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ABSTRACT

Let $G$ be an algebraic real reductive group and $Z$ a real spherical $G$-variety, that is, it admits an open orbit for a minimal parabolic subgroup $P$. We prove a local structure theorem for $Z$. In the simplest case where $Z$ is homogeneous, the theorem provides an isomorphism of the open $P$-orbit with a bundle $Q \times L$. Here $Q$ is a parabolic subgroup with Levi decomposition $L \rtimes U$, and $S$ is a homogeneous space for a quotient $D = L/L_n$ of $L$, where $L_n \subseteq L$ is normal, such that $D$ is compact modulo center.

1. Introduction

Let $G_C$ be a complex reductive group and $B_C \triangleleft G_C$ a fixed Borel subgroup. We recall that a normal $G_C$-variety $Z_C$ is called spherical provided that $B_C$ admits an open orbit. The local nature of a spherical variety is given in terms of the local structure theorem [BLV86, Kno94]. In its simplest form, namely applied to a homogeneous space $Z_C = G_C/H_C$ for which $B_C H_C$ is open, it asserts that there is a parabolic subgroup $Q_C > B_C$ with Levi decomposition $Q_C = L_C \ltimes U_C$ such that the action of $Q_C$ on $Z_C$ induces an isomorphism of $(L_C/L_C \cap H_C) \times U_C$ onto $B_C H_C$.

The purpose of this paper is to continue the geometric study of real spherical varieties begun in [KS13]. We let $G$ be an algebraic real reductive group and $Z$ a normal real algebraic $G$-variety. Then $Z$ is called real spherical provided a minimal parabolic subgroup $P \triangleleft G$ has at least one open orbit on $Z$. With this assumption on $Z$ we prove a local structure theorem analogous to the one above. In particular, when applied to a homogeneous real spherical space $Z = G/H$ with $P H$ open, it yields a parabolic subgroup $Q > P$ with Levi decomposition $Q = L \rtimes U$ such that

$$L_n \lhd Q \cap H < L.$$ 

Here $L_n \lhd L$ denotes the product of all non-compact non-abelian normal factors of $L$. Furthermore, the action of $Q$ induces a diffeomorphism of $(L/L \cap H) \times U$ onto $P H$.

Our proof of the real local structure theorem is based on the symplectic approach of [Kno94]. Our investigations also show the number of $G$-orbits on a real spherical variety is finite. Combined with the main result of [KS13], it implies that the number of $P$-orbits on a real spherical variety is finite.

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2. Homogeneous spherical spaces

Lie groups in this paper will be denoted by upper-case Latin letters, \( A, B \ldots \), and their associated Lie algebras with the corresponding lower-case Gothic letter \( a, b \ldots \).

For a Lie group \( G \) we denote by \( G_0 \) its connected component containing the identity, by \( Z(G) \) the center of \( G \) and by \([G,G]\) the commutator subgroup.

On a real reductive Lie algebra \( g \) we fix a non-degenerate invariant bilinear form \( \kappa(\cdot,\cdot) \), for example the Cartan–Killing form if \( g \) is semisimple.

A Lie group \( G \) will be called real reductive provided that:

\( - \) the Lie algebra \( g \) is reductive;
\( - \) there exists a maximal compact subgroup \( K < G \) such that we have a homeomorphism (polar decomposition)
\[
K \times s \to G, \quad (k,X) \mapsto k \exp(X)
\]
where \( s := \mathfrak{t}^\perp \).

Observe that for a real reductive group the bilinear form \( \kappa \) can (and will) be chosen \( K \)-invariant. A real reductive group is called algebraic if it is isomorphic to an open subgroup of the group of real points \( G_\mathbb{C}(\mathbb{R}) \) where \( G_\mathbb{C} \) is a reductive algebraic group which is defined over \( \mathbb{R} \).

Now let \( G \) be a real reductive group, and let \( P \) be a minimal parabolic subgroup. The unipotent part of \( P \) is denoted \( N \). If a maximal compact subgroup \( K \) as above has been chosen, with associated Cartan involution \( \theta \) of \( G \), a maximal abelian subspace \( a \subset s \) can also be chosen. These choices then induce an Iwasawa decomposition \( G = KAN \) of \( G \) and a Langlands decomposition \( P = MAN \) of \( P \). Here \( M = Z_K(a) \). However, at present we do not fix \( K \) and \( a \).

Let \( H \) be a closed subgroup of \( G \) such that \( H/H_0 \) is finite. Recall that \( Z = G/H \) is said to be real spherical if the minimal parabolic subgroup \( P \) admits an open orbit on \( Z \). Furthermore, in this case \( H \) is called a spherical subgroup. Note that \( H \) is not necessarily reductive.

**Remark 2.1.** Here a remark on terminology is in order. Historically, spherical subgroups were first introduced by M. Krämer in the context of compact Lie groups; see [Krä79]. However, as our focus is to investigate non-compact homogeneous spaces we allow a discrepancy between the original definition and the current one. In fact with our definition every closed subgroup of \( G \) is spherical if \( G \) is compact.

We denote by \( z_0 \in Z \) the origin of the homogeneous space \( Z = G/H \).

2.1 Semi-invariant functions and the local structure theorem

Let \( G \) be a real reductive Lie group.

**Definition 2.2.** Let \( Z = G/H \) with \( H \subseteq G \) a closed subgroup.

1. A finite-dimensional real representation \((\pi,V)\) of \( G \) is called \( H \)-semispherical provided there is a cyclic vector \( v_H \in V \) and a character \( \gamma : H \to \mathbb{R}^\times \) such that
\[
\pi(h)v_H = \gamma(h)v_H, \quad \forall h \in H.
\]
2. The homogeneous space \( Z \) is called almost algebraic if there exists an \( H \)-semispherical representation \((\pi,V)\) such that the map
\[
Z \to \mathbb{P}(V), \quad g \cdot z_0 \mapsto [\pi(g)v_H]
\]
is injective.
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According to a theorem of Chevalley (see [Bor91, Theorem 5.1]), \( Z = G/H \) is almost algebraic if \( G \) and \( H \) are algebraic. In the following we always assume that \( Z = G/H \) is almost algebraic.

For a reductive Lie algebra \( \mathfrak{g} \) we write \( \mathfrak{g}_0 \) for the direct sum of the non-compact non-abelian ideals in \( [\mathfrak{g},\mathfrak{g}] \). If \( \mathfrak{g} \) is the Lie algebra of \( G \), then \( G_n \) denotes the corresponding connected normal subgroup of \( [G,G] \).

**Theorem 2.3** (Local structure theorem, homogeneous case). Let \( Z = G/H \) be an almost algebraic real spherical space, and let \( P \subseteq G \) be a minimal parabolic subgroup such that \( PH \) is open. Then there is a parabolic subgroup \( Q \supseteq P \) with Levi decomposition \( Q = LU \) such that:

(i) the map \( Q \times_L (L/L \cap H) \to Z, \ [q,l(L \cap H)] \mapsto ql \cdot z_0 \)

is a \( Q \)-equivariant diffeomorphism onto \( Q \cdot z_0 \subseteq Z \);

(ii) \( Q \cap H \subseteq L \);

(iii) \( L_n \subseteq H \);

(iv) \( (L \cap P)(L \cap H) = L \);

(v) \( QH = PH \).

**Proof.** The proof consists of an iterative procedure in which we construct a strictly decreasing sequence of parabolic subgroups \( Q_0 \supseteq Q_1 \supseteq \cdots \supseteq P \)

and corresponding Levi subgroups \( L_0 \supseteq L_1 \supseteq \ldots \), all satisfying (i). Note that (ii) is an immediate consequence of (i). After a finite number of steps a parabolic subgroup is reached which also satisfies (iii)–(v).

Let \( Q_0 = G \). It clearly satisfies (i). If \( G_n \subseteq H \) then \( PH = G \) since \( P \) contains both the center of \( G \) and every compact normal subgroup of \( [G,G] \). Hence in this case \( Q = Q_0 \) solves (i)–(v). Note also that since \( L \cap P \) is a minimal parabolic subgroup of \( L \), the argument just given, but applied to \( L \), shows that (iv) and (v) are consequences of (iii).

Assume now that \( G_n \not\subseteq H \). By our general assumption on \( Z \) there is a \( \pi \)-finite-dimensional representation \( (\pi,V) \) of \( G \) and a vector \( v_H \in V \) satisfying all the properties of Definition 2.2. The assumption on \( G_n \) implies that \( \pi(g)v_H \notin \mathbb{R}v_H \) for some \( g \in G_n \), hence \( \pi \) does not restrict to a multiple of the trivial representation of \( G_n \).

Choose a Cartan involution for \( G \) and a maximal abelian subspace \( \mathfrak{a} \subset \mathfrak{s} \), but note that these choices may be valid only for the current step of the iteration. Let \( v^* \in V^* \setminus \{0\} \) be an extremal weight vector so that the line \( \mathbb{R}v^* \) is fixed by \( AN \), say \( \pi^*(g)v^* = \chi(g)v^* \) for \( g \in AN \) and some character \( \chi : AN \to \mathbb{R}^\times \). Now we need the following lemma.

**Lemma 2.4.** Let \( G \) be a connected semisimple Lie group without compact factors, and with minimal parabolic \( P = MAN \subseteq G \). Let \( V \) be a non-trivial finite-dimensional irreducible real representation of \( G \). Then \( V^{AN} = \{0\} \).

**Proof.** Let \( \bar{N} = \theta(N) \) be the unipotent part of the parabolic subgroup \( \theta(P) \) opposite to \( P \). It follows from the representation theory of \( \mathfrak{sl}(2,\mathbb{R}) \) that vectors in \( V^{AN} \) are also fixed by \( \bar{N} \). Since \( G \) has no compact factors it is generated by \( \bar{N} \) and \( AN \), hence \( V^{AN} = V^G = \{0\} \).

By this lemma and what we have just seen, we can choose \( v^* \) such that \( \chi \) is non-trivial on \( G_n \cap A \). The matrix coefficient

\[ f(g) := v^*(\pi(g)v_H) \quad (g \in G) \]
satisfies \( f(anh) = \chi(a)^{-1} \gamma(h) f(g) \) for all \( g \in G \), \( a \in AN \) and \( h \in H \). As \( v_H \) is cyclic and \( v^* \) non-zero, and as \( PH \) is open, \( f \) is not identically zero on \( M \).

We construct a new function:

\[
F(g) := \int_M f(mg)^2 \, dm \quad (g \in G).
\]

This function is smooth, \( G \)-finite, non-negative, and satisfies

\[
F(manhh) = \chi(a)^{-1} \gamma^2(h) F(g)
\]

for all \( g \in G \), \( man \in P \) and \( h \in H \). Furthermore, \( F(e) > 0 \).

It follows from the \( G \)-finiteness, together with (2.1), that \( F \) is a matrix coefficient \( F(g) = w^*(\rho(g) w_H) \)
of a finite-dimensional representation \((\rho, W)\) of \( G \), with non-zero vectors \( w_H \in W \) and \( w^* \in W^* \) such that

\[
\rho(h) w_H = \gamma(h)^2 w_H, \quad \rho^*(man) w^* = \chi(a)^2 w^*
\]

for all \( h \in H \) and \( man \in P = MAN \). Here, \( W^* \) can be chosen to be the span of all left translates of \( F \). Since \( F \) is a highest weight vector, \( W^* \) and hence \( W \) are irreducible. Define \( \nu \in \frak{a}^* \) by

\[
e^{\nu(x)} = \chi(\exp X)^2.
\]

Then \( \nu \) is the highest \( \frak{a} \)-weight of \( \rho^* \), and it is dominant with respect to the set \( \Sigma(\frak{a}, \frak{n}) \) of \( \frak{a} \)-roots in \( \frak{n} \).

Now define a subgroup \( Q_1 = Q \subseteq G \) to be the stabilizer of \( \mathbb{R} w^* \),

\[
Q = \{ g \in G \mid \rho^*(g) w^* \in \mathbb{R} w^* \},
\]

and define a character \( \psi : Q \to \mathbb{R}^\times \) by

\[
\rho^*(g) w^* = \psi(g) w^*.
\]

In particular, we see that \( Q \) is a parabolic subgroup that contains \( P \). Moreover, \( \psi : Q \to \mathbb{R} \)
extends \( \chi^2 : AN \to \mathbb{R}^\times \). Let \( U \subseteq Q \) be the unipotent radical of \( Q \); its Lie algebra is spanned by
the root spaces of the roots \( \alpha \in \Sigma(\frak{a}, \frak{n}) \) for which \( \langle \alpha, \nu \rangle > 0 \).

Note that since \( w_H \) is cyclic, \( \rho^*(g) w^* = cw^* \) if and only if \( F(g^{-1}x) = cF(x) \) for all \( x \in G \). Hence

\[
Q = \{ g \in G \mid F(g \cdot) \text{ is a multiple of } F \}
\]

and \( F(g \cdot) = \psi(q) F \) for all \( q \in Q \). (We use the symbol \( F(g \cdot) \) for the function \( x \mapsto F(g x) \) on \( G \).)

We note that \( Q \cap G_n \) is a proper subgroup of \( G_n \), for otherwise \( \rho^* \) would be one-dimensional
spanned by \( w^* \), and this would contradict the non-triviality of its highest weight \( e^\nu = \chi^2 \) on \( G_n \cap A \).

Set \( Z_0 := QH \subseteq Z \). Then \( Z_0 \) is open since \( qPH \) is open for each \( q \in Q \). Following [Knop94, Theorem 2.3], we define a moment-type map

\[
\mu : Z_0 \to \frak{g}^*, \quad \mu(z)(X) := \frac{dF(q)(X)}{F(q)} = \left. \frac{d}{dt} \right|_{t=0} \frac{F(\exp(tX)q)}{F(q)}
\]

for \( q \in Q \) such that \( z = QH \in Z_0 \) and \( X \in \frak{g} \). Note that this map is well defined: \( F(q) \neq 0 \) for \( q \in Q \), and if \( q \cdot z_0 = q' \cdot z_0 \) then \( q = q'h \) for some \( h \in H \).

We let \( G \) act on \( \frak{g}^* \) via the co-adjoint action and record the following result.
Let $z \in Z_0$, $q \in Q$ and $Y \in g$. Then
\[
\mu(qz)(Y) = \frac{d}{dt}|_{t=0} F(\exp(tY)qz) = \frac{d}{dt}|_{t=0} F(qq^{-1}\exp(tY)qz) = \frac{d}{dt}|_{t=0} F(q\exp(tq^{-1}Y)z) = (Ad^*(q)\mu(z))(Y).
\]

Note that
\[
\mu(z)(X) = d\psi(X), \quad (X \in q)
\]
for all $z \in Z_0$. In particular, $\mu(z_1) - \mu(z_2) \in q^\perp \subseteq g^*$ for $z_1, z_2 \in Z_0$. Moreover, $\mu(z)(X + Y) = -\nu(X)$ for $X \in a$ and $Y \in m + n$.

We now identify $g^*$ with $g$ via the invariant non-degenerate form $\kappa(\cdot, \cdot)$. Then $q^\perp$ is identified with $u$, and $(m + n)^\perp$ with $a + n$. Let
\[
X_0 = \mu(z_0) \in a + n.
\]
Then $X_0 \notin n$ since $\nu \neq 0$ and hence $X_0$ is a semisimple element. Write $X_0$ for the $a$-part of $X_0$. Then the eigenvalues of $ad(X_0)$ on $n$ are the $\alpha(X)$ where $\alpha \in \Sigma(a, n)$. By the identification of $g^*$ with $g$, these are the inner products $\langle -\nu, \alpha \rangle$; in particular, they are all non-positive and on $u$ they are negative.

We conclude from the above that $im\mu \subseteq X_0 + u$. We claim equality:
\[
im\mu = X_0 + u. \tag{2.3}
\]
As $\mu$ is $Q$-equivariant, we have $im\mu = Ad(Q)X_0$. The lemma below (with $X = -ad(X_0)$) implies $Ad(U)X_0 = X_0 + u$, and then (2.3) follows.

**Lemma 2.6.** Let $u$ be a nilpotent Lie algebra and $X : u \rightarrow u$ a derivation which is diagonalizable with non-negative eigenvalues. Then in the solvable Lie algebra $g := RX \ltimes u$ the following identity holds:
\[
e^{ad}uX = X + [X, u]. \tag{2.4}
\]

**Proof.** Note that $[X, u] = u$ if all eigenvalues are positive. The inclusion $\subseteq$ in (2.4) is easy. The proof of the opposite inclusion is by induction on $dim u$, and the case $dim u = 0$ is trivial. Assume $dim u > 0$ and let $u = \sum_{\lambda > 0} u(X, \lambda)$ be the eigenspace decomposition of the operator $X : u \rightarrow u$. Let $\lambda_1 \geq 0$ be the smallest eigenvalue and set $u_1 := u(X, \lambda_1)$ and $u_2 := \sum_{\lambda > \lambda_1} u(X, \lambda)$. Note that $u_2$ is an ideal in $u$, and $u = u_1 + u_2$ as vector spaces.

By induction we have $e^{ad}u_2X = X + u_2$. If $\lambda_1 = 0$ then $[X, u] = u_2$, and we are done. Otherwise $[u_1, u_1] \subseteq u_2$ and hence
\[
e^{ad}u_1X \in X + \lambda_1U + u_2
\]
for $U \in u_1$. Note that $e^{ad}u_1$ is a group as $u_1$ is nilpotent. It follows that
\[
e^{ad}uX \supseteq e^{ad}u_1e^{ad}u_2X = e^{ad}u_1(X + u_2)
\leq \bigcup_{U \in u_1} e^{ad}U(X + u_2) = \bigcup_{U \in u_1} (e^{ad}U X + u_2) = \bigcup_{U \in u_1} (X + \lambda_1 U + u_2) = X + u.
\]

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Continuing with the proof of Theorem 2.3, we conclude that the stabilizer \( L \subseteq Q \) of \( X_0 \in q \) is a reductive Levi subgroup. Let

\[ S := \mu^{-1}(X_0) = \{ z \in Z_0 \mid \mu(z) = X_0 \}. \]

Then for \( q \in Q \) we have

\[ qz_0 \in S \Leftrightarrow \mu(qz_0) = X_0 \Leftrightarrow qX_0 = X_0 \Leftrightarrow q \in L. \] (2.5)

Hence \( L \) acts transitively on \( S \). As \( \mu \) is submersive, \( S \) is a submanifold of \( Z_0 \) and we obtain with

\[ Q \times_L S \to Z_0 \] (2.6)

a \( Q \)-equivariant diffeomorphism. As an \( L \)-homogeneous space, \( S \) is isomorphic to \( L/L \cap H \). Hence (i) is valid.

Note that (2.5) implies that \((L \cap P)H = S \cap (PH)\), which is open in \( S \). Thus \( L/L \cap H \) is a real spherical space.

If (iii) is valid, we are done. Otherwise we let \( Q_1 = Q \) and consider the real spherical space \( Z_1 = L_1/L_1 \cap H \) for \( L_1 = L \). Iterating the procedure of before yields a proper parabolic subgroup \( R \) of \( L_1 \) containing \( L_1 \cap N \) and with a Levi subgroup \( L_2 \subseteq L_1 \) such that

\[ (R \cap N) \times L_2/(L_2 \cap H) \to R \cdot z_0 \] (2.7)

is a diffeomorphism. We let \( Q_2 = RP = RU_1 \), which is a subgroup since \( R \) normalizes \( U_1 \). Note that (2.7), together with the property (i) for \( Q_1 \), implies that this property is valid also for \( Q_2 \). We continue iterations until \( H \) contains the non-compact semisimple part of some \( L_i \). This will happen eventually since the non-compact semisimple part of a Levi subgroup of \( P \) is trivial. \( \square \)

### 2.2 Z-adapted parabolics

**Definition 2.7.** Let \( Z = G/H \) be a real spherical space. A parabolic subgroup \( Q < G \) will be called \( Z \)-adapted provided that:

- (i) there is a minimal parabolic subgroup \( P \subseteq Q \) with \( PH \) open;
- (ii) there is a Levi decomposition \( Q = LU \) such that \( Q \cap H \subseteq L \);
- (iii) \( l_u \subseteq h \).

A parabolic subalgebra \( q \) of \( g \) is called \( Z \)-adapted if it is the Lie algebra of a \( Z \)-adapted parabolic subgroup \( Q \).

**Theorem 2.8.** Let \( Z = G/H \) be an almost algebraic real spherical space and \( P \) a minimal parabolic subgroup such that \( PH \) is open. Then there exists a unique parabolic subgroup \( Q \supseteq P \) with unipotent radical \( U \) such that \( u \) is complementary to \( n \cap h \) in \( n \). Moreover, this parabolic subgroup \( Q \) is \( Z \)-adapted, and it is the unique parabolic subgroup above \( P \) with that property.

**Proof.** Note first that if \( Q \supseteq P \) and \( Q = LU \) is a Levi decomposition then \( n = (n \cap l) \oplus u \). Assuming in addition (ii) and (iii) above, then \( n \cap h = n \cap l \), and hence \( n \cap h \) is complementary to \( u \). Hence every \( Z \)-adapted parabolic subgroup \( Q \supseteq P \) has this property of complementarity. In particular, this holds then for the parabolic subgroup \( Q \) constructed with Theorem 2.3.

It remains to prove that if \( Q' \supseteq P \) is another parabolic for which the unipotent radical \( u' \) is complementary to \( n \cap h \), then \( Q' = Q \). Since \( l_u \subseteq h \) we find

\[ u' \cap l \subseteq u' \cap h = \{0\}. \]

The lemma below now implies \( u \supseteq u' \). But then \( u = u' \) since both spaces are complementary to \( n \cap h \), and hence \( Q = Q' \). \( \square \)
Lemma 2.9. Let \( \mathfrak{p} \) be a minimal parabolic subalgebra, and let \( \mathfrak{q}, \mathfrak{q}' \supseteq \mathfrak{p} \) be parabolic subalgebras with unipotent radicals \( \mathfrak{u}, \mathfrak{u}' \). If there exists a Levi decomposition \( \mathfrak{q} = \mathfrak{l} + \mathfrak{u} \) such that \( \mathfrak{l} \cap \mathfrak{u}' = \{0\} \), then \( \mathfrak{q} \subseteq \mathfrak{q}' \).

Proof. This follows easily from the standard description of the parabolic subalgebras containing \( \mathfrak{p} \) by sets of simple roots. \( \square \)

2.3 The real rank of \( \mathcal{Z} \)

Let \( \mathcal{Q} \) be \( \mathcal{Z} \)-adapted, with Levi decomposition \( \mathcal{Q} = \mathcal{L} \mathcal{U} \) as in Definition 2.7. From the local structure theorem we obtain an isomorphism

\[
\mathcal{Q} \times \mathcal{L} \mathcal{L}/\mathcal{L} \cap \mathcal{H} \to \mathcal{Q} \cdot z_0 = \mathcal{P} \cdot z_0.
\]

Recall that \( \mathcal{L}_n \subseteq \mathfrak{h} \). We decompose

\[
\mathfrak{l} = \mathfrak{z}(\mathfrak{l}) \oplus [\mathfrak{l}, \mathfrak{l}] = \mathfrak{z}(\mathfrak{l}) \oplus \mathfrak{l}_c \oplus \mathfrak{l}_n,
\]

where \( \mathfrak{l}_c \) denotes the sum of all compact simple ideals in \( \mathfrak{l} \). Note that \( \mathfrak{D} = \mathcal{L}/\mathcal{L}_n \) is a Lie group with the Lie algebra \( \mathfrak{d} = \mathfrak{z}(\mathfrak{l}) + \mathfrak{l}_c \), which is compact, and that

\[
\mathfrak{l} \cap \mathfrak{h} = \mathfrak{c} \oplus \mathfrak{l}_n
\]

with \( \mathfrak{c} = \mathfrak{d} \cap \mathfrak{h} \). Let \( \mathcal{C} = (\mathcal{L} \cap \mathcal{H})/\mathcal{L}_n \subseteq \mathcal{D} \); then \( \mathcal{L}/\mathcal{L} \cap \mathcal{H} = \mathcal{D}/\mathcal{C} \), and

\[
\mathcal{U} \times \mathcal{D}/\mathcal{C} \to \mathcal{P} \cdot z_0 \quad (2.9)
\]

is an isomorphism.

Consider the refined version of (2.8),

\[
\mathfrak{l} = \mathfrak{z}(\mathfrak{l})_{np} \oplus \mathfrak{z}(\mathfrak{l})_{cp} \oplus \mathfrak{l}_c \oplus \mathfrak{l}_n,
\]

in which \( \mathfrak{z}(\mathfrak{l})_{np} \) and \( \mathfrak{z}(\mathfrak{l})_{cp} \) denote the non-compact and compact parts of \( \mathfrak{z}(\mathfrak{l}) \). Let \( \mathcal{L} = \mathcal{K}_L \mathcal{A}_L \) \( (\mathcal{L} \cap \mathcal{N}) \) be an Iwasawa decomposition of \( \mathcal{L} \), and let \( \mathcal{G} = \mathcal{K} \mathcal{A} \mathcal{N} \) be an Iwasawa decomposition of \( \mathcal{G} \) which is compatible, that is, \( \mathcal{K} \supseteq \mathcal{K}_L \) and \( \mathcal{A} = \mathcal{A}_L \). Then \( \mathfrak{a} = \mathfrak{z}(\mathfrak{l})_{np} \oplus (\mathfrak{a} \cap \mathfrak{l}_n) \).

Let \( \mathfrak{a}_h \subseteq \mathfrak{z}(\mathfrak{l})_{np} \) be the image of \( \mathfrak{c} \) under the projection \( \mathfrak{l} \to \mathfrak{z}(\mathfrak{l})_{np} \) along (2.10), and let \( \mathfrak{a}_Z \) be a subspace of \( \mathfrak{z}(\mathfrak{l})_{np} \), complementary to \( \mathfrak{a}_h \). Then

\[
\mathfrak{a} = \mathfrak{a}_Z \oplus \mathfrak{a}_h \oplus (\mathfrak{a} \cap \mathfrak{l}_n). \quad (2.11)
\]

The number \( \dim \mathfrak{a}_Z \) will be called the real rank of \( \mathcal{Z} \) in §3, where we show (under an additional hypothesis) that it is an invariant of \( \mathcal{Z} \) (it is independent of the choices of \( \mathcal{P} \) and \( \mathcal{L} \)). See Remark 3.5.

2.4 \( \mathcal{H}\mathcal{P} \)-factorizations of a semisimple group

Let \( \mathcal{Z} = \mathcal{G}/\mathcal{H} \) be real spherical. In general \( \mathcal{G}/\mathcal{P} \) admits several \( \mathcal{H} \)-orbits. Here we investigate the simplest case where there is just one orbit.

Proposition 2.10. Let \( \mathcal{G} \) be semisimple. Assume that \( \mathcal{Z} = \mathcal{G}/\mathcal{H} \) is real spherical and that \( \mathfrak{h} \) contains no non-zero ideal of \( \mathfrak{g} \). Then \( \mathcal{H}\mathcal{P} = \mathcal{G} \) if and only if \( \mathcal{H} \) is compact.
3. Real spherical varieties

All complex varieties $Z_C$ in this section will be defined over $\mathbb{R}$. Typically we denote by $Z$ the set of real points of $Z_C$. If $Z$ is Zariski dense in $Z_C$, then we call $Z$ a real (algebraic) variety.

We say that a subset $U \subset Z$ is (quasi-)affine if there exists a (quasi-)affine subset $U_C \subset Z_C$ such that $U = U_C \cap Z$.

Remark 3.1. Even if $Z_C$ is irreducible it might happen that $Z$ has several connected components with respect to the Euclidean topology. However, by Whitney’s theorem, the number of connected components is always finite. Take, for example, $Z = \text{GL}(n, \mathbb{R})$ and $Z_C = \text{GL}(n, \mathbb{C})$. Here $Z$ breaks into two connected components $\text{GL}(n, \mathbb{R})_+$ and $\text{GL}(n, \mathbb{R})_-$ characterized by the sign of the determinant; certainly it would be meaningful to call $\text{GL}(n, \mathbb{R})_+$ a real algebraic variety as well.

Let $Z_1 \cup \ldots \cup Z_n$ be the decomposition of $Z$ into connected components (with respect to the Euclidean topology). A more general notion of real variety would be to allow arbitrary unions of those $Z_j$ which are Zariski dense in $Z_C$. In fact, all the statements derived in this section for real varieties are valid in this more general setup.

In this section we let $G$ be a real algebraic reductive group and $G_C \supseteq G$ its complexification. Furthermore, $P$ is a minimal parabolic subgroup of $G$ and $P = MAN$ a Langlands decomposition of it.

By a real $G$-variety $Z$ we understand a real variety $Z$ endowed with a real algebraic $G$-action. A real $G$-variety will be called linearizable provided there is a finite-dimensional real $G$-module $V$ such that $Z$ is realized as real subvariety of $P(V)$.

An algebraic real reductive group $G$ is called elementary if $G \cong M \times A$ with $M$ compact and $A = (\mathbb{R}^+)^l$. This is equivalent to $G = P$. A real $G$-variety $Z$ will then be called elementary if $G/J$ is elementary where $J$ is the kernel of the action on $Z$. 

Proof. Assume that $HP = G$. Note that then $HgP = G$ for every $g \in G$ and hence

$$\mathfrak{h} + \text{Ad}(g)(\mathfrak{p}) =\mathfrak{g}$$

for every $g \in G$.

We first reduce to the case where $H$ is reductive in $G$. Otherwise there exists a non-zero ideal $\mathfrak{h}_u$ in $\mathfrak{h}$ which acts unipotently on $\mathfrak{g}$. By conjugating $P$ if necessary, we may assume that $\mathfrak{h}_u \subseteq \mathfrak{n}$. It then follows from $G = PH$ that $\text{Ad}(g)(\mathfrak{h}_u) \subseteq \mathfrak{n}$ for all $g \in G$, which is absurd.

Assume now that $H$ is reductive and let $H = K_H A_H N_H$ be an Iwasawa decomposition. Let $X \in \mathfrak{a}_H$ be regular dominant with respect to $\mathfrak{n}_H$, and let $\mathfrak{q}$ be the parabolic subalgebra of $\mathfrak{g}$ which is spanned by the non-negative eigenspaces of $\text{ad}X$. It follows that $\mathfrak{q} \cap \mathfrak{h}$ is a minimal parabolic subalgebra of $\mathfrak{h}$, and that $\mathfrak{n}_H$ is contained in the unipotent part $\mathfrak{u}$ of $\mathfrak{q}$. As $\mathfrak{q}$ contains a conjugate of $\mathfrak{p}$ we have $\mathfrak{q} = \mathfrak{h} + \mathfrak{q}$ and hence $\dim(\mathfrak{h}/(\mathfrak{q} \cap \mathfrak{h})) = \dim(\mathfrak{q}/\mathfrak{q})$, from which we deduce that $\mathfrak{n}_H = \mathfrak{u}$. From $\mathfrak{n}_H = \mathfrak{u}$ and $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ we deduce that $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$. Let $\mathfrak{n}_n$ be the subalgebra of $\mathfrak{h}$ generated by $\mathfrak{n}_H$ and its opposite $\mathfrak{n}_H$ with respect to the Cartan involution of $H$ associated with $H = K_H A_H N_H$. Then $\mathfrak{n}_n$ is $\mathfrak{l}$-invariant and an ideal in $\mathfrak{h}$. With $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$ we now infer that $\mathfrak{n}_n$ is an ideal in $\mathfrak{g}$, and hence it is zero. It follows that $H = K_H A_H$, where $A_H$ is central in $H$. We may assume $K_H \subseteq K$ and $A_H \subseteq A$. Then $G = HP$ implies $K = K_H M$, and hence $K$ centralizes $A_H$. This is impossible unless $A_H = \{1\}$ and then $H$ is compact.

Conversely, if $H$ is compact then the open $H$-orbit on $G/P$ is closed, and since $G/P$ is connected it follows that $HP = G$. 

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DEFINITION 3.2. A linearizable real $G$-variety $Z$ will be called real spherical provided that:

- $Z_C$ is irreducible,
- $Z$ admits an open $P$-orbit.

Remark 3.3. (a) In the definition of a (complex) spherical variety one requires in particular that the variety is normal. We now explain how this is related to our notion of real spherical.

Assume that $Z_C$ is normal. Then it follows from a theorem of Sumihiro [KKLV89, p. 64] that every point $z \in Z_C$ has a $G_C$-invariant open neighborhood $U$ which can be equivariantly embedded into $\mathbb{P}(V_C)$ where $V_C$ is a finite-dimensional representation of $G_C$. It follows that if $z \in Z$ then $U_0 := (U \cap \overline{U}) \cap Z$ is a linearizable open neighborhood of $z$. Observe that there is always a normalization map $\nu : \hat{Z} \rightarrow Z$ where $\hat{Z}$ is a normal $G$-variety and $\nu$ is proper, finite to one, and invertible over an open dense subset of $Z$.

(b) If $Z$ is a real spherical variety, then the number of open $P$-orbits is finite: As $Z_C$ is irreducible, there is exactly one open $P_C$-orbit on $Z_C$ and the real points of this open $P_C$-orbit decompose into finitely many $P$-orbits. We conclude in particular that there are only finitely many open $G$-orbits in $Z$. Let $\mathcal{O} \simeq G/H$ be one of them. Then $G/H$ is a real spherical algebraic homogeneous space which we considered before.

(c) Let $Z$ be an elementary real spherical variety. If $G = A$, then $Z$ consists of the real points of a toric variety defined over $\mathbb{R}$.

(d) Let $G = M \times A$ be an elementary algebraic real reductive group and $Z = G/H$ a homogeneous real spherical $G$-variety. Since there are no algebraic homomorphisms between a split torus and a compact group, the group $H$ splits as $H = M_0 \times A_0$ with $M_0 \subseteq M$ and $A_0 \subseteq A$. Thus $Z = M/M_0 \times A/A_0$.

3.1 Some general facts about real $G$-varieties

Let $Z$ be an irreducible real variety. We denote by $\mathbb{C}[Z]$ (respectively, $\mathbb{C}(Z)$) the ring of regular (respectively, rational) functions on $Z$, that is, $\mathbb{C}[Z]$ consists of the restrictions of the regular functions on $Z_C$ to $Z$, and likewise for $\mathbb{C}(Z)$.

As $Z$ is Zariski dense we observe that the restriction mapping $\text{Res} : \mathbb{C}(Z_C) \rightarrow \mathbb{C}(Z)$ is bijective. Next we note that both $\mathbb{C}(Z)$ and $\mathbb{C}[Z]$ are invariant under complex conjugation $f \mapsto \overline{f}$. In particular with $f \in \mathbb{C}[Z]$ (respectively, $\mathbb{C}(Z)$), we also have that $\text{Re} f$ and $\text{Im} f$ belong to $\mathbb{C}[Z]$ (respectively, $\mathbb{C}(Z)$).

If a compact real algebraic group $M$ acts on $Z$, then the $M$-average

$$f \mapsto f^M; \quad f^M(z) := \int_M f(m \cdot z) \, dm \quad (z \in Z)$$

preserves $\mathbb{C}[Z]$. This follows from the fact that the $G$-action on $\mathbb{C}[Z]$ is locally finite. Put together, we conclude

$$f \in \mathbb{C}[Z] \Rightarrow (|f|^2)^M \in \mathbb{C}[Z]^M \quad \text{with } f \neq 0 \Rightarrow (|f|^2)^M \neq 0. \quad (3.1)$$

Let us denote by $\hat{P}$ the set of real algebraic characters $\chi : P \rightarrow \mathbb{R}^\times$ such that $MN \subseteq \ker \chi$. Note that the subgroup $MN$ of $P$, and hence $\hat{P}$, is independent of the choice of a Langlands decomposition of $P$. However, when that has been chosen, there is a natural identification of $\hat{P}$ with a lattice $\Lambda \subseteq \mathfrak{a}^\times$.

For the rest of this subsection we let $Z$ be a real $G$-variety. We denote by $\mathbb{C}(Z)^{(P)}$ the set of $P$-semi-invariant functions, i.e., the rational functions $f \in \mathbb{C}(Z) \setminus \{0\}$ for which there is a $\chi \in \hat{P}$ such that $f(p^{-1} z) = \chi(p) f(z)$ for all $p \in P$, $z \in Z$ for which both sides are defined. We denote
If \( \text{rk}_R (Z) = 0 \) the set of \( P \)-invariants in \( C(Z) \). Likewise we define \( C[Z]^{(P)} \) and \( C[Z]^P \). Further, we denote by \( \mathbb{R}(Z) \) and \( \mathbb{R}[Z] \) the real-valued functions in \( C(Z) \) and \( C[Z] \).

**Lemma 3.4.** Let \( Z \) be a quasi-affine real \( G \)-variety. Then for all non-zero \( f \in \mathbb{R}(Z)^P \) there exist \( f_1, f_2 \in \mathbb{R}[Z]^{(P)} \) such that \( f = f_1/f_2 \).

**Proof.** Let \( f \in \mathbb{R}(Z)^P \). As \( Z \) is quasi-affine, we find regular functions \( h_1, h_2 \in C[Z], h_2 \neq 0 \) such that \( f = h_1/h_2 \). Consider the ideal

\[
I := \{ h \in C[Z] \mid hf \in C[Z] \}.
\]

Note that:

- \( I \neq \{0\} \) as \( h_2 \in I \);
- \( I = \mathcal{I} \) as \( f \) is real;
- \( I \) is \( P \)-invariant as \( f \) is \( P \)-fixed.

The action of \( P \) on \( C[Z] \) is algebraic, hence locally finite, and thus we find an element \( 0 \neq h \in I \) which is an eigenvector for the solvable group \( AN \). We use (3.1) to obtain with \( f_2 = (|h|^2)^M \) a non-zero element of \( I \cap \mathbb{R}[Z]^{(P)} \). Now we put \( f_1 = f_2 f \in \mathbb{R}[Z]^{(P)} \).

For \( \chi \in \hat{P} = \Lambda \) we let

\[
C[Z]_\chi := \{ f \in C[Z] \mid (\forall p \in P, z \in Z) f(p^{-1}z) = \chi(p)f(z) \},
\]

and define \( C(Z)_\chi \) likewise. We define a sub-lattice of \( \Lambda \) by

\[
\Lambda_Z := \{ \chi \in \hat{P} \mid C(Z)_{\chi} \neq \{0\} \}.
\]

With that we declare the **real rank** of \( Z \) by

\[
\text{rk}_R (Z) := \dim_Q (\Lambda_Z \otimes_Z \mathbb{Q}). \tag{3.2}
\]

It is easily seen that \( \text{rk}_R (Z) \) is independent of the choice of minimal parabolic subgroup \( P \).

**Remark 3.5.** Let \( Z = G/H \) be homogeneous. Then \( \text{rk}_R (Z) = \dim A_Z \) where \( A_Z \) is defined by (2.11). In fact, as a \( Q \)-variety, an open subset of \( Z \) is isomorphic to \( U \times L/L \cap H \). Thus \( \mathbb{R}(Z)^P = \mathbb{R}(L/L \cap H)^{(L \cap P)} \). Since \( H \) contains \( L \), the variety \( L/L \cap H \) is elementary. By Remark 3.3(d), we have \( \mathbb{R}(L/L \cap H)^{(L \cap P)} = \mathbb{R}(A/A_0)^{(A)} \) which implies the claim, as \( A/A_0 \cong A_Z \).

**Lemma 3.6.** Let \( Z \) be a linearizable irreducible real \( G \)-variety and \( Y \subseteq X \) a Zariski closed \( G \)-invariant subvariety. Then there exists a \( P \)-stable affine open subset \( Z_0 \subseteq Z \) which meets \( Y \) and such that the restriction mapping

\[
\mathbb{R}[Z_0]^{(P)} \rightarrow \mathbb{R}[Z_0 \cap Y]^{(P)}
\]

is onto.

**Proof.** If \( G \) is complex, then this is the real-points version of [Bri97, Proposition 1.1]. Further, with \( P \) replaced by \( AN \), one can literally copy the proof of [Bri97]. Finally, the additional \( M \)-invariance when moving from \( AN \) to \( P \) is obtained from (3.1).
Real spherical varieties

Denote by $\Lambda^+ \subseteq \Lambda$ the semigroup of elements dominant with respect to $P$. For all $\lambda \in \Lambda^+$ we set

$$m(\lambda) := \dim_\mathbb{C} \mathbb{C}[Z]_\lambda.$$

If we identify $\Lambda^+$ with a subset of the irreducible finite-dimensional representations of $G$, then $m(\lambda)$ is the multiplicity of the irreducible representation $\lambda$ occurring in the locally finite $G$-module $\mathbb{C}[Z]$. The following proposition is a real analogue of the Vinberg–Kimel’feld theorem [VK78].

**Proposition 3.7.** Let $Z$ be a quasi-affine irreducible $G$-variety. Then the following assertions are equivalent:

(i) $Z$ is real spherical;
(ii) $m(\lambda) \leq 1$ for all $\lambda \in \Lambda^+$.

**Proof.** (i) $\Rightarrow$ (ii) Let $z \in Z$ such that $P \cdot z$ is open in $Z$. Then two $P$-semi-invariant functions $f_1$ and $f_2$ with respect to the same character $\lambda \in \hat{P}$ satisfy $f_1|_{P \cdot z} = cf_2|_{P \cdot z}$ for some constant $c \in \mathbb{C}$. As $Z_G$ is irreducible we conclude that $f_1 = cf_2$.

(ii) $\Rightarrow$ (i) We recall that there is an open $P$-orbit on $Z$ if and only if $\mathbb{C}(Z)^P = \mathbb{C}1$. This follows from Rosenlicht’s theorem [Spr89, p. 23], applied to $Z_G$. Now let $f \in \mathbb{C}(Z)^P$. According to Lemma 3.4, there exist $f_1, f_2 \in \mathbb{C}[Z]^{\{P\}}$ such that $f = f_1/f_2$. Clearly $f_1$ and $f_2$ correspond to the same character $\lambda \in \hat{P}$. As $m(\lambda) \leq 1$, we conclude that $f_1$ is a multiple of $f_2$.

**Corollary 3.8.** Let $Z$ be a real spherical variety and $Y \subseteq Z$ a closed $G$-invariant irreducible subvariety. Then $Y$ is real spherical.

**Proof.** If $Z$ is quasi-affine, then this is immediate from the previous proposition as the restriction mapping $\mathbb{C}[Z] \rightarrow \mathbb{C}[Y]$ is onto. The more general case is reduced to that by considering the affine cone over $Z$. Recall that $Z \subseteq \mathbb{P}(V)$. The preimage of $Z$ in $V \setminus \{0\}$ will be denoted by $\tilde{Z}$. Note that $\tilde{Z}$ is quasi-affine. Moreover, $Z$ is real spherical if and only if $\tilde{Z}$ is real spherical for the enlarged reductive group $G_1 = G \times \mathbb{R}^\times$.

**Corollary 3.9.** Let $Z$ be a real spherical variety. Then the number of $G$-orbits on $Z$ is finite and each $G$-orbit is spherical.

**Proof.** In view of the preceding corollary we only need to show that there are finitely many $G$-orbits. Suppose that there are infinitely many $G$-orbits. We let $Y \subseteq Z$ be a closed irreducible $G$-subvariety of minimal dimension which admits infinitely many $G$-orbits. By Corollary 3.8, $Y$ is spherical. In particular, $Y$ admits open $G$-orbits. After deleting the finitely many open $G$-orbits from $Y$, we obtain a $G$-invariant subvariety $Y_1 \subseteq Y$ with infinitely many $G$-orbits. As $\dim Y_1 < \dim Y$ we reach a contradiction.

The main result of [KS13] was that every homogeneous real spherical space admits only finitely many $P$-orbits. With Corollary 3.9 we then deduce the following result.

**Theorem 3.10.** Let $Z$ be a real spherical variety. Then the number of $P$-orbits on $Z$ is finite.

### 3.2 The local structure theorem

Let $Z$ be a real spherical variety and $Y \subseteq Z$ a $G$-invariant closed subvariety. Our goal is to find a $P$-invariant coordinate chart $Z_0$ for $Z$ which meets $Y$. For that we may assume that $Z$ is Zariski closed in $\mathbb{P}(V)$, where $V$ is a finite-dimensional $G$-module. Moreover, we may assume that $Y \subseteq Z$
is a closed $G$-orbit. In particular, $Y$ is real spherical by Corollary 3.9, and we let $Q_Y < G$ be a $Y$-adapted parabolic.

Under these assumption on $Y$ and $Z$ there is the following immediate generalization of Lemma 3.6.

**Lemma 3.11.** Let $Z$ be real spherical variety, closed in $\mathbb{P}(V)$, and $Y \subseteq Z$ a closed $G$-orbit. Then there exists a $Q_Y$-stable affine open subset $Z_0 \subseteq Z$ which meets $Y$ and such that the restriction mapping

$$\mathbb{R}[Z_0](Q_Y) \to \mathbb{R}[Z_0 \cap Y](Q_Y)$$

is onto.

**Proof.** The proof is analogous to that of Lemma 3.6. We obtain that $Z_0$ is the non-vanishing locus of a $Q_Y$-semi-invariant homogeneous polynomial function on $V$. \hfill \Box

**Corollary 3.12.** Let $Z \subseteq \mathbb{P}(V)$ be a closed real spherical variety and $Y$ an elementary closed subvariety. Then there exists a $G$-stable affine open subset $Z_0 \subset Z$ such that $Z_0 \cap Y \neq \emptyset$.

**Proof.** One has $Q_Y = G$. \hfill \Box

We now start with the construction of $Z_0$. If $Y$ is elementary, $Z_0$ is given by Corollary 3.12. So let us assume that $Y$ is not elementary, i.e. $G_n$ does not act trivially on $Y$. Let $P = MAN$ be opposite to $P$. As $Y \subseteq \mathbb{P}(V)$ is closed, we can find a vector $y_0 \in V$ such that $[y_0] \in Y$ is $AN$-fixed, and such that $A$ acts by a non-trivial character on $y_0$. This can be seen as follows. Assume for simplicity that $V$ is irreducible. Then $Y$ contains a vector $y$ of which the $A$-weight decomposition has a non-trivial component $y_0$ in the lowest weight space of $V$. Compression of $y$ by $A^+$ then exhibits a non-zero multiple of $y_0$ as a limit of elements from $Y$.

Next we choose $v_0^* \in V^*$ such that $[v_0^*] = AN$-fixed and $v_0^*(y_0) = 1$. Let $\chi : A \to \mathbb{R}^+$ be the character defined by $a \cdot v_0^* = \chi(a)v_0^*$.

Consider the function

$$F : V \to \mathbb{R}, \quad v \mapsto \int_M v_0^*(m \cdot v)^2 \, dm$$

and note that

$$F(man \cdot v) = \psi(a)F(v)$$

for all $man \in MAN$ and $v \in V$, where $\psi = \chi^{-2}$. Further, $F$ is real algebraic and homogeneous of degree 2. Thus $\{[v] \in \mathbb{P}(V) \mid F(v) \neq 0\}$ defines an affine open set in $\mathbb{P}(V)$ and the intersection with $Z$ yields an affine open set $Z_0$. Note that $F$ is not constant and hence $Z_0$ is a proper subvariety. We define $Q \supseteq P$ to be the parabolic subgroup which fixes the line $\mathbb{R}F|_{Z_0}$, that is, $Q = \{g \in G \mid gZ_0 = Z_0\}$.

As before, we define on $Z_0$ a moment-type map

$$\mu : Z_0 \to g^*, \quad \mu(z)(X) := \frac{dF(v)(X)}{F(v)}$$

for $z = [v] \in Z \subseteq \mathbb{P}(V)$. This map is algebraic and $Q$-equivariant. Let $U < Q$ be the unipotent radical.

We claim that $\text{im } \mu$ is a $Q$-orbit. In fact for $X \in q$ we have $\mu(z)(X) = d\psi(X)$ for all $z \in Z$, and after identifying $g$ with $g^*$ we obtain, as in the previous section, that

$$\text{im } \mu = \text{Ad}(Q)X_0 = X_0 + u$$
with $X_0 = \mu([y_0])$. The stabilizer of $X_0$ determines a Levi subgroup $L < Q$. Then $S := \mu^{-1}(X_0)$ is an $L$-stable affine subvariety of $Z_0$ and we obtain an algebraic isomorphism

$$Q \times_L S \to Z_0.$$ 

The affine $L$-variety $S$ is real spherical and meets $Y$. We continue the procedure with $(L, S, S \cap Y)$ instead of $(G, Z, Y)$. The procedure will stop at the moment when $S \cap Y$ is fixed under $L_n$. We have thus shown the following result.

**Theorem 3.13 (Local structure theorem, general case).** Let $Z$ be a real spherical variety and $Y \subseteq Z$ a closed $G$-invariant subvariety. Then there is parabolic subgroup $Q \supseteq P$ with Levi decomposition $Q = LU$ with the properties that there is a $Q$-invariant affine open piece $Z_0 \subseteq Z$ meeting $Y$ and an $L$-invariant closed spherical subvariety $S \subseteq Z_0$ such that:

(i) there is a $Q$-equivariant isomorphism

$$Q \times_L S \to Z_0;$$

(ii) $S \cap Y$ is an elementary spherical $L$-variety.

4. The normalizer of a spherical subalgebra

As in the preceding section, we assume that $G$ is algebraic and let $\mathfrak{h}$ be the Lie algebra of a spherical subgroup $H < G$. We denote by $\mathfrak{h} := \mathfrak{n}_0(\mathfrak{h})$ the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$ and by $\mathfrak{H}$ the normalizer in $G$. Note that $\mathfrak{h} \triangleleft \mathfrak{h}$ is an ideal. Let $\mathfrak{p}$ be a minimal parabolic subalgebra such that $\mathfrak{p} + \mathfrak{h} = \mathfrak{g}$ and let $\mathfrak{q}$ denote the unique parabolic subalgebra above $\mathfrak{p}$, which is $\mathfrak{Z}$-adapted. Let $\mathfrak{Z} = G/\mathfrak{H}$.

**Lemma 4.1.** The parabolic subalgebra $\mathfrak{q}$ is also $\mathfrak{Z}$-adapted.

**Proof.** We write $\mathfrak{q}$ for the unique $\mathfrak{Z}$-adapted parabolic above $\mathfrak{p}$ and $\mathfrak{u}$ for its unipotent radical. Then

$$\mathfrak{n} = (\mathfrak{n} \cap \mathfrak{h}) \oplus \mathfrak{u} = (\mathfrak{n} \cap \mathfrak{h}) \oplus \tilde{\mathfrak{u}}.$$ 

It follows that $\tilde{\mathfrak{u}} \subseteq \mathfrak{u}$ and $\mathfrak{q} \subseteq \mathfrak{q}$. To obtain a contradiction we assume that $\mathfrak{q} \subsetneq \mathfrak{q}$. Then $\tilde{\mathfrak{u}} \subsetneq \mathfrak{u}$ and $\mathfrak{n} \cap \mathfrak{h} \subsetneq \mathfrak{n} \cap \mathfrak{h}$. In particular, the Lie algebra $\mathfrak{h}/\mathfrak{h}$ cannot be compact.

To conclude the proof we now show that $\mathfrak{h}/\mathfrak{h}$ is compact. Suppose first that $Z$ is quasi-affine and let $\mathbb{C}[Z] = \bigoplus_{\pi \in \hat{G}} \mathbb{C}[Z]_\pi$ be the decomposition of the $G$-module $\mathbb{C}[Z]$ into $G$-isotypical components. For each $\pi$ we choose a model space $V_\pi$ and let $\mathcal{M}_\pi := \text{Hom}_G(V_\pi, \mathbb{C}[Z])$ be the corresponding multiplicity space. Note that $\mathcal{M}_\pi$ is finite dimensional as there is a natural identification of $\mathcal{M}_\pi$ with the space of $H$-fixed elements in $V^*_\pi$.

Let $C := \mathfrak{h}/\mathfrak{h}$. Note that $C$ acts from the right on $\mathbb{C}[Z]$ and preserves each $\mathbb{C}[Z]_\pi$, thus inducing an action on $\mathcal{M}_\pi$. Since $Z$ is quasi-affine we can choose finitely many $\pi_1, \ldots, \pi_k$ so that we obtain a faithful representation of $C$ on the sum $\mathcal{M} := \bigoplus_{j=1}^k \mathcal{M}_{\pi_j}$.

Let $B < G_C$ be a Borel subgroup contained in $P_C$. For every $\pi$ we let $v_\pi$ be a $B$-highest weight vector in $V_\pi$. To every $\eta \in \mathcal{M}_\pi$ we associate the function $f_\eta(g) := \eta(\pi(g^{-1})v_\pi)$ and define an inner product on $\mathcal{M}_\pi$ by

$$\langle \eta, \eta \rangle := (|f_\eta|^2)_{M}(z_0)$$

with the notation of (3.1). As $(|f_\eta|^2)_M$ is a matrix coefficient of a representation in $\Lambda$, and as multiplicities for these are at most one by Proposition 3.7, we obtain that there is a real character $\chi_\pi : C \to \mathbb{R}^\times$ such that

$$\langle h \cdot \eta, h \cdot \eta \rangle = \chi_\pi(h) \langle \eta, \eta \rangle_{\pi}.$$
The group $C_1 := \bigcap_{j=1}^k \ker \chi_{r_j}$ acts unitarily and faithfully on $\mathcal{M}$, hence is compact. By definition $C/C_1 < (\mathbb{R}^\times)^k$, hence the Lie algebra of $C$ is compact.

Finally, we reduce to the quasi-affine case using the affine cone over $\mathbb{P}(V)$ as before; see the proof of Corollary 3.8. \hfill $\Box$

Let $Q = LU$ be a Levi decomposition as in Definition 2.7 and recall the decomposition (2.10).

**Proposition 4.2.** The normalizer $\tilde{h}$ of $h$ is of the form

$$\tilde{h} = h \oplus \tilde{c}$$

(4.1)

with $\tilde{c}$ a subalgebra of the form $\tilde{c} = \tilde{a} \oplus \tilde{m}$ where $\tilde{a} < \mathfrak{z}(l)_{np}$ and $\tilde{m} < \mathfrak{z}(l)_{cp} + L_c$.

**Proof.** From Lemma 4.1 we conclude that $\tilde{h} = h + \tilde{h} \cap L$, and we obtain (4.1) with a subspace $\tilde{c}$ of $\mathfrak{z}(l) + L_c$. It is a subalgebra because $\mathfrak{z}(l) + L_c$ is reductive and $h$ is an ideal in $\tilde{h}$.

Write $\tilde{a}$ for the orthogonal projection of $\tilde{c}$ to $\mathfrak{z}(l)_{np}$ and $\tilde{m}$ for the orthogonal projection of $\tilde{c}$ to $\mathfrak{z}(l)_{cp} + L_c$. Then $\tilde{c} \subseteq \tilde{a} + \tilde{m}$, and it remains to show equality. This will follow if we can show that both $\tilde{a}$ and $\tilde{m}$ normalize $h$. For that we decompose $X \in \tilde{c}$ as $X = X_a + X_m$ with $X_a \in \tilde{a}$ and $X_m \in \tilde{m}$. Observe that $ad X_a$ commutes with $ad X_m$. Both operators are diagonalizable with real (respectively, imaginary) spectrum. As $ad X$ preserves $h$ we therefore conclude that $ad X_a$ and $ad X_m$ preserve $h$ as well. \hfill $\Box$

**Corollary 4.3.** Let $H \subseteq G$ be real spherical. Then $N_G(H)/H$ is an elementary group.

**Corollary 4.4.** The normalizer $\tilde{h}$ is its own normalizer: $\tilde{h} = \tilde{h}$.

**Proof.** It suffices to show that the normalizer $\tilde{h}$ of $\tilde{h}$ normalizes $h$ as well. Let $\tilde{H} = N_G(h)$. Observe that $\tilde{H}/H$ is an elementary real algebraic group; in particular, it is reductive. Thus, $\tilde{h}_u = h_u$ for the nilpotent radicals. This implies that $\tilde{h}$ normalizes $h_u$ and that $\tilde{H}/H_u$ is a reductive real algebraic group. A connected group, which acts by algebraic automorphisms on a reductive Lie group, acts by inner automorphisms, hence fixes every ideal. Thus $h/h_u \subseteq \tilde{h}/h_u$ is normalized by $\tilde{h}$ as well. \hfill $\Box$

**Remark 4.5.** On the group level, the statement is wrong. For example, let $G = GL(2, \mathbb{R})$ and $H = \left(\begin{smallmatrix} * & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $N_G(H) = T = \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$. Thus $N_G(N_G(H)) = N_G(T)$ is strictly larger than $N_G(H) = T$.

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