mathcal{F}_i \) be the complete fan supported in \( C_i \) which is generated by \( C_i \). Notice that \( \hat{A}_\mathcal{Z}(\mathcal{F}_i)(\mathbb{C}) \cong \mathbb{C}^n \) is open in \( \hat{A}_\mathcal{Z}(\mathcal{F})(\mathbb{C}) \). More specifically, the embedding of \( \hat{A}_\mathcal{Z}(\mathbb{C}) \hookrightarrow \mathbb{C}^n \) is obtained by

\[
\hat{A}_\mathcal{Z}(\mathbb{C}) \ni a \mapsto (\psi_{ij}(a))_{1 \leq i, j \leq n} \in (\mathbb{C}^*)^n \subset \mathbb{C}^n.
\]

Given the definition of \( C_i \) as the negative dual cone to the \( \psi_{ij} \)’s, \( j = 1, \ldots, n \), the claim now follows.

Note that the stabilizer of \( z_s := \exp(sX) \cdot z_0 \) in \( G \) is given by \( H_s := \exp(sX)H \exp(-sX) \) and has Lie algebra \( \mathfrak{h}_s := e^{s\mathfrak{ad}X} \mathfrak{h} \). Since \( z_s \to z_X \) in the smooth manifold \( \hat{\mathcal{Z}}(\mathbb{R}) \) we obtain that the vector fields corresponding to elements of \( \lim_{s \to \infty} \mathfrak{h}_s = \mathfrak{h}_X \) vanish at \( z_X \). This shows that \( \mathfrak{h}_I \subset \mathfrak{h}_X \). Finally the property \( \mathfrak{h}_X \subset \mathfrak{h}_I + \mathfrak{a}_I \) is derived from [11, Theorem 7.3].

We end this Subsection with further remarks and explanations of the construction in the proof above.
2.4 Remark. (a) It is quite instructive to consider the special case of $Z = G = A$. Here $A_Z = A_Z = A = A_{Z,E}$ with $I = \emptyset$. Upon identifying $a_Z$ with $\mathbb{R}^n$ via the character lattice $\Xi$ there are two standard choices for the complete fan $\mathcal{F}$ generated by the cones $C_1, \ldots, C_N$. The first one is for $N = 2^n$ and the cones given by the orhtants: $C_\sigma = \sigma \mathbb{R}_+^n$ for $\sigma \in \{-1, 1\}^n$. This fan leads to $A_Z(\mathcal{F})(\mathbb{R}) \simeq \mathbb{P}^1(\mathbb{R})^n$, the $n$-fold copy of the projective line. The other standard choice is obtained via identifying $\mathbb{R}^n = \mathbb{R}^{n+1}/\mathbb{R}e$ with $e = e_1 + \ldots + e_{n+1}$, where $(e_1, \ldots, e_{n+1})$ is the canonical basis of $\mathbb{R}^{n+1}$, and has $N = n + 1$ cones given by

$$C_i = \left\{ \bigoplus_{j=1 \text{ s.t. } j \neq i}^{n+1} \mathbb{R}_{\geq 0} e_i + \mathbb{R}e \right\}, \quad 1 \leq i \leq n + 1.$$ 

This fan leads to the projective space $A_Z(\mathcal{F})(\mathbb{R}) \simeq \mathbb{P}^n(\mathbb{R})$.

(b) In the previous example we have seen that there are exactly $N$ fixed points for $G$ in the compactification $\hat{Z}(\mathbb{R})$ parametrized by the cones $C_i$ and explicitly given by limits $\hat{z}_{G,i} := \lim_{t \to \infty} (\exp(tX) \cdot z_0)$ for some $X \in \text{int} C_i$. This feature is not limited to this specific example but general: the compactification $\hat{Z}(\mathbb{R})$ features exactly $N$ closed $G(\mathbb{R})$-orbits through the various $\hat{z}_{G,i}$‘s. This is in contrast to wonderful compactifications where one has exactly one closed orbit. For wonderful compactifications one has $a_{Z,E} = \{0\}$ and $S$ is a basis of the lattice $\Xi$. If one of these two conditions fails one is in need of a further subdivision of $a_{\Xi}$ into simple simplicial cones $C_i$.

(c) If $a_{Z,E} = \{0\}$, then $\hat{z}_X = : \hat{z}_I$ does not depend on the choice of $X \in a_I^{-}$. Further $h_I = \hat{h}_I := h_I + a_I$. In general we only have $h_X = \hat{h}_I$ generically as the example in (a) shows. To be more precise, let $F$ be the smallest face of the fan which contains $X$. Then $\text{span}_\mathbb{R} F \subseteq a_I$ and $h_X = h_I + \text{span}_\mathbb{R} F$.

(d) (cf. [11, Section 11]) In case $H = N_G(H)$ is self-normalizing, one obtains a wonderful compactification $\hat{Z}(\mathbb{R})$ by closing up $Z(\mathbb{R})$ in the Grassmanian $\text{Gr}_d(\mathfrak{g})$ of $d = \dim \mathfrak{h}$-dimensional subspaces of $\mathfrak{g}$. The embedding is given by $g \cdot z_0 \mapsto \text{Ad}(g)h$ and, given the definition of $h_I$ as a limit (cf. (2.1)), one derives easily that the stabilizer $\hat{H}_I$ of $\hat{z}_I$ in $G$ has Lie algebra $\hat{h}_I = h_I + a_I$.

2.4 Proof of Proposition 2.1

We choose a smooth compactification $\hat{Z}(\mathbb{R}) = Z(\mathcal{F})(\mathbb{R})$ as constructed in the previous section. To begin with we note that the limit

$$\hat{z}_I := \lim_{s \to \infty} \hat{a}_s \cdot z_0$$

exists. Moreover $\hat{z}_I \in A_Z(\mathcal{F})(\mathbb{R})$ and the convergence is rapid. Further we deduce from the fact that $\hat{z}_I$ is fixed by $H_I(\mathbb{C})$ and $w_I = \hat{t}_I h_I$ that $\lim_{s \to \infty} w_I \hat{a}_s \cdot z_0 = \hat{t}_I \cdot \hat{z}_I \in A_Z(\mathcal{F})(\mathbb{R})$ is rapid. On the other hand $w_I \hat{a}_s \cdot z_0 = U_m \hat{b}_s \cdot z_0 = U_m \hat{b}_s \cdot z_0$ which in local coordinates as given by (2.9) translates into:

$$w_I \hat{a}_s \cdot z_0 = (U_m, [m_s, \hat{b}_s \cdot z_0]) \in U \times \left[ [M/M_H] \times F_M A_Z(\mathcal{F})(\mathbb{R}) \right].$$
Since $\lim_{s \to \infty} w_l \tilde{a}_s \cdot z_0 = (1, [1, \tilde{t} \cdot \hat{z}_t])$ is rapid, we thus deduce that $\lim_{s \to \infty} u_s = 1$ is rapid as well. Next we use the smooth projection $[M/M_H] \times F_M \xrightarrow{A_Z(F)(\mathbb{R})} M/M_H F_M$ and obtain that $m_s(M_H F_M) \xrightarrow{s \to \infty \text{ rapid}} 1(M_H F_M) \in M/M_H F_M$. In particular, we may assume that $m_s \xrightarrow{s \to \infty \text{ rapid}} m_{w_1} \in M_H F_M$. Notice that we are free to replace $m_s$ by elements of the form $m_s m_H$ with $m_H \in M_H$ as we have

$$m_s m_H \tilde{b}_s w \cdot z_0 = m_s m_H \tilde{b}_s \tilde{t} \cdot z_0 = m_s \tilde{b}_s \tilde{t} \cdot z_0 = m_s \tilde{b}_s w \cdot z_0.$$

Thus we may even assume that $m := m_{w_1} \in \tilde{F}_M$ (which was defined just before Proposition 2.1.)

We remain with showing $b_s a_s^{-1} \xrightarrow{s \to \infty \text{ rapid}} 1$. Using the techniques from above it is immediate that $d(a_s, b_s) \to 0$ rapidly for any Riemannian metric $d$ on $\tilde{Z}(\mathbb{R})$. However, the statement $a_s^{-1} b_s \to 1$ rapidly is a considerably finer assertion and difficult to obtain working with only one compactification. Thus we change the strategy of proof and work with (varying) finite dimensional spherical representations instead. The representations gives us various morphisms of $\mathbb{Z}$ into affine spaces.

We assume first that $\mathbb{Z}$ is quasi-affine. The representations we work with are finite dimensional irreducible representations $(\pi, V)$ of $\mathbb{G}(\mathbb{C})$ featuring two properties:

- The representation is $H(\mathbb{C})$-spherical, that is, there exists a vector $v_H \neq 0$ such that $\pi(h) v_H = v_H$ for all $h \in H(\mathbb{C})$.
- Each $N(\mathbb{C})$-fixed vector is fixed by $M(\mathbb{C})$.

The second property can be rephrased that the representation is $K(\mathbb{C})$-spherical (Cartan-Helgason theorem). In particular, these representations are self-dual, the highest weight $\lambda$ is an element of $\mathfrak{a}^*$ and the lowest weight is given by $-\lambda$. We write $\Lambda_Z$ for the set of highest weights of all $H(\mathbb{C})$ and $K(\mathbb{C})$-spherical irreducible representations.

Given $\lambda \in \Lambda_Z$ we let $(\pi, V)$ be an irreducible representation of $\mathbb{G}(\mathbb{C})$ of highest weight $\lambda$. Further we fix a highest weight vector $v^*$ in the dual representation of $V^*$ of $V$. From the fact that $PH$ is open in $G$ we then deduce $v^*(v_H) \neq 0$ and in particular $V^H = \mathbb{C} v_H$ is one-dimensional. Moreover it follows that $\Lambda_Z \subset \mathfrak{a}_Z^*$.

We expand $v_H$ into $a$-weight vectors

$$v_H = \sum_{\mu \in \Lambda_\pi} v_{-\lambda + \mu},$$

with $\Lambda_\pi := \{ \mu \in \mathfrak{a}^* : v_{-\lambda + \mu} \neq 0 \}$. As $v_H$ is $a_H$ fixed we have $\Lambda_\pi \subset \mathfrak{a}_Z^*$ and by [12, Lemma 5.3] we obtain:

$$\mu |_{a_Z} < 0, \quad \mu \in \Lambda_\pi \setminus \{0\} . \quad (2.11)$$

Set

$$v_{H, s} := a_s^\lambda \pi(\tilde{a}_s) v_H \quad s \geq 0.$$
and note, as \( v_H \) is \( H \)-invariant, that this expression is independent of the choice of the particular section \( s \). From the definition we then get

\[
v_{H,s} = \sum_{\mu \in \Lambda_\pi} a^\mu v_{-\lambda + \mu} .
\] (2.12)

If we define

\[
v_{H,I} := \sum_{\mu \in \Lambda_\pi \text{ s.t. } \mu(X_I) = 0} v_{-\lambda + \mu} ,
\]

then it is immediate from (2.11) and (2.12) that \( v_{H,s} \to v_{H,I} \) rapidly for \( s \to \infty \).

Recall \( v^* \in V^* \), a highest weight vector in the dual representation. Then we obtain from

\[
w_I \hat{a}_s \cdot z_0 = u_s m_s \tilde{b}_s t' \cdot z_0
\]

that

\[
v^*(\pi(w_I)v_{H,s}) = a_s^\lambda \left( v^*(\pi(u_s m_s \tilde{b}_s t)v_H) \right) = (a_s b_s^{-1})^\lambda t^{-\lambda} (v^*(v_H))
\] (2.14)

By (2.13) we thus obtain from (2.14) that

\[
(a_s b_s^{-1})^\lambda = t^\lambda \frac{v^*(\pi(w_I)v_{H,s})}{v^*(v_H)} \to t^\lambda \frac{v^*(\pi(w_I)v_{H,I})}{v^*(v_H)} \quad \text{rapidly for } s \to \infty .
\] (2.15)

We now employ [15, Lemma 3.10] for the simple convergence \( a_s b_s^{-1} \to 1 \). Thus (2.15) implies

\[
t^\lambda \frac{v^*(\pi(w_I)v_{H,I})}{v^*(v_H)} = 1 \quad \text{with}
\]

\[
(a_s b_s^{-1})^\lambda \to 1 \quad \text{rapidly for } s \to \infty , \lambda \in \Lambda_Z .
\] (2.16)

In case \( Z \) is quasi-affine, the set \( \Lambda_Z \) spans \( a^*_Z \) as a consequence of [14, Lemma 3.4 and (3.2)] and we get \( a_s b_s^{-1} \to 1 \) rapidly from (2.16).

If \( Z \) is not quasi-affine, then matters are reduced via the so-called cone-construction from algebraic geometry. We extend \( G(\mathbb{C}) \) to \( G'(\mathbb{C}) := G(\mathbb{C}) \times \mathbb{C}^* \) and for a character \( \psi : H(\mathbb{C}) \to \mathbb{C}^* \) defined over \( \mathbb{R} \) we set \( H'(\mathbb{C}) := \{ (h, \psi(h)) \mid h \in H(\mathbb{C}) \} \).

In this way we obtain a real spherical space \( Z' := G'/H' \) which projects \( G' \)-equivariantly onto \( Z \). According to [11, Corollary 6.10] there is compatibility of compression cones:

\[
a^*_Z = a^-_Z \oplus \mathbb{R} .
\] (2.17)

Further, according to Chevalley’s quasiprojective embedding theorem for homogeneous spaces, we find such a \( \psi \) such that \( Z' \) is quasi-affine and we complete the reduction to the quasi-affine case as follows: we lift the identity (2.6) to \( Z' \) and obtain

\[
w_I \tilde{a}'_s \cdot z'_0 = u_s m_s \tilde{b}'_s w \cdot z'_0 \quad s \geq s_0
\]

with \( \tilde{a}'_s \in \tilde{a}_s(1 \times \mathbb{R}^\times) \in G' \) and likewise for \( \tilde{b}'_s \in \tilde{b}_s(1 \times \mathbb{R}^\times) \in G' \). Because of (2.17) we obtain the rapid convergence \( b'_s(a'_s)^{-1} \to 1 \) in the quasi-affine environment of \( Z' \). Projecting to \( Z \) then completes this final reduction step.
2.5 Remarks about $H_I$ and the relationship between $\mathcal{W}$ and $\mathcal{W}_I$

We end this Section with some explanations on the relationship between $\mathcal{W}$ and $\mathcal{W}_I$ and the geometric meaning of $H_I$. The facts collected below are not needed in any technical aspect of the paper. Proofs of facts mentioned below will appear in the forthcoming work [6].

Pick a generic $X \in a^-_f$ and recall from Remark 2.4(c) that $\tilde{z}_I := \lim_{t \to \infty} \exp(tX) \cdot z_0$ has $G$-stabilizer $\tilde{H}_I$ with Lie algebra $\tilde{h}_I = h + a^-_L$.

Consider now the boundary component $\tilde{Z}_I(\mathbb{R}) = [G \cdot \tilde{z}_I](\mathbb{R}) = (G/\tilde{H}_I)(\mathbb{R})$ which is a submanifold of $\tilde{Z}(\mathbb{R})$. Then we draw our attention to the open $G$-orbit $\tilde{Z}_I := G \cdot \tilde{z}_I \subset \tilde{Z}_I(\mathbb{R})$ and let $\mathcal{N}(\tilde{Z}_I)$ be the corresponding normal bundle. Note that $\mathcal{N}(\tilde{Z}_I) = G \times_{\tilde{H}_I} V_I$ with $V_I := T_{\tilde{z}_I}\tilde{Z}(\mathbb{R})/T_{\tilde{z}_I}\tilde{Z}_I$ the normal fiber at $\tilde{z}_I$. As a consequence of (2.9) we obtain that $V_I \simeq \mathbb{R}^k$ is just the diagonal module for $\mathcal{A}_I(\mathbb{R}) = (\mathbb{R}^k)^k$.

We defined $H_I = (H_I(\mathbb{R}))_0$ to be the connected subgroup of $G$ with Lie algebra $h_I$, but the more correct definition of $H_I$ would be as the $G$-stabilizer of a generic point in $\tilde{Z}_I$. Let us call this stabilizer $H_{I,\text{nb}}$ with "nb" referring to normal bundle. It is then easy to see that $H_I \subset H_{I,\text{nb}}$ is an open subgroup, but typically this inclusion is proper, i.e. $H_{I,\text{nb}}$ is not connected. Notice that there is an open $G$-inclusion of $Z_{I,\text{nb}} = G/H_{I,\text{nb}}$ into $\mathcal{N}(\tilde{Z}_I)$. One can show that $H_{I,\text{nb}}$ is independent of the particular smooth compactification. Likewise it can be shown that the constant terms $f_I$ and related functionals $\eta_I$ (see the introduction) are in fact invariant under $H_{I,\text{nb}}$ and not only under $H_I$.

Next recall that the set $\mathcal{W}(\mathbb{R}) := T_2/F_M$ parametrizes the open $P$-orbits in $Z(\mathbb{R})$. We use the local structure theorem (2.9) near $\tilde{z}_I$ and obtain further that $\hat{\mathcal{W}}_I(\mathbb{R}) := T_2/T_{2,I}F_M$, with $T_{2,I} := T_2 \cap \mathcal{A}_I(\mathbb{R})$, parametrizes the open $P$-orbits in $\tilde{Z}_I(\mathbb{R})$. Hence we obtain a natural quotient map $\mathcal{W}(\mathbb{R}) \to \hat{\mathcal{W}}_I(\mathbb{R})$. If we denote by $\hat{\mathcal{W}}_I$ the set of open $P$-orbits in $\tilde{Z}_I$, we thus obtain a surjective map $\mathcal{W} \to \hat{\mathcal{W}}_I$. The set $\hat{\mathcal{W}}_I$ in turn is closely related to the set of open $P$-orbits in $Z_I$ and $Z_{I,\text{nb}}$ and we refer to [15, Section 3] or the forthcoming work [6] for more details.

3 $Z$-tempered $H$-fixed continuous linear forms and the space $\mathcal{A}_{\text{temp}}(Z)$

3.1 SF-representations of $G$

Let us recall some definitions and results of [2].

A continuous representation $(\pi, E)$ of a Lie group $G$ on a locally convex topological vector space $E$ is a representation such that the map:

$$G \times E \to E, \ (g, v) \mapsto \pi(g)v, \text{ is continuous.}$$

If $R$ is a compact subgroup of $G$ and $v \in E$, we say that $v$ is $R$-finite if $\pi(R)v$ generates a finite dimensional subspace of $E$. Let $V_{(R)}$ denote the vector space of $R$-finite vectors in $E$. 

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Let $\eta$ be a continuous linear form on $E$ and $v \in E$. Let us define the generalized matrix coefficient associated to $\eta$ and $v$ by:

$$m_{\eta,v}(g) := \langle \eta, \pi(g^{-1})v \rangle, \quad g \in G.$$ 

Let $G$ be a real reductive group. Let $\| \cdot \|$ be a norm on $G$ (cf. [21, Section 2.A.2] or [2, Section 2.1.2]). We have the notion of a Fréchet representation with moderate growth. A representation $(\pi, E)$ of $G$ is called a Fréchet representation with moderate growth if it is continuous and if for any continuous semi-norm $p$ on $E$, there exist a continuous semi-norm $q$ on $E$ and $N \in \mathbb{N}$ such that:

$$p(\pi(g)v) \leq q(v)\|g\|^N, \quad v \in E, g \in G. \quad (3.1)$$

This notion coincides with the notion of $F$-representations given in [2, Definition 2.6] for the large scale structure corresponding to the norm $\| \cdot \|$. We will adopt the terminology of $F$-representations.

Let $(\pi, E)$ be an $F$-representation. A smooth vector in $E$ is a vector such that $g \mapsto \pi(g)v$ is smooth from $G$ to $E$. The space $V^\infty$ of smooth vectors in $V$ is endowed with the Sobolev semi-norms that we define now. Fix a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ and $k \in \mathbb{N}$. Let $p$ be a continuous semi-norm on $E$ and set

$$p_k(v) = \left( \sum_{m_1 + \cdots + m_n \leq k} p(\pi(X_1^{m_1} \cdots X_n^{m_n})v)^2 \right)^{1/2}, \quad v \in E^\infty. \quad (3.2)$$

We endow $E^\infty$ with the topology defined by the semi-norms $p_k, k \in \mathbb{N}$, when $p$ varies in the set of continuous semi-norms of $E$, and denote by $(\pi^\infty, E^\infty)$ the corresponding sub-representation of $(\pi, E)$.

An $SF$-representation is an $F$-representation $(\pi, E)$ which is smooth, i.e. such that $E = E^\infty$ as topological vector spaces. Let us remark that if $(\pi, E)$ is an $F$-representation, then $(\pi^\infty, E^\infty)$ is an $SF$-representation (cf. [2, Corollary 2.16]).

Recall our fixed maximal compact subgroup $K \subset G$.

Following [2] we call an $SF$-representation $E$ admissible provided that $E_{(K)}$ is a Harish-Chandra module with respect to the pair $(\mathfrak{g}, K)$, that is a $(\mathfrak{g}, K)$-module with finite $K$-multiplicities which is finitely generated as a $\mathcal{U}(\mathfrak{g})$-module.

An admissible $SF$-representation will be called an $SAF$-representation of $G$.

It is a fundamental theorem of Casselman-Wallach (cf. [5], [22, Chapter 11] or [2]) that every Harish-Chandra module $V$ admits a unique $SF$-completion $V^\infty$, i.e. an $SF$-representation $V^\infty$ of $G$, unique up to isomorphism in the $SF$-category, such that

$$V^\infty_{(K)} \simeq_{(\mathfrak{g}, K)} V.$$

In particular, all $SAF$-representations of $G$ are of the form $V^\infty$ for a Harish-Chandra module $V$. 

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3.2 The spaces $C_{\text{temp},N}^\alpha(Z)$ and $A_{\text{temp},N}(Z)$

In the remaining of Section 3, we will assume that $Z$ is unimodular. Let $\rho_Q$ be the half sum of the roots of $a$ in $u$. Let us show that:

$$\rho_Q$$

is trivial on $a_H$.

As $I \cap h$-modules,

$$g/h = u \oplus (I/I \cap h).$$

But the action of $a_H = a \cap h$ on $(I/I \cap h)$ is trivial. Since $Z$ is unimodular, the action of $a_H$ has to be unimodular. Our claim follows.

Hence $\rho_Q$ can be defined as a linear form on $a_Z$.

We have the notion of weight functions on an homogeneous space $X$ of a locally compact group $G$ (cf. [1, Section 3.1]). This is a function $w : X \rightarrow \mathbb{R}_{>0}$ such that, for every ball $B$ of $G$ (i.e. a compact symmetric neighborhood of 1 in $G$), there exists a constant $c = c(w, B)$ such that:

$$w(g \cdot x) \leq cw(x), \quad g \in B, x \in X. \quad (3.3)$$

One sees easily that if $w$ is a weight function, then $1/w$ is also a weight function.

Let $v$ and $w$ be the weight functions on $Z$ defined by

$$v(z) = \text{vol}_Z(Bz)$$

$$w(z) = \sup_{a \in A_Z \text{ s.t. } z \in \Omega_{\alpha \omega}W \cdot z_0} \| \log(a) \|,$$

where $B$ is some ball of $G$. It is then clear that $v$ is a weight function and $w$ is a weight function by [13, Proposition 3.4]. Recall that the equivalence class of $v$ does not depend on $B$ (see loc.cit. Lemma 4.1 and beginning of Section 3 for the definition of the equivalence relation).

For any $N \in \mathbb{N}$, we define a norm $p_N$ on $C_c(Z)$ by:

$$p_N(f) = \sup_{z \in Z} \left( (1 + w(z))^{-N} v(z)^{1/2} |f(z)| \right). \quad (3.4)$$

From the polar decomposition of $Z$ (cf. (??)), one has:

$$p_N(f) = \sup_{\omega \in \Omega, a \in A_Z, \omega \in W} \left( (1 + w(\omega a \cdot z_0))^{-N} v(\omega a \cdot z_0)^{1/2} |f(\omega a \cdot z_0)| \right).$$

From the fact that $v$ and $w$ are weight functions on $Z$ and from [13, Propositions 3.4(2) and 4.3], one then sees that:

The norm $p_N$ is equivalent to the norm:

$$f \mapsto q_N(f) := \sup_{\omega \in \Omega, \omega \in W, \omega \in A_Z} \left( a^{-p_\alpha} (1 + \| \log(a) \|)^{-N} |f(\omega a)| \right). \quad (3.5)$$
Moreover, due to the fact that $v$ and $1/w$ are weight functions on $Z$, one gets that $G$ acts by left translations on $(C_c(Z), p_N)$, and, for any compact subset $C$ of $G$, by changing $z$ into $z' = g^{-1} \cdot z$ in (3.4), one sees that:

There exists $c > 0$ such that:

$$p_N(L_g f) \leq cp_N(f), \quad g \in C, \ f \in C_c(Z).$$

This is in essence what is needed to identify $C^\infty_{\text{temp},N}(Z) := \{ f \in C^\infty(Z) : p_{N,k}(f) < \infty, \ k \in \mathbb{N} \}$

as an $SF$-module for $G$. Here the $p_{N,k}$, $k \in \mathbb{N}$ are as in (3.2) with $p$ replaced by $p_N$ and $(\pi, E)$ by the $SF$-representation $(L, C^\infty_{\text{temp},N}(Z))$. Further we endow the increasing union $C^\infty_{\text{temp}}(Z) := \bigcup_{N \in \mathbb{N}} C^\infty_{\text{temp},N}(Z)$ with the inductive limit topology. We call $C^\infty_{\text{temp}}(Z)$ the space of smooth tempered functions on $Z$.

Inside of $C^\infty_{\text{temp}}(Z)$ we define $A_{\text{temp}}(Z)$ as the subspace of $Z(\mathfrak{g})$-finite functions. Likewise we define $A_{\text{temp},N}(Z)$.

### 3.3 Z-tempered functionals

Let $(\pi, E)$ be an $SF$-representation and $E'$ its strong dual. An element $\eta \in (E')^H$ will be called $Z$-tempered provided

$$\text{There exists } N \in \mathbb{N} \text{ such that, for all } v \in E, \text{ one has } m_{\eta,v} \in C^\infty_{\text{temp},N}(Z).$$

The $Z$-tempered functionals then define a subspace $(E')^H_{\text{temp}}$ of $(E')^H$. Frobenius reciprocity then asserts for an $SF$-representation $(\pi, E)$ that

$$\text{Hom}(E, C^\infty_{\text{temp}}(Z)) \simeq (E')^H_{\text{temp}}$$

which can be established as in [19, Lemma 6.5] via the Grothendieck-factorization theorem for topological vector spaces.

In case $E = V^\infty$ is an $SAF$-representation we adopt to the more common terminology $V^{-\infty} := (V^\infty)'$ and recall the finiteness result for real spherical spaces (cf. [18, Theorem 3.2]):

$$\dim (V^{-\infty})^H < \infty.$$  \hspace{1cm} (3.10)

In this context we record:

**3.1 Proposition.** Let $f \in A_{\text{temp},N}(Z)$ be a $K$-finite element. Set $E^f := \text{span}_k L(G) f$ with the closure taken in $C^\infty_{\text{temp},N}(Z)$. Then $E^f$ is an $SAF$-representation, i.e. $E^f_{(K)}$ is a Harish-Chandra module.
Proof. We consider the \((\mathfrak{g}, K)\)-module \(V^f := \mathcal{U}(\mathfrak{g}) f\). Since \(f\) was \(\mathcal{Z}(\mathfrak{g})\)-finite, the same holds for \(V^f\). Now, as a finitely generated and \(\mathcal{Z}(\mathfrak{g})\)-finite module, \(V^f\) is a Harish-Chandra module by a theorem of Harish-Chandra. Standard techniques (see [2]) then show that \(V^f\) is dense in \(E^f\) and thus \(V^f \cong_{(\mathfrak{g}, K)} E^f_{(K)}\).

3.2 Remark. We are inclined to believe that \(E^f\) is an SAF-representation for any \(f \in \mathcal{A}_{\text{temp}, N}(Z)\) without the assumption of \(f\) being \(K\)-finite. This should be related to the dimension bound (3.10).

Since we were not able to show that \(E^f\) is an SAF-representation for general \(f \in \mathcal{A}_{\text{temp}, N}(Z)\) we adopt a slightly weaker terminology which is equally suitable for this paper.

An SF-representation will be called an SFF-representation provided it is annihilated by an ideal \(\mathcal{I} \subset \mathcal{Z}(\mathfrak{g})\) of finite co-dimension. Then it is clear that \(E^f\) is an SFF-representation for any \(f \in \mathcal{A}_{\text{temp}, N}(Z)\). Likewise we obtain an SFF-representation

\[ \mathcal{A}_{\text{temp}, N}(Z : \mathcal{I}) := \{ f \in \mathcal{A}_{\text{temp}, N}(Z) \mid f \text{ is annihilated by } \mathcal{I} \} \]

and we denote by \(\mathcal{A}_{\text{temp}}(Z : \mathcal{I})\) the subspace of \(\mathcal{A}_{\text{temp}}(Z)\) annihilated by \(\mathcal{I}\).

4 Differential equation for some functions on \(Z\) wave-front and unimodular

Recall the basic notions about boundary degenerations related to subsets \(I \subset S\) of spherical roots. If \(X \in \mathfrak{a}_I^\prime\), we define

\[ \beta_I(X) = \max_{\alpha \in S \setminus I} \alpha(X) < 0 \tag{4.1} \]

and, if \(a \in \mathfrak{a}_I^\prime\) with \(a = \exp X\), we set \(a^{\beta_I} = e^{\beta_I(X)}\).

Let us fix \(I \subset S\) throughout this section.

4.1 Some estimates

4.1 Lemma. Let \(Y \in \mathfrak{h}_I\) and \(N \in \mathbb{N}\). There exists a continuous semi-norm on \(C_{\text{temp}, N}^\infty(Z)\), \(p\), such that

\[ |(L_Y f)(a)| \leq p(f) a^{\theta q + \beta_I} (1 + \| \log a \|)^N, \quad a \in \mathfrak{a}_I^\prime, f \in C_{\text{temp}, N}^\infty(Z). \]

Proof. On one hand, if \(Y \in \mathfrak{I} \cap \mathfrak{h}\),

\[ (L_Y f)(a) = 0, \quad a \in \mathfrak{I}. \]

Hence the conclusion of the Lemma holds for \(Y \in \mathfrak{I} \cap \mathfrak{h}\).
On the other hand, from the definition of $T_I$ (cf. beginning of Section 2.1), $I \cap \mathfrak{h}$ and the elements
\[ Y_{-\alpha} = X_{-\alpha} + T_I(X_{-\alpha}) \in \mathfrak{h}_I, \]
for $\alpha$ varying in $\Sigma_\mathfrak{g}$ and $X_{-\alpha}$ in $\mathfrak{g}^{-\alpha}$, generate $\mathfrak{h}_I$ as a vector space. By linearity, it then remains to get the result for $Y = Y_{-\alpha}$.

Let $a \in A_I$ and $\bar{a} = s(a)$ (cf. (1.4) for the definition of $s$). Then let us show that:
\[ \text{Ad}(\bar{a})Y_{-\alpha} = \bar{a}^{-\alpha}Y_{-\alpha}. \]
One has $\text{Ad}(\bar{a})X_{-\alpha} = \bar{a}^{-\alpha}X_{-\alpha}$ and $\text{Ad}(\bar{a})X_{\alpha,\beta} = \bar{a}^\beta X_{\alpha,\beta}$. But $\alpha + \beta \in I$. Hence $\bar{a}^{\alpha+\beta} = 1$, as $a \in A_I$. Our claim follows.

Let us get the statement for $(L_{Y_{-\alpha}}f)(a) \in A_I^{--}$ and $f \in \mathcal{A}_{\text{temp},N}(Z)$. One has:
\[ (L_{Y_{-\alpha}}f)(a) = (L_{\bar{a}^{-1}}(L_{Y_{-\alpha}}f))(z_0) = \bar{a}^{\alpha}(L_{Y_{-\alpha}}L_{\bar{a}^{-1}}f)(z_0). \]

Recall that $\mathcal{M}$ is the monoid in $\mathbb{N}_0[\Sigma_\mathfrak{g}]$ defined in (1.9) and $\langle I \rangle$ denotes the monoid in $\mathbb{N}_0[S]$ generated by $I$. Let us notice that:
\[ Y_{-\alpha} + \sum_{\beta \in \Sigma_\mathfrak{g} \cup \{0\} \text{ s.t. } \alpha + \beta \notin \langle I \rangle} X_{\alpha,\beta} \in \mathfrak{h}. \]

Hence one has:
\[ (L_{Y_{-\alpha}}f)(a) = -\bar{a}^{\alpha} \sum_{\beta \in \Sigma_\mathfrak{g} \cup \{0\} \text{ s.t. } \alpha + \beta \notin \langle I \rangle} (L_{X_{\alpha,\beta}}L_{\bar{a}^{-1}}f)(z_0) = -\sum_{\beta \in \Sigma_\mathfrak{g} \cup \{0\} \text{ s.t. } \alpha + \beta \notin \langle I \rangle} \bar{a}^{\alpha+\beta}(L_{\bar{a}^{-1}}L_{X_{\alpha,\beta}}f)(z_0). \]

But $\bar{a}^{\alpha+\beta} = a^{\alpha+\beta}$ as $a \in A_I \subset A_Z$ and $\alpha + \beta \in S$. Then, as $(L_{\bar{a}^{-1}}L_{X_{\alpha,\beta}}f)(z_0) = L_{X_{\alpha,\beta}}f(a)$, one has:
\[ (L_{Y_{-\alpha}}f)(a) = -\sum_{\beta \in \Sigma_\mathfrak{g} \cup \{0\} \text{ s.t. } \alpha + \beta \notin \langle I \rangle} a^{\alpha+\beta}(L_{X_{\alpha,\beta}}f)(a). \tag{4.2} \]

If $\alpha + \beta \notin \langle I \rangle$ as above and $L_{X_{\alpha,\beta}}f \neq 0$, one has $\alpha + \beta \in \mathcal{M}\setminus \langle I \rangle$ and, from the definition of $\beta_I$ (cf. (4.1)):
\[ a^{\alpha+\beta} \leq a^{\beta_I}, \quad a \in A_I^{--}. \]

Then
\[ |(L_{Y_{-\alpha}}f)(a)| \leq a^{\beta_I} \sum_{\beta \in \Sigma_\mathfrak{g} \cup \{0\} \text{ s.t. } \alpha + \beta \notin \langle I \rangle} |(L_{X_{\alpha,\beta}}f)(a)|. \]

Hence we get the inequality of the Lemma for $Y = Y_{-\alpha}$ by taking
\[ p = \sum_{\beta \in \Sigma_\mathfrak{g} \cup \{0\} \text{ s.t. } \alpha + \beta \notin \langle I \rangle} p_{X_{\alpha,\beta},N}. \]
Let us recall (cf. e.g. [15, Section 5.1]) that $Z$ is said wavefront if

$$a_Z^ \perp = (a^- + a_H)/a_H.$$  

We will now make the following hypothesis on $Z$:

**Let us assume from now, unless specified, that $Z$ is wavefront and unimodular.**

$(\mathcal{H})$

Recall that wavefront spaces are defined by the property that the closed Weyl chamber $A^-$ maps onto the compression cone $A^\perp_Z$ under the natural quotient map $A_0 \to A_Z \simeq A_0/(A_H)_0$. This implies in particular that $A^\perp_Z$ acts by contraction on all unipotent radicals of parabolics containing $P$. This property will be used repeatedly in the sequel.

Let $F_Q$ be the subset of the set of simple roots $\Pi$ of $a$ in $n$, such that $Q$ is the parabolic subgroup of $G$ corresponding to the roots $\Sigma_n \setminus \langle F_Q \rangle$. Let us recall some results of [15, Corollary 5.6]. As $Z$ is wavefront, there exists a minimal set $F_I \subset \Pi$ which contains $F_Q$ and such that:

$$\langle F_I \rangle \cap \mathbb{N}_0[S] = \langle I \rangle.$$  

Moreover, if $Q_I$ denotes the parabolic subgroup of $G$ containing $Q$ and corresponding to the roots $\Sigma_n \setminus \langle F_I \rangle$, and $Q_I = L_I \dot{U}_I$ is its Levi decomposition with $A \subset L_I$, one has:

$$(L_I \cap H)_0 U^{-}_I \subset H_I \subset Q^{-}_I,$$

where $Q^{-}_I$ is the parabolic subgroup of $G$ opposite to $Q_I$ containing $A$. Let us denote by $u_I^{-}$ the nilradical of the parabolic subalgebra $q_I^{-}$.

**4.2 Lemma.** Let $X \in u_I^{-}$ and $u \in U(g)$. There exists a continuous semi-norm on $C_{\text{temp},N}^\infty(Z)$, $q$, such that, for all $f \in C_{\text{temp},N}^\infty(Z)$,

$$|(L_X L_u f)(a_Z a_I)| \leq q(f)(a_Z a_I)^{\rho_{\alpha} a_I^\beta I} (1 + \| \log a_Z \|) (1 + \| \log a_I \|)^N,$$

$$a_Z \in A^{-}_Z, a_I \in A^{-}_I.$$  

**Proof.** As $L_u$ is a continuous operator on $C_{\text{temp},N}^\infty(Z)$, it is enough to prove the Lemma for $u = 1$. By linearity, we can assume that $X = X_{-\alpha}$ is a weight vector in $a$ for the weight $-\alpha$, where $\alpha$ is a root of $a$ in $u_I$.

As $X_{-\alpha} \in \mathfrak{h}_I$, $T_I(X_{-\alpha}) = 0$ and $Y_{-\alpha} = X_{-\alpha}$. In particular, $\text{Ad}(\tilde{a})Y_{-\alpha} = \tilde{a}^{-\alpha}Y_{-\alpha}$ for $a \in A_Z$ (recall that in the proof of Lemma 4.1, this equality was true only for $a \in A_I$). Hence (4.2) is true for $a \in A_Z$ and:

$$(L_{Y_{-\alpha}} f)(a) = \sum_{\beta \in \Sigma_{\alpha \cup \langle I \rangle}} a^{\alpha + \beta} (L_{X_{\alpha}, \beta} f)(a), \quad a \in A_Z.$$  

Let us assume $a = a_Z a_I$ with $a_Z \in A^{-}_Z, a_I \in A^{-}_I$. Then, as $a_Z^{\alpha + \beta} \leq 1$, and as $a_I \in A^{-}_I$ implies $a_I^{\alpha + \beta} \leq a_I^{\beta I}$, by definition of $\beta_I$ (cf. (4.1)), one gets $a^{\alpha + \beta} \leq a_I^{\beta I}$. Moreover, as elements
of $\mathcal{U}(\mathfrak{g})$ act continuously on $C^\infty_{\text{temp},N}(Z)$, there exists a continuous semi-norm $p$ on $C^\infty_{\text{temp},N}(Z)$ such that, for all $\beta \in \Sigma_u \cup \{0\}$,

$$|(L_{X_\alpha,\beta}f)(a_Za_I)| \leq p(f)(a_Za_I)^{\rho_q}(1 + \|\log a_Z\|)^N(1 + \|\log a_I\|)^N, \quad f \in C^\infty_{\text{temp},N}(Z).$$

To get this inequality, we have used that:

$$\|\log(a_Za_I)\| \leq \|\log a_Z\| + \|\log a_I\|.$$ 

The Lemma follows. \qed

### 4.2 Algebraic preliminaries

Let $A_{L_I}$ be the maximal abelian subgroup of the center of the Levi subgroup $L_I$ of $Q_I$ contained in $A$. Then (cf. [15, Corollary 5.6])

$$\mathfrak{a}_{L_I}/\mathfrak{a}_{L_I} \cap \mathfrak{a}_H \simeq \mathfrak{a}_I \subset \mathfrak{a}_Z.$$ 

Let $\mathfrak{c}_I$ be the center of $I_I$ and $^0I_I = [I_I, I_I] + \mathfrak{c}_I \cap \mathfrak{k}$. One has:

$$I_I = ^0I_I \oplus \mathfrak{a}_{L_I}.$$ 

Let $pr_I$ be the projection of $I_I$ on $\mathfrak{a}_{L_I}$ parallel to $^0I_I$. Let $\rho_{Q_I}$ denote the half sum of the roots in $\Sigma_n\langle F_I \rangle$, i.e. the roots of $\mathfrak{a}$ in $u_I$. From [15, equation (3.12)] and the fact that $\mathfrak{a}_{L_I} \subset \mathfrak{a}$, one has $\mathfrak{a}_{L_I} \cap \mathfrak{h}_I = \mathfrak{a}_{L_I} \cap \mathfrak{h}$. Let us show that:

$$\rho_{Q_I} \text{ is trivial on } \mathfrak{a}_{L_I} \cap \mathfrak{h}_I. \quad (4.3)$$ 

From [15, Lemma 3.12], $Z_I$ is also unimodular and, as (left) $I_I \cap \mathfrak{h}_I$-modules,

$$\mathfrak{g}/\mathfrak{h}_I = u_I \oplus (I_I/I_I \cap \mathfrak{h}_I).$$ 

In fact, the action of $\mathfrak{a}_{L_I} \cap \mathfrak{h}_I$ on $I_I/I_I \cap \mathfrak{h}_I$ is trivial. Hence the action of $\mathfrak{a}_{L_I} \cap \mathfrak{h}_I$ on $u_I$ has to be unimodular. Our claim follows. Let us define a function $d_{Q_I}$ on $L_I$ by:

$$d_{Q_I}(l) = (\det(\text{Ad} l|_{u_I}))^{1/2}, \quad l \in L_I.$$ 

In particular

$$d_{Q_I}(a) = a^{\rho_{Q_I}}, \quad a \in A_{L_I}.$$ 

Let us notice that, from (4.3), one sees that:

$$d_{Q_I} \text{ is trivial on } A_{L_I} \cap A_H. \quad (4.4)$$

We define an automorphism of $\mathcal{U}(I_I)$:

$$\sigma_I : \mathcal{U}(I_I) \to \mathcal{U}(I_I)$$
such that:
\[ L_{\sigma_I(X)} = d_{Q_I}^{-1} \circ L_X \circ d_{Q_I}, \quad X \in \mathfrak{I}, \]  
(4.5)
i.e. \( \sigma_I(X) = X - \rho_{Q_I}(p r_I(X)), \) \( X \in \mathfrak{I}. \) If \( v \in \mathcal{U}(\mathfrak{I}), \) we will also denote
\[ v' := \sigma_I^{-1}(v). \]  
(4.6)
Hence
\[ L_{v'} = d_{Q_I} \circ L_v \circ d_{Q_I}^{-1}. \]  
(4.7)
We define also a map \( \mu_I : \mathcal{Z}(\mathfrak{g}) \to \mathcal{Z}(\mathfrak{I}) \) characterized by:
\[ z - \mu_I(z) \in \mathfrak{U}(\mathfrak{g}), \quad z \in \mathcal{Z}(\mathfrak{g}). \]  
(4.8)
Then \( \gamma_I := \sigma_I \circ \mu_I : \mathcal{Z}(\mathfrak{g}) \to \mathcal{Z}(\mathfrak{I}) \) is the so-called Harish-Chandra homomorphism and one has:
\[ L_{\gamma_I(z)} = d_{Q_I}^{-1} \circ L_{\mu_I(z)} \circ d_{Q_I}, \quad z \in \mathcal{Z}(\mathfrak{g}). \]  
(4.9)
One knows that \( \mathcal{Z}(\mathfrak{I}) \) is a free module of finite rank over \( \gamma_I(\mathcal{Z}(\mathfrak{g})). \) Hence there exists a finite dimensional vector subspace \( W \) of \( \mathcal{Z}(\mathfrak{I}) \) containing 1 such that the map:
\[ \gamma_I(\mathcal{Z}(\mathfrak{g})) \otimes W \to \mathcal{Z}(\mathfrak{I}) \]
\[ u \otimes v \mapsto uv \]
is a linear bijection.

Let \( \mathcal{I} \) be a finite co-dimensional ideal of \( \mathcal{Z}(\mathfrak{g}) \) and let \( \mathcal{J} = \gamma_I(\mathcal{I}). \) Let \( V \) be a finite dimensional vector subspace of \( \gamma_I(\mathcal{Z}(\mathfrak{g})) \) containing 1 such that \( \gamma_I(\mathcal{Z}(\mathfrak{g})) = \mathcal{J} \oplus V. \) Hence:
\[ \mathcal{Z}(\mathfrak{I}) = (\mathcal{J} \oplus V)W \]
\[ = \mathcal{J}W \oplus WV, \]
where \( \mathcal{J}W \) (resp. \( \mathcal{J}V \)) is the linear span of \( \{uv : u \in \mathcal{J}, v \in W\} \) (resp. \( \{uv : u \in V, v \in W\} \)).

We set \( W_{\mathcal{I}} := VW. \) Let us notice that:
\[ \mathcal{J}W = \mathcal{J}\gamma_I(\mathcal{Z}(\mathfrak{g}))W = \mathcal{J}\mathcal{Z}(\mathfrak{I}). \]
We see that, if \( \mathcal{I} \) is the kernel of a character \( \chi \) of \( \mathcal{Z}(\mathfrak{g}), \) one may and will take \( V = C1, \) hence \( W_{\mathcal{I}} = W. \) One has:
\[ \mathcal{Z}(\mathfrak{I}) = W_{\mathcal{I}} \oplus \mathcal{J}W. \]
Let \( s_{\mathcal{I}}, \) resp. \( q_{\mathcal{I}}, \) be the linear map from \( \mathcal{Z}(\mathfrak{I}) \) to \( W_{\mathcal{I}}, \) resp. \( \mathcal{J}W, \) deduced from this direct sum decomposition. The algebra \( \mathcal{Z}(\mathfrak{I}) \) acts on \( W_{\mathcal{I}} \) by a representation \( \rho_{\mathcal{I}} \) defined by:
\[ \rho_{\mathcal{I}}(u)v = s_{\mathcal{I}}(uv), \quad u \in \mathcal{Z}(\mathfrak{I}), v \in W_{\mathcal{I}}. \]  
(4.10)
In fact:

The representation \( (\rho_{\mathcal{I}}, W_{\mathcal{I}}) \) is isomorphic to the natural representation of \( \mathcal{Z}(\mathfrak{I}) \) on \( \mathcal{Z}(\mathfrak{I})/\mathcal{Z}(\mathfrak{I})\mathcal{J}. \)
We notice that:

\[ uv = \rho_I(u)v + q_I(uv) \]  

(4.11)

Let \((v_i)_{i=1,...,n}\) be a basis of \(W\). Then:

\[ q_I(uv) = \sum_{i=1}^{n} \gamma_I(z_i(u, v, I))v_i, \]

(4.12)

where the \(z_i(u, v, I)\) are in \(I\). Let us recall that:

\[ \gamma_I(z_i(u, v, I)) = d_{Q_I}^{-1} \circ L_{\mu_I(z_i(u, v, I))} \circ d_{Q_I} \]

and that:

\[ \mu_I(z_i(u, v, I)) \in z_i(u, v, I) + u_I^+U(\mathfrak{g}). \]

Let us take a basis \((u_{IJ}^\pm)_{j=1,...,p}\) of \(u_I^+\). We may assume that each \(u_{IJ}^+\) is a weight vector for \(\mathfrak{a}\) with weight \(\alpha_j\). Then

\[ \mu_I(z_i(u, v, I)) = z_i(u, v, I) + \sum_{j=1}^{p} u_{IJ}^+ v_{i,j}(u, v, I), \]

(4.14)

where \(v_{i,j}(u, v, I) \in U(\mathfrak{g})\).

Let \(\mathfrak{g_C}\) be a complex Cartan subalgebra of \(\mathfrak{g_c}\) of the form \(\mathfrak{t_c} \oplus \mathfrak{a_c}\), where \(\mathfrak{t}\) is a maximal abelian subalgebra of \(\mathfrak{m}\), the centralizer of \(\mathfrak{a}\) in \(\mathfrak{t}\). Let \(W(\mathfrak{g_c}, \mathfrak{g_c})\) be the corresponding Weyl group.

One has \(\mathfrak{a} = \mathfrak{a}_{L_I} \oplus (\mathfrak{a} \cap {^0L_I})\). Hence one has natural inclusions:

\[ \mathfrak{a}_{L_I}^* \subset \mathfrak{a}^* \text{ and } \mathfrak{a}_{C}^* \subset \mathfrak{j}_C^*. \]  

(4.15)

If \(\Lambda \in \mathfrak{j}_C^*\), let \(\chi_{\Lambda} = \chi_{\Lambda}^\mathfrak{g}\) be the character of \(\mathcal{Z}(\mathfrak{g})\) corresponding to \(\Lambda\) via the Harish-Chandra isomorphism \(\gamma\) from \(\mathcal{Z}(\mathfrak{g})\) onto \(S(\mathfrak{j}_C)^{W(\mathfrak{g_c}, \mathfrak{g_c})}\). More precisely,

\[ \chi_{\Lambda}(u) = (\gamma(u))(\Lambda), \quad u \in \mathcal{Z}(\mathfrak{g}). \]

According to our choice of \(\mathfrak{j}_C\), it is also a Cartan subalgebra of \(\mathfrak{l}_{C, C}\). We define similarly the character \(\chi_{\Lambda}^\mathfrak{l}\) of \(\mathcal{Z}(\mathfrak{l}_I)\).

When \(I = I_\Lambda := \ker \chi_{\Lambda}\), we take, as we have already said, \(W_I = W\) and we write \(s_\Lambda\) instead of \(s_I\), \(q_\Lambda\) instead of \(q_I\), \(\rho_\Lambda\) instead of \(\rho_I\) and \((u, v, \Lambda)\) instead of \((u, v, I)\). Let us show that, for \(u \in \mathcal{Z}(\mathfrak{l}_I)\), \(s_\Lambda(u)\) and \(q_\Lambda(u)\) are polynomial in \(\Lambda\). It is enough to prove this for \(u = \gamma_I(z)v\) where \(z \in \mathcal{Z}(\mathfrak{g})\) and \(v \in W\). Then \(u = (\gamma_I(z) - \chi_\Lambda(z))v + \chi_\Lambda(z)v\). Hence \(q_\Lambda(u) = (\gamma_I(z) - \chi_\Lambda(z))v \in \mathcal{Z}(\mathfrak{l}_I)\mathcal{J}\) and \(s_\Lambda(u) = \chi_\Lambda(z)v \in W\). Our claim follows. It implies easily that:

\[ z_i(u, v, \Lambda) \text{ in (4.12) depends polynomially on } \Lambda. \]

This implies, as \(\mu_I\) is linear, that:

\[ v_{i,j}(u, v, \Lambda) \text{ in (4.14) depends polynomially on } \Lambda. \]  

(4.16)
Using Harish-Chandra isomorphisms, one sees that:

Each simple subquotient of the representation $\rho_\Lambda$ of $\mathcal{Z}(I)$ is given by some character of the form $\lambda^l_\mu$, where $\mu$ varies in $W(g_{\mathbb{C},j\mathbb{C}})\Lambda$. 

(4.17)

Let us notice that $\chi^l_\mu = \chi^l_{w\mu}$, where $w \in W(I,\mathbb{C},j\mathbb{C})$.

4.3 The function $\Phi_f$ on $L_I$ and related differential equations

Recall the section $s$ be the section introduced in (1.4). From [15, Corollary 5.6], we have that $a_I$ is equal to the projection of $a_{L_I}$ on $a_I$. Hence:

One may and will choose the section $s$ such that $s(a_I) \subset a_{L_I}$.

Recall that the map $s$ is also denoted by $X \mapsto \tilde{X}$ or $a \mapsto \tilde{a}$ for the corresponding morphism of Lie groups.

Recall that $A_Z = A_{Z_I}$. Let $\rho_{L_I \cap Q}$ be the half sum of the roots of $a$ in $I \cap u$. In particular $\rho_Q = \rho_{L_I \cap Q} + \rho_{Q_I}$ on $a$.

Let $N \in \mathbb{N}$ and $\mathcal{I}$ be a finite co-dimensional ideal in $\mathcal{Z}(g)$. Recall that $(\rho_\mathcal{I}, W_\mathcal{I})$ is the finite dimensional $\mathcal{Z}(I_I)$-module defined in (4.10). For any $f \in \mathcal{A}_{temp, N}(Z : \mathcal{I})$, let us define a function $\Phi_f : A_Z \to W_\mathcal{I}^*$ by:

$$< \Phi_f(a_Z), v > = a_Z^{-\rho_Q}(L_w f)(a_Z), \quad v \in W_\mathcal{I}, a_Z \in A_Z,$$

(4.18)

where $v^I$ is defined as in (4.6). By definition, $\Phi_f$ does not depend on the choice of the section $s$. Hence, for $X \in a_I$, $L_X \Phi_f$ is equal to $L_{\tilde{X}} \Phi_f$, where $\tilde{X} = s(X) \in a_{L_I}$.

If $z \in \mathcal{Z}(g)$, it follows from (4.9) that $\gamma_I(z) = \mu_I(z)$ and, if $z \in \mathcal{I}$, $\gamma_I(z) \in \mathcal{I} + u_I^{-1} \mathcal{U}(g)$. Altogether, one gets, for $X \in a_I$ and $v \in W_\mathcal{I}$, $(\tilde{X} v)^I \in (\rho_\mathcal{I}(\tilde{X}) v)^I + u_I^{-1} \mathcal{U}(g)$ and

$$L_{\tilde{X}} \Phi = \rho_\mathcal{I}(\tilde{X}) \Phi_f - \Psi_{f,X},$$

(4.19)

where $\Psi_{f,X} : A_Z \to W_\mathcal{I}^*$ satisfies for $X \in a_I$, $v \in W_\mathcal{I}$ and $a_Z \in A_Z$:

$$< \Psi_{f,X}(a_Z), v > = -a_Z^{-\rho_Q} \sum_{i,j} (L_{u_{i,j}^{-1}} L_{v_i} f)(a_Z),$$

(4.20)

for some elements $u_{i,j}^{-1} \in u_I^{-1}$ and $v_i \in \mathcal{U}(g)$ depending on $X$ and $v$.

One sets:

$$\Gamma_{\mathcal{I}}(X) = -\rho_\mathcal{I}(\tilde{X}), \quad X \in a_I.$$

(4.21)

Hence, one has the important relation:

$$L_X \Phi_f = -\Gamma_{\mathcal{I}}(X) \Phi_f - \Psi_{f,X}, \quad X \in a_I.$$

(4.22)

We notice that $\Gamma_{\mathcal{I}}$ is a representation of the abelian Lie algebra $a_I$ on $W_\mathcal{I}^*$. For $\lambda \in a_{I,C}^*$, one denotes by $W_{\mathcal{I},\lambda}$ the space of joint generalized eigenvectors of $W_\mathcal{I}^*$ by the endomorphisms
Let $\Gamma_I(X), X \in a_I$, for the eigenvalue $\lambda$. Let $Q_I$ be the (finite) subset of $\lambda \in a_{I,C}^*$ such that $W_{I,\lambda}^* \neq \{0\}$. One has:

$$W_{I}^* = \bigoplus_{\lambda \in Q_I} W_{I,\lambda}^*.$$  

If $\lambda \in Q_I$, let $E_\lambda$ be the projector of $W_{I}^*$ onto $W_{I,\lambda}^*$ parallel to the sum of the other $W_{I,\mu}^*$'s. We endow $W_{I}^*$ with a scalar product and if $T \in \text{End}(W_{I}^*)$, we denote by $\|T\|$ its Hilbert-Schmidt norm. It is clear that $E_\lambda$ commutes with the operators $\Gamma_I(X), X \in a_I$. We set

$$\Phi_f,\lambda = E_\lambda \Phi_f.$$  

The proofs of the following results (Lemma 4.3 up to Proposition 4.14) follow closely the work of Harish-Chandra (cf. [8, Section 22]). Here $M_1^+$ is replaced by $A_Z$ and $M_1$ by $A_{Z_I}$.

**4.3 Lemma.** Let $f \in \mathcal{A}_{\text{temp}}(Z : I)$. One has, for all $a_Z \in A_Z$, $T \in \mathbb{R}$, $X_I \in a_I$, $\lambda \in Q_I$,

(i)  

$$\Phi_f(a_Z \exp(T X_I)) = e^{T \Gamma_I(X_I)} \Phi_f(a_Z) + \int_0^T e^{(T-t) \Gamma_I(X_I)} \Psi_{f,X_I}(a_Z \exp(t X_I)) \, dt.$$  

(ii)  

$$\Phi_{f,\lambda}(a_Z \exp(T X_I)) = e^{T \Gamma_I(X_I)} \Phi_{f,\lambda}(a_Z) + \int_0^T E_\lambda e^{(T-t) \Gamma_I(X_I)} \Psi_{f,X_I}(a_Z \exp(t X_I)) \, dt.$$  

**Proof.** The equality (i) is an immediate consequence of (4.22). Indeed, we apply the elementary result on first order linear differential equation to the function $t \mapsto F(t) = \Phi_f(a_Z \exp(t X_I))$, whose derivative is $F'(t) = -L_{X_I} \Phi_f(a_Z \exp(t X_I))$ satisfies

$$F'(t) = \Gamma_I(X_I) F(t) + \Psi_{f,X_I}(a_Z \exp(t X_I)).$$  

The equality (ii) follows by applying $E_\lambda$ to both sides of the equality of (i). \qed

For $\lambda \in Q_I$, let

$$E_\lambda(X_I) := E_\lambda \exp(\Gamma_I(X_I) - \lambda(X_I))$$  

for $X_I \in a_I$. Since $E_\lambda(\Gamma_I(X_I) - \lambda(X_I))$ is nilpotent, one has:

**4.4 Lemma.** Let $\lambda \in Q_I$. We can choose $c \geq 0$ such that:

$$\|E_\lambda(X_I)\| \leq c(1 + \|X_I\|)^{N_I}, \quad X_I \in a_I,$$

where $N_I$ is the dimension of $W_{I}$.  

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For any \( f \in A_{\text{temp}}(Z : \mathcal{I}) \) and \( X_I \in \text{a}_I \), the function \( \Psi_{f,X_I} \) is a function on \( AZ \) and one is interested in its derivatives along \( Y \in \text{a}_Z \). On one hand, one has:

\[
L_Y a_Z^{-\rho q} = \rho q(Y)a_Z^{-\rho q}, \quad a_Z \in AZ.
\]

One the other hand, one has:

\[
\tilde{Y} u_{I,j} = [\tilde{Y}, u_{I,j}] + u_{I,j} \tilde{Y} = \alpha_j(\tilde{Y}) u_{I,j} + u_{I,j} \tilde{Y}.
\]

Hence \( L_Y \Psi_{f,X_I} \) and more generally \( L_u \Psi_{f,X_I} \), \( u \in S(\text{a}_Z) \), is a function of the same type than \( \Psi_{f,X_I} \) (see (4.20)).

**4.5 Lemma.** Let \( N \in \mathbb{N} \) and fix \( u \in S(\text{a}_Z) \).

(i) There exists a continuous semi-norm on \( C_{\text{temp},N}^{\infty}(Z) \), \( p_u \), such that:

\[
\| L_u \Phi_f(a_Z \exp X_I) \| \leq p_u(f)(1 + \| \log a_Z \|)^N(1 + \| X_I \|)^N,
\]

\( a_Z \in A_Z^{-}, X_I \in \text{a}_I^{-} \), \( f \in A_{\text{temp},N}(Z : \mathcal{I}) \).

(ii) There exists a continuous semi-norm on \( C_{\text{temp},N}^{\infty}(Z) \), \( q_u \), such that:

\[
\| L_u \Psi_{f,X_I}(a_Z \exp X_I) \| \leq q_u(f)e^{\beta_I(X_I)}(1 + \| \log a_Z \|)^N(1 + \| X_I \|)^N,
\]

\( a_Z \in A_Z^{-}, X_I \in \text{a}_I^{-} \), \( f \in A_{\text{temp},N}(Z : \mathcal{I}) \).

**Proof.** Let us first prove (i). It is easy to see, using (4.18), that:

\[
< L_u \Phi_f(a_Z), v > = a_Z^{-\rho q}(L_u' L_u f)(a_Z), \quad v \in W_{\mathcal{I}},
\]

for some \( u' \in S(\text{a}_Z) \) with \( \deg u' \leq \deg u \). Then (i) follows from the continuity of the operator \( L_u' L_u \) on \( C_{\text{temp},N}(Z) \) and the definition of \( C_{\text{temp},N}(Z) \) (cf. (3.7)). By definition of \( \Psi_{f,X} \) (cf. (4.20)), one gets (ii) using Lemma 4.2.

We say that an integral depending on a parameter converges uniformly if the absolute value of the integrand is bounded by an integrable function independently of the parameter.

**4.6 Lemma.** Let us fix \( f \in A_{\text{temp}}(Z : \mathcal{I}) \) and \( u \in S(\text{a}_Z) \), and let \( \lambda \in Q_{\mathcal{I}} \) and \( X_I \in \text{a}_I^{-} \) such that \( \text{Re} \lambda(X_I) > \beta_I(X_I) \). Then

(i) The integral

\[
\int_0^{\infty} E_{\lambda} e^{-t^{\alpha_I}(X_I)} L_u \Psi_{f,X_I}(a_Z \exp(t X_I)) \, dt
\]

converges uniformly on any compact subset of \( A_Z^{-} \).
(ii) The map 

$$a_{Z_t} \mapsto \int_0^\infty E_\lambda e^{-t\Gamma_Z(\xi_1)} \Psi_{f,\xi_1}(a_{Z_t} \exp(t \xi_1)) \, dt$$

is a well-defined map on $A^-_{Z_t}$. Its derivative along $u \in S(a_Z)$ is given by derivation under the integral sign.

Proof. One has 

$$E_\lambda e^{-t\Gamma_Z(\xi_1)} = e^{-t\lambda(\xi_1)} E_\lambda e^{t(\lambda(\xi_1) - \Gamma_Z(\xi_1))} = e^{-t\lambda(\xi_1)} E_\lambda(-t\xi_1).$$

Hence, from Lemma 4.4, one has:

$$\|E_\lambda e^{-t\Gamma_Z(\xi_1)}\| \leq c(1 + \|t \xi_1\|^N) e^{-t \text{Re}\lambda(\xi_1)}. \quad (4.23)$$

Using Lemma 4.5(ii) and (4.23), one can show that the integral in (i) converges uniformly for $a_{Z_t} \in A^-_{Z_t}$. Let $a_{Z_t}$ be in a compact subset $C$ of $A^-_{Z_t}$. There exists $T_0 > 0$ such that, for all $z \in C$, $z \exp(T_0 \xi_1) \in A^-_{Z_t}$. Writing $\int_0^\infty = \int_0^{T_0} + \int_{T_0}^{\infty}$, $a_{Z_t} \exp(T_0 \xi_1)$ is in $A^-_{Z_t}$, and, using the uniform convergence proved above, one gets (i).

The assertion (ii) follows from (i) and the theorem on derivatives of integrals depending of a parameter. \[\square\]

Fix $N \in \mathbb{N}$, $f \in A_{\text{temp},N}(Z : T)$ and $\lambda \in Q_T$ and put, for $X_t$ as in Lemma 4.6, i.e. $X_t \in a_{Z_t}^{-\infty}$ such that Re $\lambda(X_t) > \beta_t(X_t)$:

$$\Phi = \Phi_f,$$

$$\Phi_\lambda = \Phi_{f,\lambda},$$

$$\Psi_{X_t} = \Psi_{f,\xi_1},$$

$$\Phi_{\lambda,\infty}(a_{Z_t}, X_t) = \lim_{T \to +\infty} e^{-T \Gamma_Z(\xi_1)} \Phi_{f,\lambda}(a_{Z_t} \exp(T \xi_1)), \quad a_{Z_t} \in A^-_{Z_t}.$$

It follows from Lemmas 4.3(ii) and 4.6 that this limit exists and is $C^\infty$ on $A^-_{Z_t}$. Moreover

$$L_u \Phi_{\lambda,\infty}(a_{Z_t}, X_t) = L_u \Phi_{\lambda}(a_{Z_t}) + \int_0^\infty E_\lambda e^{-t\Gamma_Z(\xi_1)} L_u \Psi_{X_t}(a_{Z_t} \exp(t \xi_1)) \, dt,$$

$$u \in S(a_Z), a_{Z_t} \in A^-_{Z_t}. \quad (4.25)$$

4.7 Lemma. For $X_t \in a_{Z_t}^{-\infty}$ such that Re $\lambda(X_t) > 0$, one has:

$$\Phi_{\lambda,\infty}(a_{Z_t}, X_t) = 0, \quad a_{Z_t} \in A^-_{Z_t}.$$

Proof. One has

$$\|e^{-T \Gamma_Z(\xi_1)} \Phi_{\lambda}(a_{Z_t} \exp(T \xi_1))\| \leq e^{-T \text{Re}\lambda(X_t)} \|E_\lambda(-T \xi_1)\| \|\Phi(a_{Z_t} \exp(T \xi_1))\|$$

and, from Lemmas 4.4 and 4.5(i), the right hand side of the inequality tends to zero as $T \to +\infty$. Hence the Lemma follows from the definition (4.24) of $\Phi_{\lambda,\infty}(a_{Z_t}, X_t)$. \[\square\]
4.8 Lemma. Let $X_1, X_2 \in \mathfrak{a}_I^-$ and suppose that
\[ \text{Re } \lambda(X_i) > \beta_I(X_i), \quad i = 1, 2. \]
Then
\[ \Phi_{\lambda, \infty}(aZ_i, X_1) = \Phi_{\lambda, \infty}(aZ_i, X_2), \quad aZ_i \in A_Z^-. \]

Proof. Same as the proof of [8, Lemma 22.8]. We give it for sake of completeness. Let $aZ_i \in A_Z^-$. Applying Lemma 4.3(ii) to $X_2$ instead of $X_I$ and $T_2$ instead of $T$, one gets:
\[
e^{-\Gamma x(T_1 X_1 + T_2 X_2)} \Phi_{\lambda}(aZ_i, \exp(T_1 X_1) \exp(T_2 X_2)) = e^{-T_1 \Gamma x(x_1)} \Phi_{\lambda}(aZ_i, \exp(T_1 X_1)) + \int_0^{T_2} E_x e^{-\Gamma x(T_1 X_1 - t_2 X_2)} \Psi_{X_2}(aZ_i, \exp(T_1 X_1 + t_2 X_2)) \, dt_2,
\]
for $T_1, T_2 > 0$. From Lemmas 4.4 and 4.5(ii) applied to $T_1 X_1 + t_2 X_2$ instead of $X_I$, one sees that:
\[
\lim_{T_1 \to +\infty} \int_0^{T_2} E_x e^{-\Gamma x(T_1 X_1 - t_2 X_2)} \Psi_{X_2}(aZ_i, \exp(T_1 X_1 + t_2 X_2)) \, dt_2
\]
tends to 0 when $T_1 \to +\infty$. Hence:
\[
\lim_{T_1 \to +\infty} e^{-T_1 \Gamma x(T_1 X_1 + T_2 X_2)} \Phi_{\lambda}(aZ_i, \exp(T_1 X_1 + T_2 X_2)) = \lim_{T_1 \to +\infty} e^{-T_1 \Gamma x(T_1 X_1)} \Phi_{\lambda}(aZ_i, \exp(T_1 X_1)) = \Phi_{\lambda, \infty}(aZ_i, X_1).
\]
Since the first limit on the above equality is symmetrical in $X_1$ and $X_2$, one then deduces that:
\[ \Phi_{\lambda, \infty}(aZ_i, X_1) = \Phi_{\lambda, \infty}(aZ_i, X_2). \]

We now decompose $Q_I$ into three disjoints subsets $Q_I^+, Q_I^0$ and $Q_I^-$ as follows:

1. $\lambda \in Q_I^+$ if $\text{Re } \lambda(X_I) > 0$ for some $X_I \in \mathfrak{a}_I^-$,
2. $\lambda \in Q_I^0$ if $\text{Re } \lambda(X_I) = 0$ for all $X_I \in \mathfrak{a}_I^-$,
3. $\lambda \in Q_I^-$ if $\lambda \notin Q_I^+ \cup Q_I^0$, i.e. for all $X_I \in \mathfrak{a}_I^-$, $\text{Re } \lambda(X_I) \leq 0$ and there exists $X_I \in \mathfrak{a}_I^-$ such that $\text{Re } \lambda(X_I) < 0$.

4.9 Lemma. Assume $\lambda \in Q_I^+$ and $X_I \in \mathfrak{a}_I^-$ such that $\text{Re } \lambda(X_I) > \beta_I(X_I)$. Then, for any $aZ_i \in A_Z^-$,
\[ \Phi_{\lambda, \infty}(aZ_i, X_I) = 0 \]
and, for any $u \in S(aZ)$,
\[ L_u \Phi_{\lambda}(aZ_I \exp(TX_I)) = - \int_T^\infty E_x e^{-(t-T)\Gamma x(x_I)} L_u \Psi_{X}(aZ_I \exp(tX_I)) \, dt, \quad T \in \mathbb{R}. \]
Proof. Since \( \lambda \in \mathcal{Q}_i^+ \), there exists \( X_0 \in \mathfrak{a}_i^- \) such that \( \text{Re}\lambda(X_0) > 0 \). Then, from Lemma 4.7, \( \Phi_{\lambda,X}(a_{Zi}, X_0) = 0 \), and, from Lemma 4.8, as \( \text{Re}\lambda(X_0) > 0 > \beta_I(X_0) \), one has \( \Phi_{\lambda,X}(a_{Zi}, X_0) \) for any \( X_1 \in \mathfrak{a}_i^- \) such that \( \lambda(X_1) > \beta_I(X_1) \). This proves the first part of the Lemma. The second part follows from (4.25) by change of variables and when we replace \( a_{Zi} \) by \( a_{Zi} \exp(TX_1) \). \( \square \)

4.10 Corollary. Let \( \lambda \in \mathcal{Q}_i^+ \) and \( X_1 \in \mathfrak{a}_i^- \) be such that \( \text{Re}\lambda(X_1) \geq \beta_I(X_1)/2 \). Then, for \( u \in S(a_Z) \), \( a_{Zi} \in A_{Zi}^- \), and \( T \geq 0 \),

\[
\|L_u\Phi_\lambda(a_{Zi} \exp(TX_1))\| \leq e^{T\beta_I(X_1)/2} \int_T^{\infty} e^{-t\beta_I(X_1)/2}\|E_\lambda((T-t)X_1)\|\|L_u\Phi_X(a_{Zi} \exp(tX_1))\| dt.
\]

Proof. Since \( \beta_I(X_1) < 0 \) and \( \text{Re}\lambda(X_1) \geq \beta_I(X_1)/2 \), one has in particular \( \text{Re}\lambda(X_1) > \beta_I(X_1) \). Then one can see from Lemmas 4.9 and 4.6 that:

\[
\|L_u\Phi_\lambda(a_{Zi} \exp(TX_1))\| \leq \int_T^{\infty} e^{-(t-T)\text{Re}\lambda(X_1)}\|E_\lambda((T-t)X_1)\|\|L_u\Phi_X(a_{Zi} \exp(tX_1))\| dt.
\]

Our assertion follows, since \( \text{Re}\lambda(X_1) \geq \beta_I(X_1)/2 \) implies that \( -(t - T)\text{Re}\lambda(X_1) \leq -(t - T)\beta_I(X_1)/2 \) for \( t \geq T \). \( \square \)

4.11 Lemma. Let \( X_1 \in \mathfrak{a}_i^- \) be such that \( \text{Re}\lambda(X_1) \leq \beta_I(X_1)/2 \). Then

\[
\|L_u\Phi_\lambda(a_{Zi} \exp(TX_1))\| \leq e^{T\beta_I(X_1)/2} \left(\|E_\lambda(TX_1)\|\|L_u\Phi(a_{Zi})\| \right.
\]

\[
+ \int_0^{\infty} e^{-t\beta_I(X_1)/2}\|E_\lambda((T-t)X_1)\|\|L_u\Phi_X(a_{Zi} \exp(tX_1))\| dt \right),
\]

where \( T \geq 0 \), \( u \in S(a_Z) \), \( a_{Zi} \in A_{Zi}^- \).

Proof. We use Lemma 4.3(ii) and the inequality \( (T-t)\text{Re}\lambda(X_1) \leq (T-t)\beta_I(X_1)/2 \) for \( t \leq T \) in order to get an analogue of the inequality of the Lemma where \( \int_0^{\infty} \) is replaced by \( \int_0^{T} \). The Lemma follows. \( \square \)

Like in [8, after the proof of Lemma 22.8], one sees that one can choose \( 0 < \delta \leq 1/2 \) such that:

\[
\text{Re}\lambda(X_1) \leq \delta\beta_I(X_1), \quad X_1 \in \mathfrak{a}_i^-, \lambda \in \mathcal{Q}_i^{-}.
\]

(4.26)

4.12 Lemma. Let \( \lambda \in \mathcal{Q}_i^{-} \) and \( X_1 \in \mathfrak{a}_i^- \). Then, for \( u \in S(a_Z) \), \( a_{Zi} \in A_{Zi}^- \), \( T \geq 0 \),

\[
\|L_u\Phi_\lambda(a_{Zi} \exp(TX_1))\| \leq e^{T\delta\beta_I(X_1)} \left(\|E_\lambda(TX_1)\|\|L_u\Phi(a_{Zi})\| \right.
\]

\[
+ \int_0^{\infty} e^{-t\beta_I(X_1)/2}\|E_\lambda((T-t)X_1)\|\|L_u\Phi_X(a_{Zi} \exp(tX_1))\| dt \right).
\]

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Proof. This is proved like Lemma 4.11, using that $\Re \lambda(X_I) \leq \delta \beta_I(X_I)$ and $0 < \delta \leq 1/2$. \[ \square \]

Remark now that, if $\lambda \in Q^0_T$, it follows from Lemma 4.8 and the definition of $\beta_I$ (cf. (4.1)) that:

For $a_{Z_I} \in A_{Z_I}^-$, $\Phi_{\beta_I}(a_{Z_I}; X_I)$ is independent of $X_I \in a_{Z_I}^-$. We will denote it by $\Phi_{\beta_I}(a_{Z_I})$.

4.13 Lemma. Assume $\lambda \in Q^0_T$ and let $X_I \in a_{Z_I}^-$. Then one has, for $u \in S(a_Z), T \geq 0$ and $a_{Z_I} \in A_{Z_I}^-$,

$$ \|L_u \Phi_\lambda(a_{Z_I} \exp(TX_I)) - L_u \Phi_{\beta_I}(a_{Z_I} \exp(TX_I))\| 
\leq e^{T \beta_I(X_I)/2} \int_0^T e^{-t \beta_I(X_I)/2} \|E_\lambda((T-t)X_I)\| L_u \Psi_X(a_{Z_I} \exp(tX_I))\| dt. $$

Proof. From (4.25), one deduces

$$ L_u \Phi_{\beta_I}(a_{Z_I} \exp(TX_I)) = L_u \Phi_\lambda(a_{Z_I} \exp(TX_I)) + \int_T^\infty E_\lambda(t-t)T_X(a_{Z_I}) L_u \Psi_X(a_{Z_I} \exp(tX_I)) dt. $$

The Lemma now follows from the fact that $(T-t)\beta_I(X_I) \geq 0$ if $t \geq T$. \[ \square \]

We define now:

$$ \Phi_{\beta_I}(a_{Z_I}) = 0, \quad a_{Z_I} \in A_{Z_I}^-, \lambda \in Q^+_T \cup Q^-_T. \quad (4.27) $$

4.14 Proposition. Let $X_I \in a_{I}^-$ and $u \in S(a_Z)$. Then, for $a_{Z_I} \in A_{Z_I}^-$, $T \geq 0$,

$$ \|L_u \Phi_\lambda(a_{Z_I} \exp(TX_I)) - L_u \Phi_{\beta_I}(a_{Z_I} \exp(TX_I))\| 
\leq e^{T \beta_I(X_I)} \left( \|E_\lambda(TX_I)\| L_u \Phi(a_{Z_I}) \right. 
+ \left. \int_0^\infty e^{-t \beta_I(X_I)/2} \|E_\lambda((T-t)X_I)\| L_u \Psi_X(a_{Z_I} \exp(tX_I))\| dt \right). $$

Proof. If $\lambda \in Q^0_T \cup Q^-_T$, our assertion follows from Lemmas 4.12 and 4.13. On the other hand, if $\lambda \in Q^+_T$, we can apply Lemmas 4.9 and 4.11, and Corollary 4.10. \[ \square \]

5 Definition of the constant term and its properties

Let us fix a subset $I$ of $S$ and a finite co-dimensional ideal $I$ in $Z(\mathfrak{g})$. 32
5.1 Some estimates

In this Subsection, we establish some estimates analogous to the ones given in [8, Section 23].

5.1 Lemma. We fix a compact set $\mathcal{C}$ in $a_I^-$ and choose $\varepsilon_0 > 0$ such that $\beta_l(X) \leq -2\varepsilon_0$ for all $X \in \mathcal{C}$. We put $\varepsilon = \delta \varepsilon_0$, where $\delta$ is given by (4.26). Let $N \in \mathbb{N}$ and $u \in S(a_Z)$. Then there exists a continuous semi-norm $q$ on $C_{\text{temp},N}^\infty(Z : \mathcal{I})$ such that, for all $\lambda \in \mathcal{Q}_Z$, $T \geq 0$, $X \in \mathcal{C}$, $a_Z \in A_Z^-$ and $f \in \mathcal{A}_{\text{temp},N}(Z : \mathcal{I})$,

$$\|L_u \Phi_{f,\lambda}(a_Z \exp(TX)) - L_u \Phi_{f,\lambda,\infty}(a_Z \exp(TX))\| \leq e^{-\varepsilon T} q(f)(1 + \| \log a_Z \|)^N.$$

Proof. As $A_Z^-$ is contained in $A_Z^{I_1}$, this follows from Proposition 4.14, Lemmas 4.5(ii) and 4.4.

5.2 Lemma. Let $\lambda \in \mathcal{Q}_Z$ and $N \in \mathbb{N}$. One has:

$$\Phi_{f,\lambda,\infty}(a_Z \exp X) = e^{\Gamma_Z(X)} \Phi_{f,\lambda,\infty}(a_{Z_I}), \quad X \in a_I, a_{Z_I} \in A_{Z_I}, f \in \mathcal{A}_{\text{temp},N}(Z : \mathcal{I}).$$

Proof. According to (4.27), one may assume $\lambda \in \mathcal{Q}_Z^0$. From Lemma 4.3(ii) applied with $T = 1$, one has, for $a_Z \in A_Z$, $X \in a_I$,

$$e^{-\Gamma_X(X)} \Phi_{\lambda}(a_Z \exp X) = \Phi_{\lambda}(a_Z) + \int_0^1 E_{\lambda} e^{-t \Gamma_X(X)} \Psi_X(a_Z \exp(tX)) \, dt.$$

Let $Y \in a_I^-$. Replacing $a_Z$ by $a_{Z_I} \exp(TY)$, with $a_{Z_I} \in A_{Z_I}$, and multiplying by $e^{-T \Gamma_Y(Y)}$, one gets:

$$e^{-\Gamma_X(X+TY)} \Phi_{\lambda}(a_{Z_I} \exp(X + TY)) = e^{-\Gamma_X(TY)} \Phi_{\lambda}(a_{Z_I} \exp(TY)),$$

$$+ \int_0^1 E_{\lambda} e^{-\Gamma_X(tX+TY)} \Psi_X(a_{Z_I} \exp(tX + TY)) \, dt.$$

One can choose $T_0 > 0$ such that $a_{Z_I} \exp(T_0 Y) \in A_Z^-$. If $T$ is sufficiently large, $tX + (T - T_0)Y \in a_I^-$ for all $t \in [0, 1]$. Recalling that $\lambda \in \mathcal{Q}_Z^0$, it follows from Lemma 4.5(ii) applied to $a_Z = a_{Z_I} \exp(T_0 Y)$ and $X_I = tX + (T - T_0)Y$ that, if $a_{Z_I} \in A_{Z_I}$, the integral in this equality tends to 0 as $T \to +\infty$. Recalling the definition of $\Phi_{f,\lambda,\infty}$ (cf. (4.24)), one gets

$$e^{-\Gamma_X(X)} \Phi_{f,\lambda,\infty}(a_{Z_I} \exp X) = \Phi_{f,\lambda,\infty}(a_{Z_I}), \quad X \in a_I, a_{Z_I} \in A_{Z_I}^-.$$

5.3 Lemma. Let $\lambda \in \mathcal{Q}_Z^0$ and $N \in \mathbb{N}$. There exists a continuous semi-norm $p$ on $C_{\text{temp},N}^\infty(Z : \mathcal{I})$ such that, for all $f \in \mathcal{A}_{\text{temp},N}(Z : \mathcal{I})$,

$$\| \Phi_{f,\lambda,\infty}(a_{Z_I}) \| \leq p(f)(1 + \| \log a_{Z_I} \|)^{N + \text{dim}W_Z}, \quad a_{Z_I} \in A_{Z_I}^-.$$
Proof. We fix $X \in a_I^-$. Let $a_{Z_I} \in A_{Z_I}^-$. If $t$ is large enough, $a_{Z_I} \exp(tX) \in A_{Z_I}^-$. More precisely, if $a_{Z_I} = \exp Y$ with $Y \in a_{Z_I}^-$, $t$ has to be such that $\alpha(Y + tX) \leq 0$ for all $\alpha \in S \setminus I$. For this, it is enough that $t \geq \frac{|a(Y)|}{|\alpha(X)|}$ for all $\alpha \in S \setminus I$. But $|\frac{a(Y)}{\alpha(X)}|$ is bounded above by $C\|Y\|$ for some constant $C > 0$. We will take:

$$ T = C\|Y\| $$ (5.1)

and write $a_{Z_I} = a_Z \exp(-TX)$ with $a_Z = a_{Z_I} \exp(TX) \in A_{Z_I}^-$. One has, from Lemma 5.2,

$$ \Phi_{f,\lambda,\infty}(a_Z \exp(-TX)) = e^{-TT_F(X)}\Phi_{f,\lambda,\infty}(a_Z). $$ (5.2)

As $\lambda \in Q_T^0$, $\|E_{\lambda}e^{-TT_F(X)}\|$ is bounded by a constant times $(1 + T\|X\|)^{N_T}$, where $N_T$ is the dimension of $W_T$ (cf. Lemma 4.4). Using (5.1) and as $X$ is fixed, one concludes that there exists $C_1 > 0$ such that:

$$ \|E_{\lambda}e^{-TT_F(X)}\| \leq C_1(1 + \|\log a_{Z_I}\|)^{N_T}. $$

We remark that $\|\log a_Z\| \leq \|\log a_{Z_I}\| + \|TX\|$ is bounded by some constant times $\|\log a_{Z_I}\|$ because $T = C\|Y\|$ and $\|X\|$ is fixed. Then, using (5.2), the Lemma follows from Lemma 5.1 for $T = 0$ and Lemma 4.5(i) for $X_I = 0$. \hfill \Box

Let $N \in \mathbb{N}$ and $f \in A_{\text{temp},N}(Z : \mathcal{I})$. Let us define

$$ \tilde{f}_I(a_{Z_I}) := \sum_{\lambda \in Q_T^0} \langle \Phi_{f,\lambda,\infty}(a_{Z_I}), 1 \rangle, \quad a_{Z_I} \in A_{Z_I}^-; $$ (5.3)

From Lemma 5.2 and as the eigenvalues of $E_{\lambda}(\Gamma_I(X))$, for any $X \in a_I$, are pure imaginary if $\lambda \in Q_T^0$, one has that:

The map $T \mapsto \tilde{f}_I(\exp(TX))$ is an exponential polynomial with unitary characters. (5.4)

**5.4 Lemma.** For any $f \in A_{\text{temp},N}(Z : \mathcal{I})$,

$$ (L_{a_I^{-1}}f)\tilde{I}(a_Z) = a_I^{\rho q} \tilde{f}_I(a_Ia_Z), \quad a_I \in A_I, a_Z \in A_Z. $$

**Proof.** Using (4.18), one sees that, for any $a_Z \in A_Z$, $a_I \in A_I$,

$$ \langle \Phi_{L_{a_I^{-1}}f}(a_Z), v \rangle = a_Z^{-\rho q}(L_{v^I}L_{a_I^{-1}}f)(a_Z). $$

But, as $v^I \in Z(I_I)$, $L_{a_I^{-1}}$ commutes with $L_{v^I}$. Hence:

$$ \langle \Phi_{L_{a_I^{-1}}f}(a_Z), v \rangle = a_Z^{-\rho q}(L_{a_I^{-1}}L_{v^I}f)(a_Z), $$

$$ = a_I^{\rho q} a_I^{-\rho q} a_Z^{-\rho q}(L_{v^I}f)(a_Ia_Z), $$

$$ = a_I^{\rho q} \langle \Phi_f(a_Ia_Z), v \rangle, $$

$$ = a_I^{\rho q} \langle (L_{a_I^{-1}}\Phi_f)(a_Z), v \rangle. $$
Hence $\Phi_{I_{a_{i}}} = a^{\rho_{Q}} L_{a_{i}^{-1}} \Phi_{f}$. Going to the definiton to $\Phi_{f,\lambda,\infty}$ (cf. (4.24) and (4.27)) and of $\hat{f}_{I}$ (cf. (5.3)), one gets the equality of the Lemma. \hfill \Box

Let $C$ be as in Lemma 5.1, $N \in \mathbb{N}$ and $f \in \mathcal{A}_{\text{temp},N}(Z : I)$. According to this Lemma, the fact that $\Phi_{f,\lambda,\infty} = 0$ for $\lambda \in Q_{T}^{+} \cup Q_{T}^{-}$ (cf. (4.27)) and the compactness of $C$ in $a_{I}^{-}$, one has:

$$\| \Phi_{f}(a_{Z} \exp(TX)) - \Phi_{f,\lambda}(a_{Z} \exp(TX)) \| \leq ce^{-\varepsilon T} q(f)(1 + \| \log a_{Z} \|)^{N},$$

$$T \geq 0, a_{Z} \in A_{Z}^{-}, X \in C,$$  \hspace{1cm} (5.5)

where $c$ is the cardinal of $Q_{Z}$. By the definition (4.18) of $\Phi_{f}$ applied with $v = 1$, one sees that:

$$< \Phi_{f}(a_{Z} \exp(TX)), 1 >= a_{Z}^{-\rho_{Q}} e^{-T \rho_{Q}(X)} f(a_{Z} \exp(TX)).$$

Using the equation above and the definition (5.3) of $\hat{f}_{I}$, one deduces from (5.5) the following Lemma:

**5.5 Lemma.** Let $C$ be as in Lemma 5.1 and $N \in \mathbb{N}$. There exist $c > 0$, $\varepsilon > 0$ and a continuous semi-norm $q$ on $\mathcal{A}_{\text{temp},N}(Z)$ such that, for $f \in \mathcal{A}_{\text{temp},N}(Z : I)$, $X \in C$, $a_{Z} \in A_{Z}^{-}$ and $T \geq 0$,

$$|(a_{Z} \exp(TX))^{-\rho_{Q}} f(a_{Z} \exp(TX)) - \hat{f}_{I}(a_{Z} \exp(TX))| \leq ce^{-\varepsilon T} q(f)(1 + \| \log a_{Z} \|)^{N},$$

Let us show that, for any $f \in \mathcal{A}_{\text{temp}}(Z : I)$ and $X \in a_{I}^{-}$,

$$\lim_{T \to \infty} \left( (a_{Z_{I}} \exp(TX))^{-\rho_{Q}} f(a_{Z_{I}} \exp(TX)) - \hat{f}_{I}(a_{Z_{I}} \exp(TX)) \right) = 0, \quad a_{Z_{I}} \in A_{Z_{I}}^{-}.$$  \hspace{1cm} (5.6)

If $a_{Z_{I}} \in A_{Z_{I}}^{-}$, it follows from Lemma 5.5. If $a_{Z_{I}} \in A_{Z_{I}}^{-}$, one writes $a_{Z_{I}} \exp(TX) = a_{Z_{I}} \exp(T_{0}X) \exp((T - T_{0})X)$, where $T_{0} > 0$ is such that $a_{Z_{I}} \exp(T_{0}X) \in A_{Z}^{-}$. Then one uses Lemma 5.5 and obtains (5.6).

**5.2 Definition of the constant term of elements of $\mathcal{A}_{\text{temp}}(Z : I)$**

Let us first start by the following general remark:

If an exponential polynomial function of one variable, $P(t)$, with unitary characters, satisfies:

$$\lim_{t \to +\infty} P(t) = 0,$$  \hspace{1cm} (5.7)

then $P \equiv 0$.

We define some linear forms $\eta$ and $\eta_{I}$ on $\mathcal{A}_{\text{temp}}(Z : I)$ by:

$$< \eta, f > = f(z_{0}),$$

$$< \eta_{I}, f > = \hat{f}_{I}(z_{0}t), \quad f \in \mathcal{A}_{\text{temp}}(Z : I).$$

Let us remark that $\eta$ is a continuous linear form on $\mathcal{A}_{\text{temp},N}(Z : I)$ for any $N \in \mathbb{N}$.
5.6 Lemma. Let \( f \in \mathcal{A}_{\text{temp}}(Z : \mathcal{I}) \). Recall that \( m_{\eta_I,f} \) denotes the generalized matrix coefficient \( g \mapsto \eta_I(L(g^{-1})f) \). One has:

\[
m_{\eta_I,f}(a) = a^{\rho_Q} \tilde{f}_I(a), \quad a \in A_I.
\]

Proof. This follows from the definition of \( \eta_I \) and Lemma 5.4 for \( a_I = a \) and \( a_Z = 1 \).

\[ \square \]

5.7 Lemma. Let \( N \in \mathbb{N} \). The linear form \( \eta_I \) is the unique linear form on \( \mathcal{A}_{\text{temp},N}(Z : \mathcal{I}) \) such that:

(i) For any \( f \in \mathcal{A}_{\text{temp},N}(Z : \mathcal{I}) \) and \( X \in a_I^{-} \),

\[
\lim_{T \to \infty} \left( \exp(TX) \right)^{-\rho_Q} \left( m_{\eta_I,f}(\exp(TX)) - m_{\eta_I,f}(\exp(TX)) \right) = 0.
\]

(ii) For any \( f \in \mathcal{A}_{\text{temp},N}(Z : \mathcal{I}) \) and \( X \in a_I \), \( T \mapsto (\exp(TX))^{-\rho_Q} m_{\eta_I,f}(\exp(TX)) \) is an exponential polynomial with unitary characters.

Moreover \( \eta_I \) is continuous on \( \mathcal{A}_{\text{temp},N}(Z : \mathcal{I}) \) and \( H_I \)-invariant.

Proof. The assertion (i) follows from Lemma 5.6 and (5.6). From Lemma 5.6 and (5.4), one gets (ii).

To prove the unicity of such an \( \eta_I \) satisfying (i) and (ii), we use (5.7). If \( \eta'_I \) is another linear form satisfying (i) and (ii), then, for any \( f \in \mathcal{A}_{\text{temp},N}(Z : \mathcal{I}) \),

\[
m_{\eta_I,f}(\exp(TX)) - m_{\eta'_I,f}(\exp(TX)) = 0, \quad X \in a_I^{-}, T \in \mathbb{R}.
\]

This equality applied to \( T = 0 \) implies that \( \eta_I = \eta'_I \).

Let us show the continuity of \( \eta_I \). By taking \( T = 0 \) and \( a_Z = 1 \) in the inequality of Lemma 5.5, one gets:

\[
|f(z_0) - \tilde{f}_I(z_0)| \leq C q(f), \text{ i.e. } | \langle \eta, f \rangle - \langle \eta_I, f \rangle | \leq C q(f).
\]

Moreover \( \eta \) is a continuous map on \( \mathcal{A}_{\text{temp},N}(Z : \mathcal{I}) \). This implies that \( \eta_I \) is continuous on \( \mathcal{A}_{\text{temp},N}(Z : \mathcal{I}) \).

It remains to get that \( \eta_I \) is \( H_I \)-invariant. From (5.6), for any \( X \in a_I^{-} \),

\[
\lim_{T \to \infty} \left( (\exp(TX))^{-\rho_Q} f(\exp(TX)) - \tilde{f}_I(\exp(TX)) \right) = 0.
\]

One applies this to \( L_Y f, Y \in \mathfrak{h}_I \), and gets:

\[
\lim_{T \to \infty} \left( \exp(TX)^{-\rho_Q} L_Y f(\exp(TX)) - (L_Y f)(\exp(TX)) \right) = 0. \tag{5.8}
\]

On the other hand, from Lemma 4.1, one has:

\[
\lim_{T \to \infty} \exp(TX)^{-\rho_Q} L_Y f(\exp(TX)) = 0. \tag{5.9}
\]
Hence, one gets from (5.8) and (5.9) that:

$$\lim_{T \to \infty} (L_Y f)_I(\exp(TX)) = 0.$$ 

But $T \mapsto (L_Y f)_I(\exp(TX))$ is an exponential polynomial with unitary characters (cf. (5.4)). Hence, from (5.7), it is identically equal to 0. This means that:

$$\eta_I(L_Y f) = 0.$$ 

Then $\eta_I$ is continuous and $h_I$-invariant, and hence $H_I$-invariant. \hfill $\square$

Let $N \in \mathbb{N}$ be fixed. For $f \in A_{\text{temp},N}(Z : \mathcal{I})$, let $f_I$ be the function on $Z_I$ defined by:

$$f_I(g \cdot z_0, I) = m_{\eta_I,f}(g), \quad g \in G.$$ 

As $\eta_I$ is an $H_I$-invariant continuous linear form on $A_{\text{temp},N}(Z : \mathcal{I})$ (cf. Lemma 5.7), $f_I$ is well-defined. Moreover,

$$(L_g f)_I = L_g f_I, \quad g \in G.$$ 

5.8 Proposition. Let $f \in A_{\text{temp},N}(Z : \mathcal{I})$. One has that $f_I$ is the unique $C^\infty$ function on $Z_I$ such that, for all $g \in G$:

(i) For $X \in a_{I}^{-}$, $\lim_{T \to \infty}(\exp(TX))^{-\rho_Q}(f(g \exp(TX)) - f_I(g \exp(TX))) = 0$,

(ii) For $X \in a_I$, $T \mapsto (\exp(TX))^{-\rho_Q} f_I(g \exp(TX))$ is an exponential polynomial with unitary characters.

Proof. The Proposition follows immediately from Lemma 5.7 applied to $L_g^{-1} f$, (5.11) and the definition (5.10) of $f_I$. Unicity follows from (5.7). \hfill $\square$

5.9 Lemma. For any $f$ as above:

$$f_I(a_{Z_I}) = a_{Z_I}^{\rho_Q} f_I(a_{Z_I}), \quad a_{Z_I} \in A_{Z_I}^-,$$ 

Proof. This follows from Proposition 5.8 and (5.6). \hfill $\square$

5.10 Lemma. Let $p$ be as in Lemma 5.3. For any $a_{Z_I} \in A_{Z_I}^-$ and $f \in A_{\text{temp},N}(Z : \mathcal{I})$,

$$|f_I(a_{Z_I})| \leq a_{Z_I}^{\rho_Q} p(f)(1 + \|a_{Z_I}\|)^{N + \dim W_I}.$$ 

Proof. The Lemma follows from Lemma 5.3, (5.3) and (5.12). \hfill $\square$
Let \( w_I \in \mathcal{W}_I \) and \( w \in \mathcal{W} \). Set \( H_{I,w_I} = w_I H_I w_I^{-1} \) and \( H_w = w H w^{-1} \). Consider the real spherical spaces \( Z_w = G/H_w \) and \( Z_{I,w_I} = G/H_{I,w_I} \), and put \( z_0^w = H_w \in Z_w \) and \( z_0^{w_I} = H_{I,w_I} \in Z_{I,w_I} \). Then (cf. \cite[Corollary 3.8]{15}) \( Q \) is \( Z_w \)-adapted to \( P \) and \( A_{Z_w} = A_Z \) with \( A_{Z_w}^{-} = A_Z^{-} \).

For \( f \in C^\infty(Z) \), let us define \( f^w \) by:

\[
f^w(g \cdot z_0^w) = f(gw \cdot z_0), \quad g \in G.
\]

In the same way, one defines \( \phi^w \) for \( \phi \in C^\infty(Z_I) \). Then \( f^w \in C^\infty(Z_w) \) and \( \phi^w \in C^\infty(Z_{I,w_I}) \).

**5.11 Proposition.** Let \( w_I \in \mathcal{W}_I \) and \( w \in \mathcal{W} \) be associated to \( w_I \) as in (2.5). Let \( f \in \mathcal{A}_{\text{temp},N}(Z : I) \). Then \( f^w \in \mathcal{A}_{\text{temp},N}(Z_w : I) \) and

\[
(f_I)^{w_I}(a_Z) = (f^w)_I(a_Z), \quad a_Z \in A_Z.
\]

Here \( f^w \in \mathcal{A}_{\text{temp}}(Z_w : I) \), \( (f^w)_I \in C^\infty(Z_w,I) \), \( f_I \in C^\infty(Z_I) \), \( f_{I}^{w_I} \in C^\infty(Z_{I,w_I}) \), and, from \cite[Proposition 3.2(5) and Corollary 3.8]{15}, one has:

\[
A_{Z_{w,I}} = A_{Z_w} = A_Z, \\
A_{Z_{I,w_I}} = A_{Z_I} = A_Z.
\]

Hence both sides of the equality are well-defined on \( A_Z \).

The proof of Proposition 5.11 is prepared by a simple technical lemma. Recall the elements \( a_s = \exp(sX_I) \) for \( X \in \mathfrak{a}_I^{-} \).

**5.12 Lemma.** Let \( (g'_s) \) be a family in \( G \) which converges rapidly to \( g \in G \). Let \( f \in \mathcal{A}_{\text{temp},N}(Z) \). Then there exist \( C > 0 \) and \( \varepsilon > 0 \) such that:

\[
|\langle L(g'_s)^{-1}f \rangle(a_s) - \langle L_{g^{-1}}f \rangle(a_s)\rangle| \leq C a_s^\rho Q e^{-\varepsilon s}, \quad s \geq s_0.
\]

**Proof.** As \( (g'_s) \) converges rapidly to \( g \) when \( s \) tends to \( +\infty \), there exists \( s'_0 \), \( C' \), \( \varepsilon' \) strictly positive and \( (X_s) \subset \mathfrak{g} \) such that, for all \( s \geq s'_0 \),

\[
g'_s = g \exp X_s \quad \text{and} \quad \|X_s\| \leq C' e^{-\varepsilon's}. \tag{5.13}
\]

As \( L_{g^{-1}} \) preserves \( \mathcal{A}_{\text{temp},N}(Z) \), one is reduced to prove, for all \( f \in \mathcal{A}_{\text{temp},N}(Z) \), that there exist \( C, \varepsilon, s_0 > 0 \) such that:

\[
|\langle f(\exp(X_s)a_s) \rangle - \langle f(a_s) \rangle| \leq C a_s^\rho Q e^{-\varepsilon s}.
\]

But, by the mean value Theorem, if \( a \in A_Z \) and \( X \in \mathfrak{g} \),

\[
|\langle f(\exp(X)a) \rangle - \langle f(a) \rangle| \leq \sup_{t \in [0,1]} |L_X f(\exp(tX)a)||X|.
\]
From (5.13), one then sees that it is enough to prove that, if $\|X\|$ is bounded by a constant $C'' > 0$, there exists a constant $C''' > 0$ such that:

$$
\sup_{r \in [0,1]} |L_X f(\exp(tX)a)| \leq C''' a^{\rho q} (1 + \|a\|)^N, \quad a \in A_Z.
$$

(5.14)

Decomposing $X$ in a basis $(X_i)$ of $\mathfrak{g}$ and using the continuity of the endomorphisms $L_X$, of $A_{\text{temp},N}(Z)$, one sees that there exists a continuous semi-norm such that:

$$
|(L_X f)(a)| \leq q(f) a^{\rho q} (1 + \|a\|)^N, \quad a \in A_{\overline{Z}}.
$$

But $f \mapsto \sup_{\|X\| \leq C''} q(L_{\exp(-tX)} f)$ is a continuous semi-norm on $A_{\text{temp},N}(Z)$. Hence, as $L_X$ and $L_{\exp(-tX)}$ commute, (5.14) follows. This achieves to prove the Lemma.

\[\square\]

**Proof of Proposition 5.11.** If $a \in A$, one has:

$$
((L_a f)^{w})_I = (L_a (f^w))_I \text{ as } (L_a f)^w = L_a f^w.
$$

Hence it is enough to prove the identity of the Proposition for $a_Z = z_0$. Then, using (5.7) and Proposition 5.8, it is enough to prove that $s \mapsto (f_I)^{w_I}(a_s)$ is an exponential polynomial with unitary characters satisfying:

$$
\lim_{s \to +\infty} a_s^{-\rho q} (f^w(a_s) - (f_I)^{w_I}(a_s)) = 0.
$$

(5.15)

But from (2.7),

$$
\tilde{a}_s w \cdot z_0 = (\tilde{a}_s \tilde{b}_s^{-1} m_s^{-1} u_s^{-1})(u_s m_s \tilde{b}_s w) \cdot z_0 = g_s w_I \tilde{a}_s \cdot z_0
$$

for $s \geq s_0$, where $g_s = \tilde{a}_s \tilde{b}_s^{-1} m_s^{-1} u_s^{-1}$. Then one has:

$$
f^w(a_s) = L_{w_I^{-1} g_s^{-1}} f(a_s).
$$

On the other hand, from [15, Lemma 3.5] for $Z = Z_I$, as $A_{Z_I,E} = A_I$ (cf. loc.cit. equation (3.13)), one has:

$$
\tilde{a}_s w_I \cdot z_{0,I} = w_I \tilde{a}_s \cdot z_{0,I},
$$

(5.16)

which implies that:

$$
(L_{w_I^{-1} f_I})(\tilde{a}_s \cdot z_{0,I}) = (f_I)^{w_I}(a_s).
$$

(5.17)

Now, according to Proposition 2.1 – this is the key ingredient! – the sequence $(g_s w_I)$ converges rapidly to $w_I$. Hence we can apply Lemma 5.12 with $g_s' = g_s w_I$ and find $C', \varepsilon', s'_0 > 0$ such that:

$$
a_s^{-\rho q} |(L_{w_I^{-1} g_s^{-1}} f)(a_s) - (L_{w_I^{-1} f_I})(a_s)| \leq C' e^{-\varepsilon' s}, \quad s \geq s'_0.
$$

(5.18)

Using Lemmas 5.5 and 5.9, one has, for some $C'', \varepsilon' > 0$,

$$
|a_s^{-\rho q} \left( (L_{w_I^{-1} f})(a_s) - (L_{w_I^{-1} f_I})(a_s) \right) | \leq C'' e^{-\varepsilon' s}, \quad s \geq s'_0.
$$

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Hence from (5.17) and (5.18), one deduces (5.15). It remains to prove that:

\[ s \mapsto (f_I)^{w_I}(a_s) = f_I(a_s w_I \cdot z_{0,I}) \]

is an exponential polynomial with unitary characters. But, from [15, Lemma 3.5] applied to \( Z_I \),

\[ (f_I)^{w_I}(a_s) = f_I(w_I a_s). \]

Hence our claim follows from (5.16). This achieves the proof of the Proposition.

5.13 Remark. Proposition 5.11 was obtained before in [15, Lemma 9.7] without the assumption of wavefront but via different, more elaborate and less geometric techniques.

Before stating the next Theorem, we recall, from Proposition 5.8, that, for \( f \in A_{\text{temp}}(Z) \), \( f_I \) is the unique \( C^\infty \) function on \( Z_I \) such that, for all \( X \in a_I^- \) and \( g \in G \),

\[
\lim_{T \to \infty} (\exp(TX))^{-\rho_Q} (f(g \exp(TX)) - f_I(g \exp(TX))) = 0 \tag{5.19}
\]

and

\[
T \mapsto (\exp TX)^{-\rho_Q} f_I(g \exp(TX)) \text{ is an exponential polynomial with unitary characters.} \tag{5.20}
\]

We see that, using Lemma 5.2 for \( \lambda \in Q^0_I \), one can replace (5.20) by the stronger condition:

\[
X \mapsto (\exp X)^{-\rho_Q} f_I(g \exp X) \text{ is an exponential polynomial on } a_I
\]

with unitary characters.

Let \( f \in A_{\text{temp}}(Z) \). Then \( f \in A_{\text{temp},N}(Z : I) \) for some \( N \in \mathbb{N} \) and some finite co-dimensional ideal \( I \) in \( Z(g) \). Hence we can define \( f_I \) as above.

5.14 Proposition. With \( f \in A_{\text{temp}}(Z) \) as above, one has that \( f_I \) does not depend on \( N \) and \( I \).

Proof. This follows from the characterization of \( f_I \) above (see (5.19) and (5.20)).

From this Proposition, we can define a linear form, still denoted \( \eta_I \), on \( A_{\text{temp}}(Z) \), by \( f \mapsto f_I(z_{0,I}) \).

5.15 Theorem.

(i) With \( N_Z = \dim W_Z \) as in Lemma 5.10, for all \( N \in \mathbb{N} \), the map \( f \mapsto f_I \) is a continuous linear map from \( A_{\text{temp},N}(Z : I) \) to \( A_{\text{temp},N + N_Z}(Z_I : I) \).

(ii) Let \( N \in \mathbb{N} \) be a compact subset of \( a_I^- \) and \( \Omega' \) be a compact subset of \( G \). Let \( w_I \in W_I \) and associated to it \( w \in W \) as in (2.5). Then there exist \( \varepsilon > 0 \) and a continuous semi-norm \( p \) on \( C^\infty_{\text{temp},N}(Z) \) such that, for all \( f \in A_{\text{temp},N}(Z : I) \),

\[
|(a_Z \exp(TX))^{-\rho_Q} (f(\omega' a_Z \exp(TX)w \cdot z_0) - f_I(\omega' a_Z \exp(TX)w_I \cdot z_{0,I}))| \\
\leq e^{-\varepsilon T} p(f)(1 + \| \log a_Z \|)^N, \quad a_Z \in A^c_Z, X \in \mathcal{C}, \omega' \in \Omega', T \geq 0.
\]
Proof. In view of (3.5), to get (i), it is enough to prove that, for any \( w \in \mathcal{W}_I \) and any compact subset \( \Omega' \) of \( G \), there exists a continuous semi-norm \( p \) on \( \mathcal{A}_{\text{temp},N}(Z : I) \) such that:

\[
\sup_{\omega \in \Omega', a \in \mathcal{A}^*_Z} |a^{-\rho q}(1 + \log |a|)^{-\left(N + N_I\right)} f_I(\omega a w_I)| \leq p(f), \quad f \in \mathcal{A}_{\text{temp},N}(Z : I).
\]

Using (3.6), one is reduced to prove this for \( \Omega' \) reduced to 1. For \( w = 1 \), one can take \( w = 1 \) and our claim follows from Lemma 5.10. For general \( w \), one uses Proposition 5.11 to get \( (L_{a w_I} f_I)(a Z_I w_I) = (f^w)_I(a Z_I) \) and the above inequality for \( H^w \) instead of \( H \). This shows (i).

One reduces easily to prove (ii) for \( \Omega' = \{1\} \), by using (3.6). Then, using Proposition 5.11, one is reduced to prove (ii) with \( \Omega' = \{1\} \) and \( w_I = w = m_{w_I} = 1 \) by changing \( H \) into \( H_w \). In that case, (ii) follows from Lemma 5.1. \( \square \)

5.3 Constant term of tempered \( H \)-fixed linear forms

Let \( I \) be a subset of \( S \).

5.16 Proposition. Let \( (\pi, V^\infty) \) be an SAF-representation of \( G \). If \( \xi \) is a \( Z \)-tempered continuous linear form on \( V^\infty \), then there exists a unique \( Z_I \)-tempered continuous linear form \( \xi_I \) on \( V^\infty \) such that:

(i) \( \lim_{T \to \infty} (\exp(TX))^{-\rho q} (m_{\xi,v}(\exp(TX)) - m_{\xi_I,v'}(\exp(TX))) = 0, \quad v \in V^\infty, X \in a_I^- \).

(ii) For any \( v \in V^\infty \) and \( X \in a_I, \ T \mapsto (\exp(TX))^{-\rho q} m_{\xi_I,v}(\exp(TX)) \) is an exponential polynomial with unitary characters.

Proof. Let

\[
< \xi_I, v > := (m_{\xi,v})_I(z_{0,I}), \quad v \in V^\infty.
\]

Then

\[
m_{\xi_I,v}(g) = < \xi_I, \pi(g^{-1})v > = (m_{\xi,\pi(g^{-1})v})_I(z_{0,I}) = (L_{g^{-1}} m_{\xi,v})_I(z_{0,I}).
\]

As \( f \mapsto f_I \) is a \( G \)-morphism (cf. Theorem 5.15), one then obtains that:

\[
m_{\xi_I,v}(g \cdot z_{0,I}) = (m_{\xi,v})_I(g \cdot z_{0,I}).
\]

From the properties of \( (m_{\xi,v})_I \), one sees that (ii) is satisfied. Furthermore, from Theorem 5.15, one sees that \( (m_{\xi,v})_I \in \mathcal{A}_{\text{temp},N}(Z_I) \) for some integer \( N \). Hence \( \xi_I \) satisfies the required properties. Unicity is clear using (5.7). \( \square \)
5.4 Application to the relative discrete series for $Z$

As $Z$ is wavefront and $\rho_q \in a_Z$, one has $\rho_q|_{a_Z} \leq 0$. Hence $\rho_q|a_{Z,E} = 0$.

Let $\chi$ be a unitary character of $A_{Z,E}$, we recall that, if $a \in A_{Z,E}$ and $w \in \mathcal{W}$, $\tilde{aw} = wa$ (cf. [15, Lemma 3.5]). This implies that $\tilde{aw} w^{-1}$ only depends on $a$ and thus will be denoted $aH_w$. As $A_{Z,E}$ normalizes $H$, there is a right action $(a, z) \mapsto z \cdot a$ of $A_{Z,E}$ on $Z$. Let $C^\infty(Z, \chi)$ be the space of $C^\infty$ functions on $Z$ such that:

$$f(z \cdot a) = \chi(a) f(z), \quad a \in A_{Z,E}, z \in Z.$$ 

If $f \in C^\infty(Z, \chi)$, $u \in \mathcal{U}(g)$ and $N \in \mathbb{N}$, let

$$r_{N,u}(f) = \sup_{\omega \in \Omega, a \in A_Z/A_{Z,E}, w \in \mathcal{W}} \left| a^{-\rho_q} (1 + \| a \|)^N (L_u f)(\omega a w) \right|,$$

and we define:

$$\mathcal{C}(Z, \chi) = \{ f \in C^\infty(Z, \chi) : r_{N,u}(f) < \infty, N \in \mathbb{N}, u \in \mathcal{U}(g) \}.$$

Since $A_{Z,E}$ normalizes $H$, we obtain a closed subgroup $\hat{H} := HA_{Z,E}$ and a real spherical space $\hat{Z} = G/\hat{H}$. If $\chi$ is a character of $A_{Z,E}$, we extend it trivially to $H$ on a character of $\hat{H}$ still denoted by $\chi$. Let us define $L^2(\hat{Z}; \chi)$ as in [15, Section 8.1]. Let $w \in \mathcal{W}$. We recall that $H_w = w H w^{-1}$ and $Z_w = G/H_w$. Let $f$ be in $C^\infty(Z, \chi)$. Recall that $f_w$ defined by $f_w(g) = f(gwH)$, $g \in G$, is right $H_w$-invariant and defines an element of $C^\infty(Z_w)$ and even of $C^\infty(Z_w, \chi)$ by using the relation above. This element will still be denoted $f_w$. Moreover, by “transport de structure”, if $f$ is $Z$-tempered, $f_w$ is $Z_w$-tempered.

Let $\eta$ be a $Z$-tempered $H$-fixed linear form on $V^\infty$. Let $w \in \mathcal{W}$. Then $a_{Z_w} = a_Z$, and $w\eta$ is fixed by $H_w$ and is $Z_w$-tempered by “transport de structure”. By [15, Corollary 3.8], $Q$ is $Z_w$-adapted to $P$. Moreover, the set of (simple) spherical roots for $Z_w$ is equal to $S$ (see [15, equation (3.2), definition of $S$ in the beginning of Section 3.2 and Lemma 3.7]). Hence one can define $(w\eta)_f$, $w \in \mathcal{W}$.

5.17 Theorem. Let $(\pi, V^\infty)$ be an SAF-representation of $G$ and $\eta$ be a $Z$-tempered continuous linear form on $V^\infty$ which transforms under a unitary character $\chi$ of $A_{Z,E}$. Then the following assertions are equivalent:

(i) For all $v \in (V^\infty)_{(\pi)}$, $m_{\eta,v} \in L^2(\hat{Z}; \chi)$.

(ii) For all proper subsets $I$ of $S$ and $w \in \mathcal{W}$, $(w\eta)_I = 0$.

(iii) For all $v \in V^\infty$, $m_{\eta,v} \in \mathcal{C}(Z, \chi)$.

Proof. Let us assume (i). Let $S = \{ \sigma_1, \ldots, \sigma_s \}$ and $\omega_1, \ldots, \omega_s \in a_Z$ be such that:

$$\sigma_i(\omega_j) = \delta_{i,j}, \quad i, j = 1, \ldots, s$$

$$\omega_i \perp a_{Z,E}, \quad i = 1, \ldots, s.$$
Here we use the scalar product on $\mathfrak{a}_Z$ defined before (1.8). From [15, Theorem 8.5], the linear form $\Lambda_{V,\eta}$ on $\mathfrak{a}_Z$, defined in loc.cit. (6.20), satisfies
\[(\Lambda_{V,\eta} - \rho_Q)(\omega_j) > 0, \quad j = 1, \ldots, s.\] (5.21)
Then it follows from loc.cit. Theorem 7.6 used for a fixed $X \in \mathfrak{a}_Z^-$ of norm 1, $\Omega = \{\exp(-X)\}$, $w = 1$ and $t = 1$, that there exists a $d \in \mathbb{N}$ and a continuous semi-norm $p$ on $V^\infty$ such that:
\[|m_{\eta,v}(a)| \leq p(v)a^{\Lambda_{V,\eta}}(1 + \|\log a\|)^d, \quad a \in A_Z^-, v \in V,\] (5.22)
where $V = (V^\infty)(K)$. Let $I$ be a proper subset of $S$ and $X_I = -\sum_{i \text{ s.t. } \sigma_i \in S \setminus I} \omega_i \in \mathfrak{a}_T^-$. From (5.21), one deduces that one can choose $\beta > 0$ such that:
\[(\Lambda_{V,\eta} - \rho_Q)(X_I) < -\beta.\]
Hence, one deduces from (5.22) that, for each $v \in V$,
\[|m_{\eta,v}(\exp(tX_I))| \leq p(v)e^{t\rho_Q(X_I)}(1 + t\|X_I\|)^d e^{-t\beta}, \quad t \geq 0.\]
As $\beta > 0$, this implies that:
\[\lim_{t \to +\infty} (\exp(tX_I))^{-\rho_Q}m_{\eta,v}(\exp(tX_I)) = 0.\]
From the definition of the constant term $\eta_I$ of $\eta$ (cf. Proposition 5.16) and from (5.7), one deduces $\eta_I(v) = 0$ for any $v \in V$. As $\eta_I$ is continuous on $V^\infty$ and $V$ is dense in $V^\infty$, one concludes $\eta_I = 0$. We get (ii) for $w = 1$. For general $w$, (ii) is obtained by applying the same arguments to $Z_w$. This achieves to prove that (i) implies (ii).
Let us assume that (ii) holds. Let $I$ be an ideal of $\mathcal{Z}(\mathfrak{g})$ which annihilates $V$ or $V^\infty$. It is of finite co-dimension. Since $\eta$ is $Z$-tempered, there exists $N_0 \in \mathbb{N}$ such that, for all $v \in V^\infty$, $m_{\eta,v} \in A_{\temp,N_0}(Z : I)$ (cf. (3.8)). Let $v \in V^\infty$ and set $f = m_{\eta,v}$. Then one can apply Theorem 5.15 to $Z_w$ and $f_w$ for $w$ equals to 1. Let $I \subseteq S$. Let $\mathcal{C}$ be a compact subset of $\mathfrak{a}_T^-$, $\Omega_1$ be a compact subset of $G$ and $u \in \mathcal{U}(\mathfrak{g})$. Hence there exists a continuous semi-norm $p$ on $C^\infty_{\temp,N_0}(Z)$, $\varepsilon > 0$ such that:
\[|(a_Z \exp(TX))^{-\rho_Q}(L_u f)(\omega a_Z \exp(TX)w \cdot z_0)| \leq e^{-\varepsilon T}p(f)(1 + \|\log a_Z\|)^N, \quad a_Z \in A_Z^E/A_{Z,E}, X \in \mathcal{C}, \omega \in \Omega_1, w \in \mathcal{W}, T \geq 0.\] (5.23)
From this, we will deduce that $f \in \mathcal{C}(Z, \chi)$. Let $S_1$ be the unit sphere on $a_Z/a_{Z,E}$ and let $X_0 \in S_1 \cap a_Z/a_{Z,E}$. Let $\Omega_0$ be an open neighborhood of $X_0$ in $S_1 \cap a_Z/a_{Z,E}$ such that, for all $X \in \Omega_0$, $\alpha(X) \leq \alpha(X_0)/2$, $\alpha \in S$. Let $I$ be the set of $\alpha \in S$ such that $\alpha(X_0) = 0$. One has $I \neq S$. Then one has $X_0 \in \mathfrak{a}_T^-$. Let $Y \in \Omega_0$ and $t \geq 0$. Then $t(Y - X_0/2) \in a_Z^-$ and $\exp(tY) = \exp t(Y - X_0/2) \exp(tX_0/2)$. Using (5.23) for $X = X_0/2$, $a_Z = \exp t(Y - X_0/2)$ and $T = t$, one gets:
\[|(\exp(tY))^{-\rho_Q}(L_u f)(\omega \exp(tY)w \cdot z_0)| \leq e^{-\varepsilon T}p(f)(1 + t\|Y - X_0/2\|)^N \leq c(1 + t)^N, \quad Y \in \Omega_0, \omega \in \Omega_1, w \in \mathcal{W}, t \geq 0,\]
for some $c > 0$ and any $N \in \mathbb{N}$. One deduces easily from this that, for any $u \in U(g)$ and $N \in \mathbb{N}$:

$$\sup_{\omega \in \Omega, a \in \mathcal{W}, \omega \exp(R + \Omega_0)} a^{-\rho a} (1 + \|a\|)^{N} |(L_u f)(\omega a w \cdot z_0)| < +\infty.$$ 

Using a finite covering of the compact set $S_1 \cap a Z_1/a Z, E$, one deduces from this that $f \in C(Z, \chi)$. This achieves to prove that (ii) implies (iii).

To prove that (iii) implies (i), one proceeds as in the proof that (ii) implies (i) in [15, Theorem 8.5].

\[\square\]

### 6 Transitivity of the constant term

Let us notice that if $Z$ is wavefront, then, for $J \subset S$, $Z_J = G/H_J$ is not necessarily wavefront. However the (non-unimodular) space $\hat{Z}_J = G/H_J A_J$ is wavefront. This allows us to define the constant term $f_I$ for $I \subset J$ and $f \in \mathcal{A}_{\text{temp}}(Z_J)$ in a similar manner as before.

In more detail, the characterization of $f_I$ will be given by the analogue of Proposition 5.8, say Proposition 5.8', with $Z$ changed in $Z_J$ and $a_I^-$ changed in $a_{I,J}^- = \{X \in a_I : \alpha(X) < 0, \alpha \in \epsilon J \mid I\}$. One has also an analogue of Theorem 5.15 (say Theorem 5.15'). To see this, one gets the analogues of Lemmas 4.1 and 4.5 where $Z$ is changed in $Z_J$, $a_I^-$ in $a_{I,J}^-$ and $\beta_I$ is changed in $\beta_{I,J}$ with:

$$\beta_{I,J}(X) = \max_{\alpha \in \epsilon J \mid I} \alpha(X), \quad X \in a_{I,J}^-.$$ 

In the proof one changes $\alpha + \beta \notin \langle I \rangle$ by $\alpha + \beta \in \langle J \rangle, \alpha + \beta \notin \langle I \rangle$. The rest of the proof is then entirely similar to the proof of Proposition 5.8 and Theorem 5.15. Let us notice that here we use Proposition 2.1 for a non wavefront spherical space.

### 6.1 Proposition. Let $I \subset J$ be two subsets of $S$. Then, if $f \in \mathcal{A}_{\text{temp}}(Z)$,

$$f_I = (f_J)_I.$$

**Proof.** By $G$-equivariance of the maps:

$$\mathcal{A}_{\text{temp}}(Z) \rightarrow \mathcal{A}_{\text{temp}}(Z_I) \quad \text{and} \quad \mathcal{A}_{\text{temp}}(Z_J) \rightarrow \mathcal{A}_{\text{temp}}(Z_I),$$

it is enough to show that, if $f \in \mathcal{A}_{\text{temp}}(Z)$, $f_I(z_{0,I}) = (f_J)_I(z_{0,I})$. Recall that $a_{Z_J} = a_{Z}$ and

$$a_I^- = \{X \in a_I : \alpha(X) < 0, \alpha \in \epsilon S \mid I\}, \quad a_{I,J}^- = \{X \in a_I : \alpha(X) < 0, \alpha \in \epsilon J \mid I\}.$$ 

As $a_I = \{X \in a_Z : \alpha(X) = 0, \alpha \in \epsilon I\}$ and $a_J = \{X \in a_Z : \alpha(X) = 0, \alpha \in \epsilon J\}$, one has:

$$a_J \subset a_I, \quad a_I^- \subset a_Z^-, \quad a_{I,J}^- \subset a_{Z}^-.$$ 

One remarks that $a_I^- \subset a_{I,J}^-$. Let $X \in a_I^-$ and $Y \in a_I^-$. Then $X + Y \in a_I^-$. 44
Using Theorem 5.15(ii) applied successively to \((Z, I, f, X+Y, 1)\) and \((Z, J, f, X, \exp(TY))\) instead of \((Z, I, f, X, aZ)\), and finally the analogue Theorem 5.15' (ii) of Theorem 5.15(ii) for \((Z, I, f, J, Y, \exp(TX))\), one gets that there exist \(C > 0\) and \(\varepsilon > 0\) such that, for all \(T \geq 0\),

\[
\begin{align*}
\alpha_T \| f(\exp(T(X + Y))) - f_I(\exp(T(X + Y))) \| & \leq C e^{-\varepsilon T} \\
\alpha_T \| f(\exp(TY) \exp(TX)) - f_J(\exp(TY) \exp(TX)) \| & \leq C e^{-\varepsilon T} \\
\alpha_T \| f_J(\exp(TX) \exp(TY)) - (f_J)_I(\exp(TX) \exp(TY)) \| & \leq C e^{-\varepsilon T},
\end{align*}
\]

where \(\alpha_T = e^{-T \rho_Q(X+Y)}\). Hence one concludes from the three inequalities above that:

\[
\alpha_T \| f_I(\exp(T(X + Y))) - (f_J)_I(\exp(T(X + Y))) \| \leq 3C e^{-\varepsilon T}, \quad T \geq 0.
\]

Hence \(\alpha_T f_I(\exp(T(X + Y))) - \alpha_T (f_J)_I(\exp(T(X + Y)))\) tends to zero when \(T\) goes to \(+\infty\). But each term of this difference is an exponential polynomial in \(T\) with unitary characters. Hence, according to (5.7), the difference of the two occurring exponential polynomials is identically zero. It implies, taking \(T = 0\), that \(f_I(z_0,J) = (f_J)_I(z_0,J)\). \(\square\)

### 7 Uniform estimates

The goal of this section is to obtain a parameter independent version of the main result Theorem 5.15: the bounds become uniform if we restrict ourselves to ideals \(\mathcal{I}\) of co-dimension one. The crucial ingredient is a recent result that infinitesimal characters of tempered representations have integral real parts (see [16] and summarized in Lemma 7.4 below.)

We recall the maximal torus \(t \subset \mathfrak{m}\) and \(j = it \oplus \mathfrak{a}\). Note that \(j_{\mathbb{C}}\) is a complex Cartan subalgebra of \(\mathfrak{g}_{\mathbb{C}}\).

For \(\Lambda \in j_{\mathbb{C}}^*\) we let \(\mathcal{I}_{\Lambda}\) be the kernel of the character \(\chi_{\Lambda}\) of \(Z(\mathfrak{g})\) given by the composition of the Harish-Chandra isomorphism from \(Z(\mathfrak{g})\) onto \(S(j_{\mathbb{C}})W_{(\mathfrak{g},\mathfrak{c})}\) with the evaluation at \(\Lambda\). Likewise we write \(W_\Lambda := W_{\mathcal{I}_{\Lambda}}\) and \(Q_\Lambda \subset \mathfrak{a}_{\mathfrak{I}_{\mathbb{C}}}^*\) for the spectrum of the \(\mathfrak{a}_{\mathfrak{I}}\)-module \(W_\Lambda^*\) under the representation

\[
\mathfrak{a}_{\mathfrak{I}} \to \text{End}(W_\Lambda^*), \quad X \mapsto \Gamma_\Lambda(X) := \rho_\Lambda(X)
\]

(see (4.10) and after (4.15) for the definition of \(\rho_\Lambda\)). Notice we identified \(W_\Lambda\) with the fixed space \(W\) in Subsection 4.2.

The section \(s\) we use in the sequel is the one where we identify \(\mathfrak{a}_{\mathfrak{I}}\) with the subspace \(\mathfrak{a}_{\mathfrak{I}}^{\perp \mathcal{L}} \subset \mathfrak{a}_{\mathfrak{L}}\), the orthogonal being taken with respect to the form \(B\) introduced at the beginning of Subsection 1.2. Let \(\mathcal{J}(\mathbb{C}) \subset \mathcal{G}(\mathbb{C})\) be the Cartan subgroup with Lie algebra \(j_{\mathbb{C}}\) and \(\mathcal{L} := \text{Hom}(\mathcal{J}(\mathbb{C}), \mathbb{C}^*)\) be its character group. In the sequel we identify \(\mathcal{L}\) with a lattice in \(j^*\).

We call a subspace \(U \subset j^*\) rational provided that \(U = \mathbb{R}(U \cap \mathcal{L})\). Likewise we call a discrete subgroup \(\Gamma \subset (j^*,+)\) rational if \(\Gamma = \Gamma \cap \mathbb{Q}\mathcal{L}\). Using the dual lattice \(\mathcal{L}^\vee \subset j\) we obtain a notion of rationality for subspaces and discrete subgroups of \(j\) as well.

Finally we may and will request that \(B|_{j^{\times}_{j=1}}\) is rational, i.e. with respect to a basis of \(j\) which lies in \(\mathcal{L}^\vee\), the matrix entries are rational.
7.1 Lemma. The following subspaces of $\mathfrak{g}$ are all rational: $\mathfrak{a}_H, \mathfrak{a}_Z$ and $\mathfrak{a}_I$ for $I \subset S$.

Proof. The subspace $\mathfrak{a}_H$ is rational as it corresponds to the Lie algebra of the sub-torus $(A_I \cap H)_{\mathbf{0}} \subset J$. Since the form $B_{\mid \mathfrak{a}_H}$ is rational, we obtain that $\mathfrak{a}_Z \subset \mathfrak{a} \subset \mathfrak{g}$ is rational as well. Finally we recall that $S \subset \mathbb{Q}L$ and this gives us the rationality of $\mathfrak{a}_I$ for any $I \subset S$. □

We recall that

$$\mathcal{Q}_\lambda = \{-w\Lambda\mid \mathfrak{a}_I \mid w \in W(\mathfrak{g}_C, \mathbf{i}_C)\} \quad (7.1)$$

where we identify $\mathfrak{a}_I$ as a subset of $\mathfrak{a}$ as above. For $\lambda \in \mathcal{Q}_\lambda$ we recall the projectors $E_{\lambda} : W_{\Lambda,\lambda} \rightarrow W_{\Lambda,\lambda}^*$ to the generalized common eigenspace along the complementary generalized eigenspaces.

In the sequel we abbreviate and write $A_{\text{temp}}(Z : \Lambda) : \mathfrak{a}_Z$ instead of $A_{\text{temp}}(Z : I_{\Lambda})$.

The key to obtain uniform estimates for the constant term approximation is at the core related to polynomial bounds for the truncating spectral projections $E_{\lambda}$.

7.2 Proposition. There exist constants $C, N > 0$ such that for all $\Lambda \in \mathfrak{g}_C^*$ with $A_{\text{temp}}(Z : \Lambda) \neq \{0\}$ one has

$$\|E_{\lambda}\| \leq C(1 + \|\Lambda\|)^N \quad (\lambda \in \mathcal{Q}_\lambda).$$

The proof of the proposition is preceded by two lemmas:

7.3 Lemma. Let $0 < \nu \leq 1$, $N \in \mathbb{N}$ and $A \in \text{Mat}_N(\mathbb{C})$ with $\text{Spec}(A) = \{\lambda_1, \ldots, \lambda_r\}$ such that $\text{Re} \lambda_1 \leq \ldots \leq \text{Re} \lambda_r$. For every $1 \leq j \leq r$ let $V_j \subset \mathbb{C}^n$ be the generalized eigenspace of $A$ associated to the eigenvalue $\lambda_j$. For every $1 \leq k \leq r$ we let $E_k = \bigoplus_{j=k+1}^r V_j$ and $P_k : C^N \rightarrow E_k$ be the projection along $\bigoplus_{j=k+1}^r V_j$. Suppose for some $1 \leq k \leq r-1$ that $\text{Re} \lambda_{k+1} - \text{Re} \lambda_k \geq \nu$. Then there exists a constant $C = C(\nu, N) > 0$ such that

$$\|P_k\| \leq C(1 + \|A\|)^N.$$  

Proof. [17, Lemma 6.4]. □

7.4 Lemma. There exists a $W(\mathfrak{g}_C, \mathbf{i}_C)$-stable rational lattice $\Xi_Z$ in the vector space $\mathfrak{g}_C^*$ such that

$$\text{Re} \Lambda \in \Xi_Z \quad (7.2)$$

for all $\lambda \in \mathfrak{g}_C^*$ with $A_{\text{temp}}(Z : \Lambda) \neq \{0\}$.

Proof. Let $0 \neq f \in A_{\text{temp}}(Z : \Lambda)$ be a $K$-finite element which generates an irreducible Harish-Chandra module, say $V$. According to [15, Theorem 9.11], $V$ embeds into a twisted discrete series of some $L^2(G/H_I)$. Now we apply [16, Theorem 1.1] and obtain a $W(\mathfrak{g}_C, \mathbf{i}_C)$-invariant lattice $\Xi_Z$, called $\Lambda_Z$ in [16], with property (7.2). The lattice is indeed rational by [16, Theorem 8.3], combined with [16, Lemma 3.4]. □
Proof of Proposition 7.2. According to Lemma 7.1, \( a_I \) is a rational subspace of \( a \subset j \). Now we keep in mind the following general fact: if \( U \subset j \) is a rational subspace and \( \Xi \subset j^* \) is a rational lattice, then \( \Xi|_U \) is a rational lattice in \( U^* \). In particular, it follows that \( \Xi_{Z,I} := \Xi|_{a_I^*} \) is a lattice in \( a_I^* \). Next observe that Lemma 7.4 combined with (7.1) implies that \( \text{Re} \mathcal{Q}_\Lambda \subset \Xi_{Z,I} \) for all tempered infinitesimal characters \( \Lambda \). Denote by \( \Xi_{Z,I}^\vee \) the dual lattice. Since \( a_I^* \) is a rational cone, we find elements \( X_1, \ldots, X_k \) of \( a_I^* \) such that

\[ a_I^* = \sum_{j=1}^k \mathbb{R}_{\geq 0}X_j. \]

Let \( \lambda \in \mathcal{Q}_\Lambda \). We define matrices \( A_i := \Gamma_\Lambda(X_i) \) and write \( E_{\lambda,i} \) for the spectral projection to the generalized eigenspace of \( A_i \) with eigenvalue \( \lambda(X_i) \). Since the matrices \( A_i \) commute with each other and the \( X_i \) span \( a_I \) we obtain that

\[ E_{\lambda} = E_{\lambda,1} \circ \ldots \circ E_{\lambda,k}. \]  

Hence we are reduced to prove a polynomial bound for each \( E_{\lambda,i} \). As \( \text{Spec}(A_i) = -(W(\mathfrak{g}_C, j_C)\Lambda)(X_i) \) we get \( \text{Re} \text{Spec}(A_i) \subset \mathbb{Z} \). Hence we can apply Lemma 7.3 to the matrices \( A_i \) with \( \nu = 1 \) and obtain \( \| E_{\lambda,i} \| \leq C(1 + \| \Lambda \|)^N \). Now we recall from Subsection 4.2 (see the paragraph around (4.16) ) that

\[ \| \Gamma_\Lambda(X) \| \leq C\| X \|(1 + \| \Lambda \|)^N \]

after possible enlargement of \( C \) and \( N \). This gives the asserted norm bound for \( \| E_{\lambda,i} \| \) and then for \( E_{\lambda} \) via (7.3).

For \( \lambda \in \mathcal{Q}_\Lambda \) we recall the notation

\[ E_{\lambda}(X) = e^{-\lambda(X)}E_{\lambda}(e^{\Gamma_\Lambda(X)}), \quad X \in a_I \]

and recall from Lemma 4.3(ii) the starting identity

\[ \Phi_{f,\lambda}(a_Z \exp(TX_I)) = e^{TT_\Lambda(X_I)}\Phi_{f,\lambda}(a_Z) \]

\[ + \int_0^1 E_{\lambda}e^{(T-t)\Gamma_\Lambda(X_I)}\Psi_{f,X}(a_Z \exp(tX_I)) \, dt, \]

\[ a_Z \in A_{Z,I}, X_I \in a_I. \]

7.5 Lemma. Let \( X \in a_I \). There exist a continuous semi-norm \( q \) on \( C_{\text{temp},N}^\infty(Z) \) and \( m \in \mathbb{N} \) such that, for all \( f \in A_{\text{temp},N}(Z : \Lambda), \Lambda \in j_C^* \),

\[ \| \Psi_{f,X}(a_Z \exp X_I) \| \leq q(f)(1 + \| \log a_Z \|)^N(1 + \| X_I \|)^N(1 + \| \Lambda \|)^m, \quad a_Z \in A_{Z,I}^{-}, X_I \in a_I^{-}. \]

Proof. The proof is the same than the proof of Lemma 4.5(ii), the factor \( (1 + \| \Lambda \|)^m \) coming from (4.16).
Having said all that, it is now clear that all bounds from Sections 4 and 5 become uniform at the cost of an extra polynomial factor in $\|\Lambda\|$. Polynomial behavior in $\|\Lambda\|$ can be subsumed in raising the Sobolev order of the corresponding norms. In more detail, if $p$ is a continuous norm on an $SF$-module $V^\infty$ with infinitesimal character $\Lambda$, then there exists $C > 0, k \in \mathbb{N}$ independent of $p$ and $V$ such that $(1 + \|\Lambda\|)p(v) \leq C p_k(v)$ for all $v \in V$ where $p_k$ denotes the $k$-th Sobolev norm of $p$ with respect to a fixed basis of $\mathfrak{g}$. This simply follows from the fact that

$$|\chi_\Lambda(z)|p(v) = p(zv) \leq C_z p_{\deg z}(v), \quad v \in V^\infty$$

for all $z \in \mathcal{Z}(\mathfrak{g})$ and constants $C_z > 0$. We only have to test against finitely many $z$, namely a choice of Chevalley generators of the polynomial algebra $\mathcal{Z}(\mathfrak{g}) \simeq S(j_{\mathbb{C}})^W(g_{\mathbb{C}}; l) \simeq S(j_{\mathbb{C}})$ and the maximal degree of such a generator will serve as $k$.

The preceding reasoning now implies the following parameter independent version of Theorem 5.15:

**7.6 Theorem.** Let $N \in \mathbb{N}$, $C$ be a compact subset of $\mathfrak{a}^-\mathfrak{t}$ and $\Omega'$ be a compact subset of $G$. Let $w_1 \in \mathcal{W}_1$ and associated to that $w \in \mathcal{W}$ as in Section 2. Then there exist $\varepsilon > 0$ and a continuous semi-norm $p$ on $C_{\text{temp},N}(Z)$ such that, for all $f \in \mathcal{A}_{\text{temp},N}(Z : \Lambda), \Lambda \in i^*_C$:

$$(a_Z \exp(TX))^{-\rho q} |f(\omega' a_Z \exp(TX) - z_{0,l}) - f(\omega' a_Z \exp(TX) w_1) - z_{0,l}|$$

$$\leq e^{-\varepsilon T} p(f)(1 + \|\log a_Z\|)^N \quad a_Z \in A_{\mathfrak{Z}}, X \in C, \omega' \in \Omega', T \geq 0.$$

Moreover, let $q$ be a continuous semi-norm on $C_{\text{temp},N+\dim(W)}(Z_t)$. Then there exists a continuous semi-norm $p$ on $C_{\text{temp},N}(Z)$ such that

$$q(f_t) \leq p(f) \quad f \in \mathcal{A}_{\text{temp},N}(Z : \Lambda), \Lambda \in i^*_C.$$

### A Rapid convergence

**A.1 Definition.** Let $a \geq 0$ and $(x_s)$ be a family of elements of a normed vector space with $s \in [a, +\infty[$. One says that $(x_s)$ converges rapidly to $l$ if

there exist $\varepsilon > 0, C > 0, s_0 \in [a, +\infty[$ such that, for any $s \geq s_0$

$$\|x_s - l\| \leq C e^{-\varepsilon s}.$$

To shorten, we will write $x_s \xrightarrow{\text{rapid}}_{s \to \infty} l$.

**A.2 Lemma.** Let $a \geq 0$, $E$, $F$ be two Euclidean spaces and $l \in E$. Let $\phi$ be an $F$-valued map of class $C^1$ on a neighborhood $U$ of $l$ and such that the differential $\phi'(l)$ of $\phi$ at $l$ is injective. If $(x_s)_{s \in [a, +\infty[}$ is a family of elements of $E$ such that $\phi(x_s) \xrightarrow{\text{rapid}}_{s \to \infty} \phi(l)$ and $(x_s)$ converges to $l$ when $s$ tends to $+\infty$, then

$$x_s \xrightarrow{\text{rapid}}_{s \to \infty} l.$$
**Proof.** Let $G$ be a supplementary of the image of $\phi'(l)$ in $F$ and consider the map:

$$\Phi : U \times G \to F$$

$$(x, z) \mapsto \phi(x) + z.$$ 

As $\phi'(l)$ is injective, $\Phi'(l, z)$ is injective and $\dim(E \times G) = \dim(F)$. Hence $\Phi'(l, z)$ is invertible for any $z \in G$. From the local inversion theorem, $\Phi$ is then bijective on its image and of class $C^1$ on a neighborhood $V \times W$ of $(l, z)$ contained in $U \times G$. Consider the restriction $\tilde{\Phi}$ of $\Phi$ to $V \times W$. Then $\tilde{\Phi}$ is well-defined and of class $C^1$. Applying the Taylor expansion of $\tilde{\Phi}^{-1}$ at $\Phi(l, 0) = \phi(l)$, one has for $s$ large enough such that $x_s \in V$:

$$\|x_s - l\| = \|\tilde{\Phi}^{-1}(\phi(x_s)) - \tilde{\Phi}^{-1}(\phi(l))\| \leq \|((\tilde{\Phi}^{-1})'(\phi(l))) \| \|\phi(x_s) - \phi(l)\| + o(\|\phi(x_s) - \phi(l)\|).$$

Our claim follows from the rapid convergence of $(\phi(x_s))$. 

---

**A.3 Definition.** Let $a \geq 0$, $X$ be a $d$-dimensional smooth manifold and $(x_s)_{s \in [a, +\infty[}$ be a family of elements of $X$. One says that $(x_s)$ converges rapidly in $X$ if there exist $l \in X$ and a chart $(U, \phi)$ around $l$ such that

$$(\phi(x_s))$$ converges rapidly to $\phi(l)$.

**A.4 Remark.** This notion is independent of the choice of the chart $(U, \phi)$. Indeed, let $(\tilde{U}, \tilde{\phi})$ be another chart around $l$. Then, from Lemma A.2, $((\phi \circ \tilde{\phi}^{-1})^{-1}(\phi(x_s)))$ converges rapidly to $\tilde{\phi}(l)$ which means that $(\tilde{\phi}(x_s))$ converges rapidly to $\tilde{\phi}(l)$. Also if $\Psi : X \to Y$ is a differentiable map between $C^\infty$ manifolds and $(x_s)$ converges rapidly to $x$ in $X$, then $\Psi((x_s))$ converges rapidly to $\Psi(x)$ in $Y$.

**B Real points of elementary group actions**

We assume that $G$ is a reductive group defined over $\mathbb{R}$ and let $H$ be an $\mathbb{R}$-algebraic subgroup of $G$. We form the homogeneous space $Z = G/H$ and our concern is to what extent $Z(\mathbb{R})$ coincides with $G(\mathbb{R})/H(\mathbb{R})$.

We say that $G$ is anisotropic provided $G(\mathbb{R})$ is compact and recall from [11, Proposition 13.1] the following fact:

**B.1 Lemma.** If $G$ is anisotropic, then $Z(\mathbb{R}) = G(\mathbb{R})/H(\mathbb{R})$.

In the sequel we assume that $G$ is a connected elementary group (defined over $\mathbb{R}$), that is

- $G = MA$ for normal $\mathbb{R}$-subgroups $A$ and $M$,
- $M$ is anisotropic,
• \( A \) is a torus, i.e. \( A(\mathbb{R}) \simeq (\mathbb{R}^\times)^n \).

Consider now \( Z = G/H \) with \( G \) elementary. We set \( \underline{M_H} := M \cap H \) and likewise \( \underline{A_H} := A \cap H \). Further we set \( \underline{A_Z} := \underline{A}/\underline{A_H} \) and \( \underline{M_Z} := \underline{M}/\underline{M_H} \) which we view as subvarieties of \( Z \). From Lemma B.1 we already know that \( \underline{M_Z}(\mathbb{R}) = \underline{M(\mathbb{R})}/\underline{M_H(\mathbb{R})} \). Consider now the fiber-bundle

\[
A_Z \to Z \to G/HA
\]

and take real points

\[
A_Z(\mathbb{R}) \to Z(\mathbb{R}) \to (G/HA)(\mathbb{R}). \tag{2.1}
\]

We claim that the natural map

\[
\underline{M_Z}(\mathbb{R}) \times A_Z(\mathbb{R}) \to Z(\mathbb{R}) \tag{2.2}
\]

is surjective. In fact, observe that \( G/HA \simeq \underline{M}/(\underline{M} \cap (HA)) \) is homogeneous for the anisotropic group \( \underline{M} \). Hence \( (G/HA)(\mathbb{R}) \simeq \underline{M(\mathbb{R})}/(\underline{M} \cap (HA))(\mathbb{R}) \) and our claim follows from (2.1).

We remain with the determination of the fiber of the map (2.2). Since \( \underline{M} \) and \( \underline{A} \) commute we obtain with

\[
\underline{M_H} := \{ m \in \underline{M} \mid mH \in A_Z \subset Z \}
\]

a closed \( \mathbb{R} \)-subgroup of \( \underline{M} \) which acts on \( A_Z \) by morphisms (translations). The kernel of this action is \( \underline{M_H} \) and this identifies \( \underline{M_H} \) as a normal subgroup of \( \underline{M_H} \). In particular, we obtain an embedding \( \underline{M_H}/\underline{M_H} \to A_Z \) and taking real points we obtain, as \( \underline{M} \) is anisotropic and \( \underline{M_H} \) is closed in \( \underline{M} \), a closed embedding

\[
F_{\underline{M}(\mathbb{R})} := \underline{M_H}(\mathbb{R})/\underline{M_H}(\mathbb{R}) \to A_Z(\mathbb{R}).
\]

The image of \( F_{\underline{M}(\mathbb{R})} \) is compact, hence a 2-group of \( A_Z(\mathbb{R}) \simeq (\mathbb{R}^\times)^k \). In summary, we have shown:

**B.2 Proposition.** Let \( Z = G/H \) be a homogeneous space for an elementary group \( G = MA \) with respect to an \( \mathbb{R} \)-algebraic subgroup \( H \). Then \( F_{\underline{M}(\mathbb{R})} \) is a finite 2-group and the map

\[
[\underline{M}(\mathbb{R})/\underline{M_H}(\mathbb{R})] \times F_{\underline{M}(\mathbb{R})} A_Z(\mathbb{R}) \to Z(\mathbb{R}), \quad [m\underline{M_H}(\mathbb{R}), a_Z] \mapsto ma_Z
\]

is an isomorphism of real algebraic varieties.

References


